ISOBARIC MASS FORMULA IN THE 1p SHELL*

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Three topics of current interest prompt this examination of Coulomb energies in light nuclei; all involve mass relationships within isobaric multiplets: (I) the charge independence of the $^{31}S_0$ force[1]; (II) high-T light nuclei such as n^4, H^5, He^8, etc.[2]; (III) $T=2$ states in $T_Z=0,+1$ nuclei[3] and the discovery of $T_Z=-\frac{3}{2}$ nuclei.[4]

This report considers all states that we believe are satisfactorily represented to first order by $(1s)^4(1p)^{A-4}$ through the third excited state of a system of any T. We look for the mean behavior presented by "nuclei-as-they-are" and do not correct or reject states because of Thomas or other shifts, while recognizing that such shifts may be of critical importance when we deal with specific cases. We avoid as much as possible questions of detailed structure and attempt to bring out the properties of "average light nuclear matter." We believe that this is the least biassed approach to Topic I and the best current basis for predictions in Topics II and III where we must necessarily handle nuclei as they come. This philosophy is orthogonal to that of the detailed state-by-state computations made if we wish to check the predictions of definite models.[5]

We carry out the discussion in terms of the general formula relating the masses within an isobaric multiplet:

$$M=a+bT_Z+cT_Z^{\,2},$$

which, as recently pointed out,[6] Topic III now enables us to test for the first time. This formula is correct provided that any departures from charge independence may be treated as perturbations that do not change the nucleon wave functions; this is true not only of the Coulomb contribution but also of any charge dependence of the specifically nuclear force provided that this is of a two-body character. Now write $b=b_{\rm CI}+b_\mu+b_{\rm CD}$ and similarly for c. $b_{\rm CI}$ is the charge-independent value of the coefficient, b_μ represents the charge dependence of the magnetic moment interactions, and $b_{\rm CD}$ other possible charge dependencies.

We wish, as far as possible, to avoid details but the phenomenon of pairing is so powerful that attention must be given to it and we write $b_{\rm CI}=b_{\rm CI}{}^m+b_{\rm P}$ and similarly for c. The Coulomb energy of two paired protons we take to be higher than the average by an amount P, which we treat as a constant except for what in jj coupling would be $1p_{1/2}$ protons for which we use fP (and of course for s protons where $P=0$). We now expect $b_{\rm CI}{}^m, c_{\rm CI}{}^m$ to be smooth functions of A. We compute the pairing corrections using the simplest pairing model: $2T$ nucleons carry the isotopic spin and are paired as far as possible; any remaining may pair into the $T=0$ "core." This gives

$$\begin{aligned} b_{\rm P} &= -\tfrac{1}{2}P && (A \text{ even}),\\ &= -[(2T-1)/4T]P && (A \text{ odd};\ A/2-T \text{ even}),\\ &= -[(2T+1)/4T]P && (A \text{ odd};\ A/2-T \text{ odd});\\ c_{\rm P} &= [2(2T-1)]^{-1}P && (A \text{ even}),\\ &= (1/4T)P && (A \text{ odd}). \end{aligned}$$

These corrections are applied to all states even though we should expect them to be a strongly state dependent. The authority for this is the behavior in the $(2s, 1d)$ shell where the Coulomb pairing phenomenon is dramatically displayed ($P=0.21$ MeV) and where it is found just as strongly in excited states as in ground states.

In general, we cannot write down any $b_{\rm CI}$-$c_{\rm CI}$ relationship. In one case, however, this is possible; namely, the multiplet beginning $n^A, {\rm H}^A, \cdots$, where we must have

$$M_{\rm CI}=\text{constant}+Nm_n+Zm_{\rm H}+\text{constant}\times Z(Z-1),$$

the antisymmetrization being absorbed into the constants. Setting $\Delta=m_n-m_{\rm H}$, we have

$$\Delta-b_{\rm CI}=c_{\rm CI}(A-1),$$

and

$$\Delta-b_{\rm CI}{}^m=c_{\rm CI}{}^m(A-1). \qquad \text{(i)}$$

Now expression (i) is certainly valid only for $T=A/2$ but we may take it as a guide and, until proved wrong, assume that it has practical validity for all values of T.[7] We should test this assumption directly if possible but we may also examine its consistency by seeing whether the plot of $\Delta-b_{\rm CIe}{}^m$ vs A passes through $A=1$ ($b_{\rm CIe}{}^m=b_{\rm expt}-b_\mu-b_{\rm P}$).

We use only complete isobaric doublets and triplets, finding a total of 24 multiplets in the $1p$ shell. The magnetic moment corrections have been computed using intermediate-coupling wave functions; they range from a few keV for the lightest nuclei to about 70 keV at the heavy end of the shell. We now fit the 24 data points to a quadratic minimizing $\sum\delta_i^2$, where δ_i is the departure of the datum from the line, in the five-parameter space that includes P and f. We find $\sum\delta_i^2 = 0.120$ MeV^2. To ask if the quadratic is demanded we repeat using a linear fit and find $\sum\delta_i^2 = 0.135$ MeV^2. The Fisher-Snedecor variance-ratio test shows that the quadratic is unnecessary. This best linear fit cuts the axis at $A = 1.341$ which is very close to the value $A = 1$ expected on our crude model. We therefore ask for the best linear fit that passes through $A = 1$ and find $\sum\delta_i^2 = 0.144$ MeV^2. The variance-ratio test shows that this fit is as good as the free linear fit or the free quadratic and so we have some internal evidence as to the effective T independence of the relation (i). We find $P = 0.210$ MeV, $f = 0.73$ (see Fig. 1).

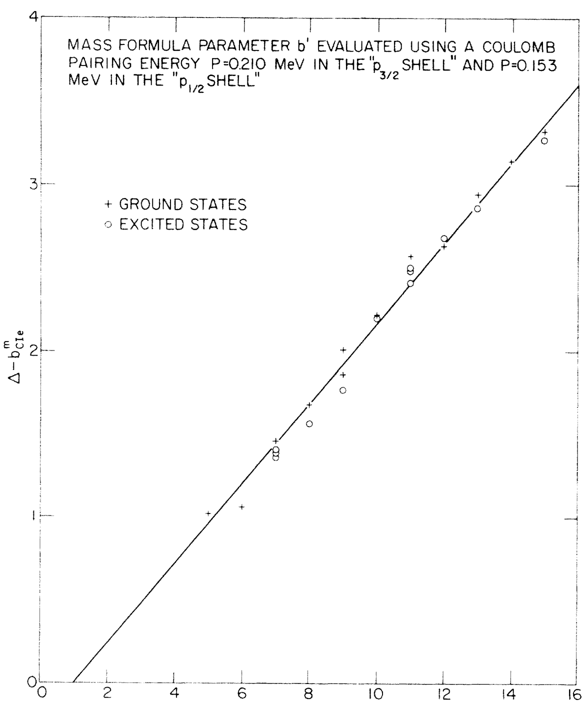

FIG. 1. $\Delta - b_{\text{CIe}}{}^m$ as a function of mass number for states that belong in first order to $(1s)^4(1p)^{A-4}$. Of the two ground-state points at $A = 9$ the lower refers to that of the $T = \frac{1}{2}$ system and the upper to that of the $T = \frac{3}{2}$ system. A Coulomb pairing energy of $P = 0.210$ MeV is found for the "$1p_{3/2}$ shell" and 0.153 MeV for the "$p_{1/2}$ shell".

The fact that the data are well fitted by a straight line passing through $A = 1$ gives a strikingly simple interpretation to relationship (i), namely that the $c_{\text{CIe}}{}^m$ should be independent of A and equal to the slope of the $\Delta - b_{\text{CIe}}{}^m$ vs A line (0.240 MeV) if charge independence holds. Figure 2 shows the situation using the above P, f values to get $c_{\text{CIe}}{}^m$ from c_{expt}. The experimental points indeed show no significant trend with A and have a mean value of $c_{\text{CIe}}{}^m = 0.230 \pm 0.017$ MeV so that $c_{\text{CD}} = -0.010 \pm 0.017$ MeV (assuming charge symmetry so that $b_{\text{CD}} = 0$). If we accept the temporary premise that the relationship (i) is T independent we should interpret this value of c_{CD} literally. We do this through the Slater integral K which has a typical value of about 1.0 MeV throughout the $1p$ shell. Using an appropriate degree of intermediate coupling and the Inglis-Kurath mixture of 0.8 space and 0.2 spin exchange[8] we find, as an average value, $c_{\text{CD}} \approx 6\delta K$ where δK is the change in K as between the n-p and the n-n, p-p force. Taking $c_{\text{CD}} \lesssim 0.05$ MeV we find $\delta K/K \lesssim 0.008$. It appears that, failing a lusus naturae, the effective residual ${}^{31}S_0$ interaction that dominates the specification of the $T = 1$ multiplets is charge independent to better than 1%.

The straight line of Fig. 1 cannot, of course, be taken to imply that all $1p$-shell nuclei are of

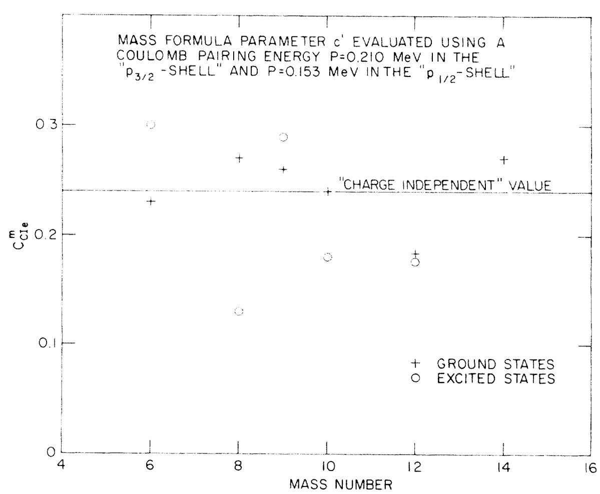

FIG. 2. $c_{\text{CIe}}{}^m$ as a function of mass number for the $T = 1$ and $T = \frac{3}{2}$ multiplets of Fig. 1. The line labeled "'charge independent' value" is the slope of the line of Fig. 1. The excited-state points at $A = 6$ and 9 do not figure in Fig. 1 which uses only complete doublets and triplets. These points are derived from incomplete triplets and for each we have used the b value that is the mean of that found for the ground-state point at the same mass number as the value read off the line of Fig. 1.

the same size, because of the A dependence of the Coulomb exchange integral, but that integral is small compared with the direct term and it is clear that the change in size is small. This provides some a posteriori justification for setting P constant.

The correctness of the assumption that $b_{\mathrm{CI}}{}^m$, $c_{\mathrm{CI}}{}^m$ do not depend significantly on T is not proved by the demonstration that the $b_{\mathrm{CIe}}{}^m$ vs A line passes through $A=1$ but is at least consistent with it. Direct information comes from analyzing the data for the $T=\frac{1}{2}$ and $T=1$ multiplets separately. When this is done the $b_{\mathrm{CIe}}{}^m$ vs A relationships for $T=\frac{1}{2}$ and $T=1$ are the same within the errors of about 3%. Data on $T=\frac{3}{2}$ states are scanty; such as are available suggest that the $b_{\mathrm{CIe}}{}^m$ and $c_{\mathrm{CIe}}{}^m$ values are there the same as for $T=\frac{1}{2}, 1$ and $T=1$ states, respectively, to within about 0.1 MeV and 0.05 MeV, respectively. There are no $T=2$ data in the $1p$ shell but for $A=16$ and 20 it is found[3] that $(b_{\mathrm{CIe}}{}^m - c_{\mathrm{CIe}}{}^m)_{T=2} - (b_{\mathrm{CIe}}{}^m - c_{\mathrm{CIe}}{}^m)_{T=1} = 0.05 \pm 0.15$ MeV and -0.03 ± 0.15 MeV, respectively (using $P=0.21$ MeV), so that the assumption is again upheld. (Unfortunately we cannot yet test the assumption separately for b and c in the $T=2$ cases. F^{16} is particle-unstable so cannot be used to tell us the coefficients for the $T=1$ states owing to the probably large Thomas shift; without proper correction for this shift any attempt to use the F^{16} data to test the present assumption would be worthless. The mass of Na^{20} is poorly determined.)

When we apply these results to Topic II we are unlikely to make errors of greater than about 0.3 MeV since the recipe of Fig. 1 predicts the Coulomb energy of even He^3 to within that amount. As an illustration consider the possible $T=2$ state[9] of Li^4; if this is the ground state of the $T=2$ system we find that n^4 is unstable by 1.34 MeV.

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[1]See contributions to Section I of the Congrès International de Physique Nucléaire: Paris, July 1964, particularly Abstract 1/C104 by H. P. Noyes; D. H. Wilkinson, Phil. Mag. 1, 1031 (1956).

[2]V. I. Goldanskii, Zh. Eksperim. i Teor. Fiz. 38, 163 (1960) [translation: Soviet Phys.–JETP 11, 1179 (1960)]; A. I. Baz, V. I. Goldanskii, and Ya. B. Zeldovich, Usp. Fiz. Nauk 72, 211 (1960) [translation: Soviet Phys.–Usp. 3, 729 (1961)]; B. M. K. Nefkens, Phys. Rev. Letters 10, 55 243 (1963); A. Schwarzschild, A. M. Poskanzer, G. T. Emery, and M. Goldhaber, Phys. Rev. 133, B1 (1964).

[3]G. T. Garvey, J. Cerny, and R. H. Pehl, Phys. Rev. Letters 12, 726 (1964); and private communication.

[4]R. Barton, R. McPherson, R. E. Bell, W. R. Frisken, W. T. Link, and R. B. Moore, Can. J. Phys. 41, 2007 (1963); V. A. Karnaukhov, G. M. Ter-Akopian, L. A. Petrov, and V. G. Subbotin, Zh. Eksperim. i Teor. Fiz. 45, 1280 (1963) [translation: Soviet Phys.–JETP 18, 879 (1964)]; R. McPherson, J. C. Hardy, and R. E. Bell, Phys. Letters 11, 65 (1964).

[5]B. C. Carlson and I. Talmi, Phys. Rev. 96, 436 (1954); S. Sengupta, Nucl. Phys. 21, 542 (1961); W. M. Fairbairn, Proc. Phys. Soc. (London) 77, 599 (1961); Nucl. Phys. 45, 437 (1963).

[6]D. H. Wilkinson, Phys. Letters 11, 243 (1964); and to be published.

[7]This is a refinement of the assumption pursued in reference 6; namely, that the b, c themselves are T independent.

[8]D. Kurath, Phys. Rev. 101, 216 (1956).

[9]M. J. Beniston, B. Krishnamurthy, R. Levi-Setti, and M. Raymund, Phys. Rev. Letters 13, 553 (1964).