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CONVERGENT CORRELATION FUNCTION FOR A TWO-COMPONENT PLASMA

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A satisfactory account of the short-range attractive interaction in a two-component high-temperature plasma requires a quantum-mechanical treatment. The inadequacy of the classical theory is exhibited[1] by the singular behavior of the electron-ion correlation function for particle separations less than the Landau length $(=e^2/kT)$.

The purpose of the present note is to outline a quantum-mechanical analysis of the short-range part of the electron-ion correlation function. The starting point of the calculation is a two-particle Wigner distribution function. The corresponding radial distribution function is then written in the form given by Goldberger and Adams.[2] As expected, it is found that the finite value of Planck's constant provides a natural short-range cutoff for the divergence appearing in the classical calculation. The distance at which this effect takes place is on the order of $(l\lambda^2)^{1/3}$, where l is the Landau length and λ is the thermal de Broglie wavelength.

For a one-component plasma, it has been shown[3] that particle correlations are due to short-range two-body encounters for particles closer than the distance $r_0=(lL/2)^{1/2}$, where l is again the Landau length and L the Debye length $[=(kT/4\pi ne^2)^{1/2}]$. For particle separations greater than r_0, the correlation becomes dominated by collective effects. The transition from two-body to collective interaction may also be shown to take place at this distance in a two-component plasma. Since the quantum-mechanical effects will occur only at distances much less than r_0, only this inner region need be considered and only a two-particle Wigner distribution function need be calculated.

Only the radial distribution function, the result of integrating the Wigner distribution function over particle momenta, is of interest. This takes the standard form

$$n(r)=\sum_{\vec{k}}\psi^*_{\vec{k}}(\vec{r})e^{-\beta H}\psi_{\vec{k}}(\vec{r}), \tag{1}$$

where $\vec{r}$ is the relative-position vector of the two particles, $\beta=1/kT$, H is the energy operator for the two-particle system having an interaction potential $V(r)$, and the $\psi_{\vec{k}}(\vec{r})$ are eigenfunctions of the operator H. In terms of plane waves this may be rewritten[2,4]

$$n(r)=\int\frac{d^3\vec{p}}{(2\pi)^3}\exp(-p^2)\psi_{\vec{p}}^*(\vec{r}) \times\exp[-(\lambda^2\nabla^2+U)]\psi_{\vec{p}}(\vec{r}), \tag{2}$$

where $\vec{p}$ is the relative particle momentum measured in units of $(2mkT)^{1/2}$, $\lambda=(\hbar^2/2mkT)^{1/2}$ is the thermal de Broglie wavelength, and $U=\beta V$.

As shown in reference 2, a sequence of transformations may be employed to simplify Eq. (2). The first transformation converts Eq. (2) to the form

$$n(r)=\pi^{-3/2}\int d^3\vec{p}\exp(-p^2)\times\exp[-\mathcal{V}(\vec{r}+2i\lambda\vec{p},1)]v_1(1), \quad (3)$$

which has been normalized to reduce to unity for a perfect gas. The function $\mathcal{V}$ is related to $U(r)$ by

$$\mathcal{V}(\vec{r},1)\equiv\int_0^1 ds_1\,U(\vec{r}-2i\lambda\vec{p}s_1). \quad (4)$$

In Eq. (3), the function $v_1(1)$ contains functions of r which are of order λ^2. As is shown in reference 2, it is possible to extract from $v_1(1)$ another exponential times a function $v_2(1)$ containing functions of r of order λ^4, and so on. It is found that the short-range divergence of the classical theory is eliminated after the first transformation. Additional quantum corrections to this result will be neglected. The function $v_1(1)$ may, therefore, be set equal to unity, and Eq. (3) written as

$$n(r)=\pi^{-3/2}\int d^3\vec{p}\exp(-p^2)\exp[-\mathcal{V}(\vec{r}+2i\lambda\vec{p},1)]. \quad (5)$$

Evaluating Eq. (4) for the Coulomb potential $V(r)=e_1e_2/|\vec{r}|$, one finds

$$\mathcal{V}(\vec{r}+2i\lambda\vec{p},1)=-i\nu\{\sinh^{-1}[\cot\theta+i(a/\nu)\csc\theta]-\sinh^{-1}(\cot\theta)\}, \quad (6)$$

where $\cos\theta=\vec{r}\cdot\vec{p}/rp$, $\nu=l/2\lambda p$, and $a=l/r$. The constant l, which is now equal to e_1e_2/kT, may be of either sign. When Eqs. (5) and (6) are combined, the highly transcendental nature of the result may be eliminated with the help of the identity

$$\exp[n\sinh^{-1}b]=[b+(b^2+1)^{1/2}]^n. \quad (7)$$

The angular integration in Eq. (5) is then found to be amenable to considerable algebraic simplification. After some simple transformations Eq. (5) may be reduced to the form

$$n(r)=2\pi^{-1/2}\int_0^\infty p^2dp\exp(-p^2)\frac{\nu}{ia}\int_{z^*}^{z}tdt\left(\frac{t+z}{t+z^*}\right)^{i\nu}, \quad (8)$$

$$n(r)=8\pi^{-1/2}\int_0^\infty p^2dp\exp(-p^2)\,\mathrm{Re}[zH(z,\nu)], \quad (9)$$

where $z=1+ia/\nu$, and Re signifies the real part of the quantity in brackets. The function $H(z,\nu)$ is defined by

$$H(z,\nu)\equiv\int_{1/z^*}^{z}dy\,y^{i\nu}(1-y)^{-3}. \quad (10)$$

For a high-temperature plasma, $|l|$ is much larger than λ, and since the p integration cuts off sharply for $p>1$, only values of $\nu\gg1$ are important in Eq. (10). It is therefore sufficient to develop the integral in Eq. (10) into an asymptotic series by partial integration. Retaining only the first term in this expansion yields

$$H(z,\nu)\simeq-(\nu^2/a^3)[z^{i\nu+1}-(z^*)^{-i\nu+2}]. \quad (11)$$

For $r>\lambda$, a/ν is less then unity, although a^2/ν may be large. Writing $z=\exp(\ln z)$ in Eq. (11), expanding the logarithm in powers of a/ν, and retaining only the first two terms, one finds

$$\mathrm{Re}[zH(z,\nu)]=(\nu/a^2)e^{-a}\sin(a^2/2\nu). \quad (12)$$

Since one does not seek detailed information at distances on the order of λ, but merely requires a theory which converges as $r\to0$, this approximation is adequate.

With this result the p integration in Eq. (9) is readily performed. The final expression for the radial distribution function at short distances is found to be

$$n(r)=\exp\left(-\frac{l}{r}-\frac{l^2\lambda^2}{4r^4}\right),\quad r<r_0. \quad (13)$$

The second term in the exponential is independent of the sign of l and for negative l it provides a natural cutoff for small separation distances. When l is negative the maximum value of $n(r)$ is found to occur at $r=(|l|\lambda^2)^{1/3}$. For $r>r_0$, the usual Debye shielding result must, of course, be used.

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[1]E. A. Frieman and D. L. Book, Phys. Fluids 6, 1700 (1963).
[2]M. L. Goldberger and E. N. Adams, II, J. Chem. Phys. 20, 240 (1952).
[3]G. L. Lamb, Jr., and B. Burdick, Phys. Fluids 7, 1087 (1964).
[4]J. Mayer and W. Band, J. Chem. Phys. 15, 141 (1947).