NUCLEAR OPTICAL MODEL FOR VIRTUAL PIONS*

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Cross sections for incident virtual pions are sometimes needed, for example, in the one-pion-exchange approximation.[1] A particularly nice example has been given recently by Adler.[2] He studies neutrino reactions

$$\nu + \alpha \rightarrow l + \alpha^*,$$

where l is a charged lepton and α^* an aggregate of strongly interacting particles. Assuming conserved vector and almost conserved axial-vector currents, he finds that for forward-going leptons l ($q^2 \approx 0$) the differential cross section has the form

$$\frac{\partial^2\sigma}{\partial q^2 \partial W^2} = K\frac{2p(W)W}{M_\alpha}\sigma(W, -q^2). \qquad (1)$$

Here q is the difference of four-momenta between ν and l, W is the mass of the aggregate α^*, K is a given kinematical factor, and M_α is the target mass; finally $\sigma(W, m_\pi^2)$ would be the total cross section for the reaction

$$\pi + \alpha \rightarrow \alpha^*$$

at total center-of-mass energy W, with $p(W)$ the corresponding center-of-mass pion momentum. If the difference between $q^2 \approx 0$ and m_π^2 can be ignored in Eq. (1), we have a relation between observable quantities. Such a result would be particularly valuable for neutrino reactions, where nuclei are much more convenient targets than nucleons, because the complications of nuclear physics are apparently eliminated by the use of observed cross sections for pions on nuclei. However, the following difficulty presents itself: Because of absorption, pion cross sections depend on the size of large nuclei roughly as $A^{2/3}$. But neutrinos penetrate to all parts of nuclei; for them cross sections should contain at least a part proportional to A. This indicates for large nuclei a critical dependence of $\sigma(W, -q^2)$ on q^2. It could be discussed in terms of anomalous thresholds. However, we use here a more traditional method of the nuclear many-body problem, the optical model potential.

As in a previous discussion[3] of the nuclear optical model, we ignore the motion of the heavy target α and regard it as fixed in space. Then in terms of the renormalized pion field operator at zero time, $\pi(\vec{x}, 0)$, we have

$$\Gamma(q_0, -q^2) = \frac{2p(W)W}{M_\alpha}\sigma(W, -q^2) = (m_\pi^2 + q^2)^2$$
$$\times \sum_n |\langle n| \int d\vec{x}\, e^{i\vec{q}\cdot\vec{x}}\pi(\vec{x}, 0)|\alpha\rangle|^2 2\pi\delta(E_n - E_\alpha - q_0), \qquad (2)$$

where the summation is over all final states $|n\rangle$. Equivalently,

$$\Gamma = 2(m_\pi^2 + q^2)^2$$
$$\times \operatorname{Re}\int d\vec{x}d\vec{y}\, e^{-i\vec{q}\cdot\vec{x}} G(\vec{x}, \vec{y}, q_0) e^{i\vec{q}\cdot\vec{y}}, \qquad (3)$$

where[4]

$$G(\vec{x}, \vec{y}, q_0)$$
$$= \int dt \exp(-iq_0 t)\langle \alpha | T[\pi(\vec{x}, 0), \pi(\vec{y}, t)]|\alpha\rangle. \qquad (4)$$

As usual the propagator G satisfies an integral equation

$$G(q_0) = G_0(q_0) - iG_0(q_0)V(q_0)G(q_0), \qquad (5)$$

where space arguments and integrations are suppressed in a conventional way. G_0 is the propagator in vacuo, and the integral operator V represents the effect of nuclear matter. We will approximate[5] G_0 by the pole term

$$G_{00}(\vec{x}, \vec{y}, k_0)$$
$$= -i\int \frac{d\vec{k}}{(2\pi)^3} e^{i\vec{k}\cdot(\vec{x}-\vec{y})}(m_\pi^2 + k^2 - i\epsilon)^{-1}, \qquad (6)$$

where $k^2 = \vec{k}^2 - k_0^2$. The optical model wave function defined by

$$\Psi(\vec{x}) = i(m_\pi^2 + q^2)\int d\vec{y}\, G(\vec{x}, \vec{y}, q_0)e^{i\vec{q}\cdot\vec{y}} \qquad (7)$$

then satisfies

$$\Psi(\vec{x}) = e^{i\vec{q}\cdot\vec{x}} + (i\epsilon + \nabla^2 + q_0^2 - m_\pi^2)^{-1} V\Psi. \qquad (8)$$

From Eqs. (3), (5), (6), and (8)

$$\Gamma = -2 \operatorname{Im}\int d\vec{x}d\vec{y}\, e^{-i\vec{q}\cdot\vec{x}} V(\vec{x}, \vec{y}, q_0)\Psi(\vec{y}). \qquad (9)$$

This relates the total transition rate to an off-the-mass-shell forward-scattering amplitude. It can further be shown[6] that Γ is the sum of "elastic" and "absorptive" parts:

$$\Gamma = \Gamma_e + \Gamma_a,$$

with

$$\Gamma_e = \int \frac{d\vec{k}}{(2\pi)^3} \left| \int d\vec{x}\, e^{-i\vec{k}\cdot\vec{x}} V(\vec{x},\vec{y},q_0)\Psi(\vec{y}) \right|^2 \times 2\pi\delta(m^2 + \vec{k}^2 - q_0^2), \tag{10}$$

$$\Gamma_a = -2\,\mathrm{Im} \int d\vec{x}d\vec{y}\, \Psi^*(\vec{x}) V(\vec{x},\vec{y},q_0)\Psi(\vec{y}). \tag{11}$$

The integrand of Γ_e gives the relative probability for final states consisting of a pion of momentum $\vec{k}$ and the original unexcited nucleus; Γ_a gives the probability for all other states. When the four-vector q is on the mass shell, Eqs. (8), (9), (10), and (11) are equivalent to those customary in the optical model, with a potential V that is in general nonlocal and energy dependent. They are seen to remain relevant off the mass shell.

The critical effect of going off the mass shell is revealed by writing Eq. (8) as a differential equation:

$$(\nabla^2 - m_\pi^2 + q_0^2)\Psi = V\Psi - (m_\pi^2 + q^2)e^{i\vec{q}\cdot\vec{x}}. \tag{12}$$

On the mass shell, $m_\pi^2 + q^2 = 0$, there is no inhomogeneous term, and the wave is absorbed in the surface of the nucleus toward the incident beam. But off the mass shell there is a driving term carrying the wave to all parts of the nucleus. Deep enough inside a sufficiently large nucleus the solution is

$$\Psi = \frac{m_\pi^2 + q^2}{m_\pi^2 + q^2 + \tilde{V}} e^{i\vec{q}\cdot\vec{x}}, \tag{13}$$

where

$$\tilde{V} = \int d(\vec{x} - \vec{y})\, e^{i\vec{q}\cdot(\vec{x}-\vec{y})} V(\vec{y},\vec{x}). \tag{14}$$

Thus the vacuum propagator is replaced by that for nuclear matter. For $m_\pi^2 + q^2 \neq 0$ substitution of Eq. (13) in Eqs. (9), (10), and (11) gives volume-proportional terms which are equal for Γ and Γ_a and zero for Γ_e.

To obtain a first orientation let us make the approximation, which appears to work fairly well for real pions of high energy,[7]

$$2\,\mathrm{Im}\,\tilde{V} = -\rho\Gamma_1 \tag{15}$$

where ρ is nucleon density and Γ_1 the value of Γ appropriate to a single-nucleon target. From Eqs. (11) and (13) the volume contribution is then

$$\Gamma = \Gamma_a = \left| \frac{m_\pi^2 + q^2}{m_\pi^2 + q^2 + \tilde{V}} \right|^2 A\Gamma_1. \tag{16}$$

If $\tilde{V}$ could be ignored in the first factor we would have simply the summed cross sections of individual nucleons. This would be correct for sufficiently dilute matter, but not for nuclear matter. Giving Γ_1 its mass shell value $2(q_0^2 - m_\pi^2)^{1/2}\sigma_1 = 2k\sigma_1$,

$$\left| \frac{m_\pi^2 + q^2}{m_\pi^2 + q^2 + \tilde{V}} \right|^2 \lesssim \left| \frac{m_\pi^2}{k\sigma_1\rho} \right|^2 \tag{17}$$

for $q^2 \lesssim m_\pi^2$. With

$$\sigma \gtrsim 3\times10^{-26}\ \mathrm{cm}^2, \quad \rho \approx 1.7\times10^{38}\ \mathrm{cm}^{-3},$$

one finds

$$\left| \frac{m_\pi^2 + q^2}{m_\pi^2 + q^2 + \tilde{V}} \right|^2 \lesssim 2\left(\frac{m_\pi}{k}\right)^2. \tag{18}$$

This decreases with increasing virtual pion energy, and is already small compared with unity at a few hundred MeV.

Estimation of surface effects depends on a more accurate solution of Eq. (8) or Eq. (12). However, the above conclusion about the volume effect is already of interest. As applied to neutrinos, by Eq. (1), it implies that highly inelastic events with small q^2 should be much less probable than might be expected by counting nucleons. It is interesting to consider if other contributions to the neutrino reaction (e.g., ρ exchange) should show similar effects at appropriately higher energies.

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[1]S. D. Drell, Rev. Mod. Phys. 33, 458 (1961); E. Fer-

rari and F. Selleri, Nuovo Cimento Suppl. 24, 453S (1962). These papers give earlier references.

[2]S. L. Adler, unpublished.

[3]J. S. Bell and E. J. Squires, Phys. Rev. Letters 3, 96 (1959); J. S. Bell, Lectures on the Many Body Problem, edited by E. R. Caianiello (Academic Press, Inc., New York, 1962), p. 91.

[4]Instead of the time-ordered product, the retarded commutator could be used. This is advantageous for the energy averaging problem (see references 2 and 3).

[5]This approximation can be avoided by accepting the existence of short-range integral operators M and M^{-1} such that $MG_0M = G_{00}$. Then $G' = MGM$ satisfies

$$G' = G_{00} - iG_{00}V'G',$$

with $V' = M^{-1}VM^{-1}$. The subsequent development proceeds with G' and V' replacing G and V. Equations (9), (10), and (11) finally contain V' instead of V and their right-hand sides are multiplied by $M^{-2}(q^2)$, a factor which for $q^2 = -m_\pi^2$ is unity and for $q^2 > -m_\pi^2$ is real, positive, nonzero, nonsingular, and independent of nuclear size.

[6]To prove this "off-the-mass-shell optical theorem" write $\chi = \Psi - \exp(i\vec{q}\cdot\vec{x})$ so that Eq. (12) becomes

$$(\nabla^2 - m_\pi^2 + q_0^2)\chi = V\Psi,$$

whence

$$2\int d\vec{s}\ \mathrm{Im}\chi^*\nabla\chi = 2\,\mathrm{Im}\int d\vec{x}(\Psi^*V\Psi - e^{-i\vec{q}\cdot\vec{x}}V\Psi)$$

where the first integral is over a distant surface and the second over the enclosed volume. The last term is Γ of Eq. (9); the other term on the right is $-\Gamma_a$ of Eq. (11); on using the asymptotic form of χ the right-hand side is seen to equal Γ_e of Eq. (10).

[7]J. W. Cronin, R. Cool, and A. Abashian, Phys. Rev. 107, 1121 (1957); M. J. Longo and B. J. Moyer, Phys. Rev. 125, 701 (1962).

DIFFERENTIAL AND TOTAL CROSS SECTION FOR THE REACTION $p+p \to d+\pi^+$*

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The differential cross section for the reaction $p+p \to d+\pi^+$ has been measured with scintillation counters for incident proton kinetic energies of 1.3, 1.5, 1.7, 2.0, 2.5, and 2.8 BeV, using the external proton beam of the Cosmotron of the Brookhaven National Laboratory. Values of $d\sigma/d\Omega$ with a total uncertainty of $\pm 11\%$ are given for barycentric deuteron angles, θ, for $0 \leq \cos\theta \leq -0.97$ in small intervals of $\cos\theta$. The primary interest of the authors in this reaction is that it is one of the few two-body reactions in high-energy particle physics, and therefore might provide both a stimulus and a test for dynamical theories of particle interactions. Perl, Jones, and Ting[1] have pointed out that the simplest Feynman diagram for this reaction is a one-nucleon exchange, while Chahoud, Russo, and Selleri[2] and Yao[3] have studied the reaction in terms of a one-pion-exchange model. A further interest stems from the results of Turkot, Collins, and Fujii[4] and Cocconi et al.[5] who measured a single point in the differential cross section near 0° at various energies. Their combined results when plotted versus energy[5] show a peak in the forward differential cross section at 2.5-BeV incident kinetic energy.

The reaction occurred in a three-inch-long liquid hydrogen target. The detection apparatus consisted of two scintillation-counter telescopes, one consisting of two counters to detect the pion, and the other of six counters (two parallel channels of three each) to detect the deuteron. The solid angle subtended from the target was determined by the first counter in the pion telescope. Depending on this counter size and its distance from the target, the solid angle varied from 0.035 to 0.87 millisteradians. The deuteron telescope channel included a magnet providing a 10° deflection of the emergent deuterons, and a flight path of 53 feet from the target to the last counter. Both counter telescopes and the magnet were movable to allow them to be set for any desired pion and deuteron angle. In the deuteron channel the magnetic momentum selection combined with the time-of-flight selection separated the deuterons from pions and protons up to 2.4 BeV/c. This selection, combined with the required pion channel coincidence, identified the desired events which comprise 0.05% to 0.5% of the total pp cross section in this energy interval.

Numerous checks to verify that the events detected indeed corresponded to the π^+d final state