EXACT BOOTSTRAPS IN SOME STATIC MODELS*

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We have studied the known solutions[1,2] of some exactly soluble static models of meson-baryon scattering (to wit: neutral, charged, and symmetric scalar, and neutral pseudoscalar theories in the one-meson approximation) to see whether any of them possesses bootstrap solutions. We define such a solution to be any solution of the dispersion equations that satisfies Levinson's theorem[3]:

$$\delta_\alpha(\infty)-\delta_\alpha(1)=-\pi b_\alpha, \qquad (1)$$

where $\delta_\alpha(\omega)$ is the phase shift in the channel α, ω is the meson energy, and b_α is the number of stable particles in the channel α.

All these theories satisfy the intuitive requirement for possessing bootstrap solutions: Exchange of a particle in the u channel produces an attractive potential in the s channel. In our static models, s becomes ω and u becomes $-\omega$.

We try to take into account effects of closed channels and unknown high-energy behavior by introducing a cutoff function and arbitrary subtractions in the dispersion relations. The cutoff function is chosen to be $v(\omega)=\kappa^{2c}/(q^2+\kappa^2)^c$, where $q=(\omega^2-1)^{1/2}$, so defined that $\mathrm{Im}q \geqslant 0$ on the physical sheet of the complex ω plane, and $c=0, 1, 2, \cdots$.

Our results are as follows:

(1) For all of these models no bootstrap solutions exist that satisfy unsubtracted dispersion relations. Bootstrap solutions do exist that satisfy subtracted dispersion relations. A subtraction can only be made if a cutoff function is also introduced. It is interesting that these models differentiate so sharply between cutoff functions of arbitrary power on the one hand, and subtractions on the other. This suggests that the existence and properties of bootstrap solutions for the complete S matrix must be extremely sensitive to the finer details of high-energy phenomena.

(2) The only bootstrap solution to the charged scalar theory is equivalent to the neutral scalar theory (for which there is a unique boostrap solution):

$$f(\pi^+p)=f(\pi^-p)=v(\omega)/[g-L(\omega)], \qquad (2)$$

where f is the scattering amplitude, g is an arbitrary positive constant, and

$$L(\omega)=\left(\frac{2}{\pi}\right)\omega^2\int_1^\infty d\omega'\frac{v(\omega')(\omega'^2-1)^{1/2}}{[\omega'^2(\omega'^2-\omega^2-i\epsilon)]}. \qquad (3)$$

Evidently f/v satisfies a once-subtracted dispersion relation, with g acting as the effective subtraction constant. For $g>L(1)$ there is no bound state. For $g<L(1)$ there is one bound state. For $g=0$ the bound state is at $\omega=0$. The solution is labeled by three arbitrary parameters: the cutoff power $c\geqslant 1$, the cutoff momentum $\kappa>1$, and the effective subtraction constant $g\geqslant 0$.

(3) Scattering in the symmetric scalar theory proceeds through the $I=\frac{1}{2}$ and $I=\frac{3}{2}$ channels. The number of bootstrap solutions satisfying a once-subtracted dispersion relation is greater than one. These solutions have either $c=2$ or $c=1$. If we apply the physical requirement that the meson-baryon coupling constant must not vanish (i.e., the target baryon must appear as a bound state in the appropriate channel), then we must have $c=1$, and the bootstrap solution is unique. The S-matrix elements for that case are given by

$$S_{1/2}=[B/(B-1)][(B-2)/(B+1)]D,$$
$$S_{3/2}=[B/(B-1)]D, \qquad (4)$$

where

$$B(\omega)=\tfrac{1}{2}+i[\pi^{-1}\ln(\omega+q)-(\omega/q)\beta_0] \quad (\beta_0\neq 0), \qquad (5)$$

$$D(\omega)=\frac{(1-iq)(1-iq/\kappa)(1-iq/s_2)(1+iq/s_1)}{(1+iq)(1+iq/\kappa)(1+iq/s_2)(1-iq/s_1)} \quad (\beta_0<0),$$

$$=\frac{(1-iq)(1-iq/\kappa)(1-iq/s_0)(1+iq/s_1)}{(1+iq)(1+iq/\kappa)(1+iq/s_0)(1-iq/s_1)} \quad (\beta_0>0), \qquad (6)$$

with $s_\gamma\equiv(1-\omega_\gamma^2)^{1/2}$, where ω_γ is the unique root of the equation $B(\omega_\gamma)=\gamma$ satisfying the following conditions: For $\gamma=0$, either $0<\omega_0<1$, or ω_0 is pure imaginary; for $\gamma\geq 1$, $0<\omega_\gamma<1$.

For $\beta_0<0$ there is a bound state in the $I=\frac{1}{2}$ channel at $\omega=0$, which represents the target baryon, and a bound state in the $I=\frac{3}{2}$ channel at $\omega=\omega_2$. The respective squared coupling constants are

$$\lambda_1=3\left(\frac{1}{\pi}-\beta_0\right)\frac{(1-s_1)(1+s_2)}{(1+s_1)(1-s_2)}\left(1+\frac{1}{\kappa}\right)^2,$$

$$\lambda_3=\frac{2s_2(1+s_2)(s_1-s_2)}{\omega_2(1-s_2)(s_1+s_2)}\left(1-\frac{1}{\kappa^2}\right)\frac{\kappa+s_2}{\kappa-s_2}. \tag{7}$$

They are always positive because $0<s_2<s_1<1$.

For $\beta_0>0$ there is a bound state in the $I=\frac{1}{2}$ channel at $\omega=0$, and no bound state in the $I=\frac{3}{2}$ channel. The meson-baryon squared coupling constant is in this case

$$\lambda_1=3\left(\frac{1}{\pi}-\beta_0\right)\frac{(1+s_0)(1-s_1)}{(1-s_0)(1+s_1)}\left(1+\frac{1}{\kappa}\right)^2, \tag{8}$$

and is again always positive because $0<s_1<1$, and ω_0 is real for $0<\beta_0<\pi^{-1}$, pure imaginary for $\beta_0>\pi^{-1}$.

The solution is labeled by two arbitrary parameters: the cutoff momentum $\kappa>1$, and the effective subtraction constant $\beta_0\neq 0$.

(4) Scattering in the neutral pseudoscalar theory proceeds through the $J=\frac{1}{2}$ and $J=\frac{3}{2}$ channels. The number of bootstrap solutions satisfying a once-subtracted dispersion relation is again greater than one. These solutions have either $c=3$ or $c=2$. If we again apply the physical requirement that the meson-baryon coupling constant must not vanish, and further require that no bound state can have a smaller mass than the target baryon (i.e., no inelastic threshold lower than the elastic threshold), then we must have $c=2$, and the bootstrap solution is unique. The S-matrix elements are given by

$$S_{1/2}=[B/(B-1)][(B-2)/(B+1)]D,$$

$$S_{3/2}=[B/(B-1)]D, \tag{9}$$

where

$$B(\omega)=\tfrac{1}{2}+i[\pi^{-1}\ln(\omega+q)-(\omega/q^3)(\beta_0+\beta_1\omega^2)]\quad(\beta_0+\beta_1\neq 0). \tag{10}$$

The two constants β_0 and β_1 are related to each other by a threshold condition given below. The function $D(\omega)$ is a real analytic rational function of q, such that $D(\omega=1)=1$, and $|D(\omega)|=1$ for $\omega\geq 1$. It is uniquely specified by the requirements that

$D(\omega)$ have double poles at $\pm i(\kappa^2-1)^{1/2}$,

simple poles at all the roots of $B(\omega)=0$,

simple zeros at all the roots of $B(\omega)=-1$. (11)

The threshold condition is

$$D(\omega)\xrightarrow[\omega\to 1]{} 1+O(q^3),$$

which relates β_0 and β_1 for a given value of κ. For all β_0, β_1, and κ, the target baryon is the only bound state in the $J=\frac{1}{2}$ channel, and there are no bound states in the $J=\frac{3}{2}$ channel. The solution is labeled by two arbitrary parameters: the cutoff momentum $\kappa>1$, and the effective subtraction constant β_0 (or β_1).

The explicit form of $D(\omega)$ depends on the location of the roots of $B(\omega)=\gamma$, and hence on β_0 and β_1. For $\beta_0>0$, $\beta_1>0$, $B(\omega)=0$ has five roots at 0, $\pm x$, $\pm x^*$, where x is complex; $B(\omega)=-1$ has three roots at $-\omega_1$, y, y^*, where $0<\omega_1<1$, and y is complex, with $\mathrm{Re}\,y>1$. Letting $s_1=(1-\omega_1^2)^{1/2}$, $q_0=(x^2-1)^{1/2}$, and $q_1=(y^2-1)^{1/2}$, we have

$$D(\omega)=\left(\frac{1-iq/\kappa}{1+iq/\kappa}\right)^2\frac{(1-iq)(1+iq/s_1)(1+q/q_0)(1-q/q_0^*)(1-q/q_1)(1+q/q_1^*)}{(1+iq)(1-iq/s_1)(1+q/q_0^*)(1-q/q_0)(1-q/q_1^*)(1+q/q_1)}. \tag{12}$$

The meson-baryon squared coupling constant is given by

$$\gamma_1 = 3\left(\frac{1}{\pi}+\beta_0\right)\frac{1-s_1}{1+s_1}\left(1+\frac{1}{\kappa}\right)^4. \tag{13}$$

The threshold condition reads

$$s_1^{-1}+2\,\mathrm{Im}(q_0^{-1}-q_1^{-1}) = 1+2\kappa^{-1}, \tag{14}$$

which, for $\kappa \sim 1$, approximately reduces to

$$\beta_1+\beta_0 \approx \tfrac{3}{2}(1+\kappa^{-1})^{-3}. \tag{15}$$

There is resonance in the $J=\frac{3}{2}$ channel represented by a pole of $S_{3/2}$ at $\omega = y$ on the second Riemann sheet. For $\kappa \sim 1$ the pole is near threshold, with position and width given, respectively, by the approximate expressions

$$\mathrm{Re}y \approx 1+\tfrac{1}{4}(1+\kappa^{-1})^{-2},$$

$$\mathrm{Im}y \approx \frac{\sqrt{3}}{4}(1+\kappa^{-1})^{-2}. \tag{16}$$

(5) Applying the usual bootstrap philosophy to these models, we would expect the positions ω_i and squared coupling constants λ_i of bound states (or virtual states, or resonances) to be determined by a set of equations, which for the case of two bound states (one of them being the target baryon located at $\omega_1 = 0$) should be of the form

$$\begin{aligned} 0 &= f_1(\lambda_1,\lambda_2,\omega_2,g,\kappa),\\ \omega_2 &= f_2(\lambda_1,\lambda_2,\omega_2,g,\kappa),\\ \lambda_1 &= F_1(\lambda_1,\lambda_2,\omega_2,g,\kappa),\\ \lambda_2 &= F_2(\lambda_1,\lambda_2,\omega_2,g,\kappa), \end{aligned} \tag{17}$$

where g is the subtraction constant and κ the cutoff momentum. (The mass of the scattered meson acts as a scaling parameter, which is set equal to unity.) From (17) we should expect ω_2, λ_1, and λ_2 to be determined up to one arbitrary parameter. The results in all the models studied disagree with this counting. We may perhaps understand the nature of this disagreement by considering the present theory as the limit of a relativistic theory in which the mass of the target baryon (M) is to be inserted into Eq. (17) as follows:

$$\begin{aligned} M &= M+f_1(\lambda_1,\lambda_2,\omega_2,g,\kappa,M),\\ M+\omega_2 &= M+f_2(\lambda_1,\lambda_2,\omega_2,g,\kappa,M),\\ \lambda_1 &= F_1(\lambda_1,\lambda_2,\omega_2,g,\kappa,M),\\ \lambda_2 &= F_2(\lambda_1,\lambda_2,\omega_2,g,\kappa,M). \end{aligned} \tag{18}$$

We see that Eqs. (18) have the correct number of variables to allow two free parameters. If now a static limit is to make sense, the functions f_i and F_i must be insensitive to M, and therefore one of Eqs. (17) must be an identity.

Details of this work will be published elsewhere.

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SPIN AND UNITARY-SPIN INDEPENDENCE IN A PARAQUARK MODEL OF BARYONS AND MESONS

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Wigner's supermultiplet theory,[1] transplanted independently by Gürsey, Pais, and Radicati,[2] and by Sakita,[2] from nuclear-structure physics to particle-structure physics, has aroused a good deal of interest recently. In the nuclear supermultiplet theory, the approximate independence of both spin and isospin of those forces relevant to the energies of certain low-lying bound states (nuclei) makes it useful to classify the states according to irreducible representations of SU(4). Parallel to this, in the particle supermultiplet theory, the possible independence of both spin and unitary spin of those forces relevant to the masses of certain low-lying bound states (particles) makes it interesting to classify the states according to irreducible representations of SU(6). Three results associated with this SU(6) classification indicate its usefulness: (1) The best known baryons (in particular, the spin-$\frac{1}{2}^+$ baryon octet and the spin-$\frac{3}{2}^+$ baryon decuplet) are grouped into a supermultiplet containing 56 particles.