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SU(4) MASS FORMULA*

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(Received 28 September 1964)

There have been a number of speculations on the possibility of a symmetry scheme of fundamental particles based on the group SU(4).[1] These ideas extend the SU(3) symmetry scheme[2] in much the same way that the SU(3) scheme extends the isospin symmetry. It must be stressed, however, that unlike the search for SU(3) there is very little if any experimental motivation for this extension. In view of the success of the empirical SU(3) mass formula,[3] which says that the mass splittings transform like the $Y=0$ isospin-singlet component of the regular representation of SU(3), i.e., the $\underline{8}$, it is a natural first guess to presume that the mass splittings in the SU(4) scheme transform like a component of the regular representation of SU(4), i.e., the $\underline{15}$. In the following, we give the mass formula for an arbitrary representation of SU(4) based on the assumption that the mass splittings transform like a component of the $\underline{15}$. This general result is greatly simplified by using different SU(3) subgroups of SU(4). To emphasize the general usefulness of the various subgroups, the electromagnetic properties in SU(4) are briefly discussed also.

Since SU(4) is a group of rank 3, there are three independent conserved quantum numbers which we take to be the charge Q, the hypercharge Y,[4] and a new quantum number X. (Q is more convenient than I_3 for discussing electromagnetic interactions.) The $\underline{15}$ contains the following SU(3) submultiplets: a $\underline{3}$ with $X=1$, a $\underline{1}$ and an $\underline{8}$ with $X=0$, and a $\underline{3}^*$ with $X=-1$. The most general form of the mass splittings contained in the $\underline{15}$ compatible with Q, Y, X, and I conservation is a superposition of the SU(3) singlet and the isoscalar component of the SU(3) octet ($Q=Y=X=0$). [The SU(3) subgroup we refer to here is the one whose multiplets lie in planes orthogonal to the direction of the new quantum number X.]

Consider first the splittings which transform like the $Q=Y=X=0$ SU(3) singlet. These do not give splittings within a given SU(3) multiplet but do split apart different SU(3) multiplets. Since the $Q=Y=X=0$ SU(3) singlet is the component $\psi_4{}^4$ of the traceless tensor $\psi_\mu{}^\nu$ representing the $\underline{15}$, the general form of these mass splittings for any SU(4) representation[5] (L,M,N) is obtained by applying the Wigner-Eckart theorem for SU(4) to the matrix element of the component $T_4{}^4$ of a traceless tensor $T_\mu{}^\nu$. The general form of the matrix element of the component $T_4{}^4$ of an arbitrary traceless tensor $T_\mu{}^\nu$ in the SU(4) representation $(L,M,N,)$ turns out to be

$$T_4{}^4=a+bX+c[C_2{}^{(x)}-X^2]+d[C_3{}^{(x)}-3XC_2{}^{(x)}+X^3], \quad (1)$$

where C_2 and C_3 are the Casimir operators of the SU(3) subgroup orthogonal to the direction of X. For the SU(3) multiplet (L,M) which lies in the plane $X = L-M$ they are explicitly given by

$$C_2 = L^2 + LM + M^2 + 3L + 3M, \tag{2}$$

and

$$C_3 = (L-M)(2L + M + 3)(L + 2M + 3). \tag{3}$$

The parameters a, b, c, and d depend on the particular SU(4) representation and, of course, the space-time properties of the multiplet, e.g., J^π.

A brief derivation of Eq. (1) can be based on the following theorem: The number of independent terms in the matrix element of $T_4{}^4$ in the representation (L,M,N) is just the number of times the $\underline{15}$ occurs in the product $(L,M,N)\otimes(N,M,L)$ and this can be shown to be precisely the number of the integers L, M, and N which are not zero.[6] Consequently, such mass splittings are described by a maximum of three parameters, and fewer for certain representations. It follows that the product of the $\underline{15}$ $(1,0,1)$ with any of the tetrahedral representations $(L,0,0)$ or $(0,0,N)$ contains the same tetrahedral representation exactly once. Therefore the splitting of such a representation will be linear in X, i.e., the levels will be equally spaced. More generally there can be terms up to third order so that for an arbitrary representation we have

$$T_4{}^4 = a + bX + c(C_2 + \alpha X^2) + d(C_3 + \beta X C_2 + \gamma X^3). \tag{4}$$

Requiring that Eq. (4) reduce to a linear form for the tetrahedral representations determines α, β, and γ and gives Eq. (1).

It is simple to adjoin to Eq. (1) the terms which transform like the $Q = Y = X = 0$ isospin singlet member of the SU(3) octet contained in the $\underline{15}$. These additional splittings are obtained by applying the Wigner-Eckart theorem to the component $T_3{}^3$ of the general traceless tensor $T_\mu{}^\nu$. However, observe that $T_3{}^3$ bears the same relation to the direction of Y as $T_4{}^4$ bears to the direction of X. Therefore, using the previous result [Eq. (1)], the most general first-order mass formula is[7]

$$\begin{aligned} M = M_0 + M_1{}^{(x)}X + M_2{}^{(x)}[C_2{}^{(x)} - X^2] &+ M_3{}^{(x)}[C_3{}^{(x)} - 3XC_2{}^{(x)} + X^3] \\ &+ M_1{}^{(y)}Y + M_2{}^{(y)}[C_2{}^{(y)} - Y^2] \\ &+ M_3{}^{(y)}[C_3{}^{(y)} - 3YC_2{}^{(y)} + Y^3], \end{aligned} \tag{5}$$

where the Casimir operators $C_2{}^{(x)}$ and $C_3{}^{(x)}$ refer to the SU(3) subgroup orthogonal to the X direction while $C_2{}^{(y)}$ and $C_3{}^{(y)}$ are the corresponding Casimir operators referring to the SU(3) subgroup orthogonal to the Y direction. For any SU(3) multiplet they are given by Eqs. (2) and (3). The parameters $M_0, M_i{}^{(x)}$, and $M_i{}^{(y)}$ in Eq. (5) depend only on the SU(4) multiplet (besides space-time properties) and hence are functions of the SU(4) Casimir operators only.

If one regards Eq. (5) as an SU(3) mass formula for each value of X, then part of the dependence of the parameters on the Casimir operators appears explicitly. Thus the parameters in the SU(3) mass formula for different multiplets of particles of the same spin and parity can be related in this scheme.

In applying the SU(4) mass formula, one must bear in mind that in general $C_i{}^{(x)}$ and $C_i{}^{(y)}$ do not commute so that the eigenstates with respect to one SU(3) subgroup are not necessarily eigenstates with respect to the other. For example, in the center of the $\underline{15}$ $(Q = Y = X = 0)$ if $\psi_1{}^{(x)}$ and $\psi_8{}^{(x)}$ are, respectively, the SU(3) singlet and octet member with respect to the X direction, then with respect to the Y direction $\psi_1{}^{(y)} = \frac{1}{3}\psi_1{}^{(x)} - \frac{2}{3}\sqrt{2}\,\psi_8{}^{(x)}$ and $\psi_8{}^{(y)} = \frac{1}{3}\psi_8{}^{(x)} + \frac{2}{3}\sqrt{2}\,\psi_1{}^{(x)}$ are the SU(3) singlet and octet member, respectively.[8] Moreover, because such states are mixed, in general the physical states, which presumably are those with definite masses, are not pure eigenstates of either of the Casimir operators. The degree of mixing is determined by diagonalizing the mass matrix and depends on the phenomenological parameters of the mass formula [Eq. (5)].

We note some general features of the SU(4) mass formula. As remarked above, the tetrahedral representations $(L,0,0)$ and $(0,0,N)$ are equally spaced in both the hypercharge Y and the new quantum number X. The same holds true for representations of the type $(0,M,0)$. Since for bosons terms odd in Y and X are excluded by charge conjugation, boson multiplets of the type $(0,M,0)$ will not be split in first order. In addition, self-conjugate particles, e.g., η, π^0, ω, etc., must belong to one of the real representations (L,M,L) where M is even but L is arbitrary.

Finally we wish to emphasize the advantages of using the different subgroups of SU(4) in analyses of this symmetry scheme. In the above we have used the SU(3) subgroups orthogonal to the X and Y directions to obtain the simple form of the mass formula Eq. (5); however, these techniques are more general. For example, electromagnetic properties can easily be found by considering the subgroup whose multiplets are orthogonal to the direction of the electric charge Q. This subgroup is not necessarily SU(3) but depends on the particular model, i.e., the multiplet assignments which determine the direction of Q. But independent of the model all particles in the same such submultiplet have the same electromagnetic properties, e.g., magnetic moments. It is also independent of the model that the neutral particles in any representation $(0,M,0)$ will have no magnetic moments, except possibly due to breaking of the symmetry.

One of us (D.S.) should like to thank the members of the Physics Department of Stanford University and of the Stanford Linear Accelerator Center for the hospitality extended to him.

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*Work supported in part by the U. S. Air Force through Air Force Office of Scientific Research Contract No. AF49(638)-1389 and the U. S. Atomic Energy Commission.

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[4]It should be observed that the direction of the hypercharge Y referred to here does not lie in the plane $X=0$ of the usual SU(3) multiplets. Actually $\bar{Y}=Y+\frac{1}{3}X$ is parallel to this plane and hence orthogonal to the direction of X.

[5]Here we follow the usual convention and denote an irreducible representation of SU(4) by three non-negative integers (L,M,N) and an irreducible representation of SU(3) by two non-negative integers (L,M). This convention is not the most convenient for all purposes. It differs from the one adopted by J. P. Antoine and D. Speiser (to be published) where the integers are positive.

[6]A proof of this theorem as well as an alternative derivation of the mass formula will be given elsewhere.

[7]Some consequences of this general result for certain models can be found in reference 1.

[8]In addition to the 15 there are representations of dimensions 20, 36, 45, 60, 64, 70, and 84 as well as infinitely many of higher dimension in which not all the states are simultaneous eigenstates of both $C_i^{(x)}$ and $C_i^{(y)}$. In general, to relate the pure states of one SU(3) subgroup to the pure states of another SU(3) subgroup write out the irreducible tensors of the SU(4) representation (these are symmetric with respect to the various SU(3) subgroups) in terms of both sets of states and solve the resulting system of linear equations.