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MASS FORMULAS IN THE SU(6) SYMMETRY SCHEME*

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Recently it was proposed by Gürsey and Radicati[1] and Pais[2] that the SU(6) symmetry scheme incorporating spin and unitary spin may have important consequences in particle physics. They discussed, among other things, a possible mass formula and applied it to some low-dimensional representations. In this note[3] we propose that the SU(6) symmetry is broken analogously as in SU(3),[4,5] namely, the primary symmetry-breaking term in the Hamiltonian transforms like the $I=0, Y=0, J=0$ member of the $\underline{35}$ representation. The major result of this assumption is that in a given SU(6) representation, states with the same I, Y, and J belonging to different SU(3) multiplets are mixed in a definite way.

The 36 traceless operators $B_\mu{}^\nu$ of SU(6) are defined such that their representation in the six-dimensional vector space C_6 are given by

$$(B_\nu{}^\mu)_{ij} = \delta_{\mu j}\delta_{\nu i} - \tfrac{1}{6}\delta_{\mu\nu}\delta_{ij}, \tag{1}$$

$(\mu, \nu, i, j = 1, 2, \cdots, 6)$. These operators satisfy the commutation relations

$$[B_\nu{}^\mu, B_\beta{}^\alpha] = \delta_\beta{}^\mu B_\nu{}^\alpha - \delta_\nu{}^\alpha B_\beta{}^\mu. \tag{2}$$

The symmetry-breaking term is proposed to be $T_3{}^3 + T_6{}^6$, where

$$[B_\nu{}^\mu, T_\beta{}^\alpha] = \delta_\beta{}^\mu T_\nu{}^\alpha - \delta_\nu{}^\alpha T_\beta{}^\mu. \tag{3}$$

Note that the hypercharge operator Y is $-(B_3{}^3 + B_6{}^6)$. It can be shown that[6]

$$T_\nu{}^\mu = a_0\delta_\nu{}^\mu + a_1 B_\nu{}^\mu + a_2(B\cdot B)_\nu{}^\mu + a_3(B\cdot B\cdot B)_\nu{}^\mu + a_4(B\cdot B\cdot B\cdot B)_\nu{}^\mu + a_5(B\cdot B\cdot B\cdot B\cdot B)_\nu{}^\mu, \tag{4}$$

where the a_i's are constants depending only on the five Casimir operators of the group.

For the few low-dimension representations discussed below, only the first three terms in Eq. (4) are needed. Therefore, for those SU(6) supermultiplets we can write down the following mass formula:

$$M = M_0 + aY + b\{(B\cdot B)_{SU(4)} - 2Q(Q+1) - \tfrac{1}{2}Y^2\}. \tag{5}$$

For mesons mass squared is to be used in Eq. (5). The symbol $(B\cdot B)_{SU(4)}$ denotes the quadratic Casimir operator of the SU(4) subgroup which is considered by Gürsey, Pais, and Radicati.[7] $\vec{Q}$ is an angular momentum vector with components

$$Q_3 = \tfrac{1}{2}(B_3{}^3 - B_6{}^6),$$
$$Q_+ = B_3{}^6,$$
$$Q_- = B_6{}^3. \tag{6}$$

In the quark language,[8] $Q=\frac{1}{2}$ for the $S=\pm1$ quarks and $Q=0$ for the $S=0$ quarks. In Table I we shall give all the eigenvalues of $(B\cdot B)_{SU(4)}$ and Q of all the particles in the $\underline{20}$, $\underline{35}$, $\underline{56}$, and $\underline{70}$ representations.

Now let us discuss the $\underline{35}$ representation $[\underline{35} = (\underline{8}, \underline{1}) + (\underline{8}, \underline{3}) + (\underline{1}, \underline{3})]$ which has as members the pseudoscalar-meson octet and the vector-meson nonet. Since

$$\underline{35}\otimes\underline{35} = \underline{1}\oplus\underline{35}\oplus\underline{35}\oplus\underline{189}\oplus\underline{280}\oplus\underline{280}^*\oplus\underline{405}, \tag{7}$$

the matrix element

$$\langle\underline{35}|T_\nu{}^\mu|\underline{35}\rangle = a_0 + a_1\langle\underline{35}|B_\nu{}^\mu|\underline{35}\rangle + a_2\langle\underline{35}|(B\cdot B)_\nu{}^\mu|\underline{35}\rangle. \tag{8}$$

Table I. SU(4) multiplets in SU(6).

Particles	No. of states	Representation of SU(4)	$(B\cdot B)_{\mathrm{SU}(4)}$	Q
		$\underline{35}$		
ρ, ω, π	15	$\underline{15}$	8	0
K^*, K	8	$\underline{4}$	4	$\frac{1}{2}$
$\bar{K}^*, \bar{K}$	8	$\underline{4}^*$	4	$\frac{1}{2}$
φ	3	$\underline{1}$	0	1
η	1	$\underline{1}$	0	0
		$\underline{56}$		
N^*, N	20	$\underline{20}$	16	0
Y_1^*, Σ, Λ	20	$\underline{10}$	9	$\frac{1}{2}$
Ξ^*, Ξ	12	$\underline{4}$	4	1
Ω	4	$\underline{1}$	1	$\frac{3}{2}$
		$\underline{70}$		
$N_{3/2}{}^{1/2}, N_{1/2}{}^{1/2}, N_{1/2}{}^{3/2}$	20	$\underline{20}$	10	0
$Y_1{}^{1/2}, Y_0{}^{1/2}, Y_0{}^{3/2}$	12	$\underline{6}$	5	$\frac{1}{2}$
$Y_1{}^{1/2}, Y_0{}^{1/2}, Y_1{}^{3/2}$	20	$\underline{10}$	9	$\frac{1}{2}$
$\Xi_{1/2}{}^{1/2}, \Xi_{1/2}{}^{3/2}$	12	$\underline{4}$	4	1
$\Xi_{1/2}{}^{1/2}$	4	$\underline{4}$	4	0
$\Omega_0{}^{1/2}$	2	$\underline{1}$	1	$\frac{1}{2}$
		$\underline{20}$		
$Y_0{}^{1/2}, Y_1{}^{1/2}, Y_0{}^{3/2}$	12	$\underline{6}$	5	$\frac{1}{2}$
$N_{1/2}{}^{1/2}$	4	$\underline{4}^*$	4	0
$\Xi_{1/2}{}^{1/2}$	4	$\underline{4}$	4	0

This simplification from Eq. (4) is the result that in Eq. (7) $\underline{35}$ occurs only twice in $\underline{35}\otimes\underline{35}$. The immediate consequence of Eq. (5) is as follows:

(1) For vector mesons we get the familiar result

$$m_\omega^{\ 2} = m_\rho^{\ 2},$$

$$m_\varphi^{\ 2} + m_\rho^{\ 2} = 2m_{K^*}^{\ 2}. \tag{9}$$

(2) For pseudoscalar mesons we get the usual mass sum rule,[4,5]

$$m_K^{\ 2} = \tfrac{1}{4}(3m_\eta^{\ 2} + m_\pi^{\ 2}). \tag{10}$$

(3) We also obtain the relation

$$m_{K^*}^{\ 2} - m_\rho^{\ 2} = m_K^{\ 2} - m_\pi^{\ 2}, \tag{11}$$

which was noticed before.[2]

(4) In connection with Eq. (9), we also obtain from Eq. (8) the mixing of φ^0 and ω^0 unambiguously, such that the physical φ and ω are given by

$$\varphi = -(\tfrac{2}{3})^{1/2}\varphi^0 + (\tfrac{1}{3})^{1/2}\omega^0,$$

$$\omega = (\tfrac{1}{3})^{1/2}\varphi^0 + (\tfrac{2}{3})^{1/2}\omega^0. \tag{12}$$

(5) We further notice that the primary symmetry-breaking term $(T_3^{\ 3} + T_6^{\ 6})$ still leaves ρ and π degenerate (also K^* and K). This degeneracy can be lifted by a spin-dependent mass term which can only be a function of $J(J+1)$. We emphasize that the inclusion of this spin-dependent term will not affect the results in Eqs. (9)-(12). (See below for more details.)

We next come to a discussion of the $\underline{56}$ representation $[\underline{56} = (\underline{10}, \underline{4}) + (\underline{8}, \underline{2})]$ which has as members the baryon octet and decuplet. Since

$$\underline{35}\otimes\underline{56} = \underline{56}\oplus\underline{70}\oplus\underline{700}\oplus\underline{1134}, \tag{13}$$

$\underline{56}$ occurs only once, the matrix element

$$\langle\underline{56}|T_\nu^{\ \mu}|\underline{56}\rangle = a_0 + a_1\langle\underline{56}|B_\nu^{\ \mu}|\underline{56}\rangle. \tag{14}$$

Thus the mass formula for $\underline{56}$ reduces to the simple form

$$M = M_0 + a_1 Y. \tag{15}$$

(1) Now both the decuplet and the octet are equally spaced:

$$M_\Omega - M_{\Xi^*} = M_{\Xi^*} - M_{Y_1^*} = M_{Y_1^*} - M_{N^*}, \tag{16}$$

$$M_\Xi - M_\Sigma = M_\Sigma - M_N, \tag{17a}$$

$$M_\Lambda = M_\Sigma. \tag{17b}$$

(2) Furthermore,

$$M_{\Xi^*} - M_{Y_1^*} = M_\Xi - M_\Sigma. \tag{18}$$

(3) We still have the degeneracy between Ξ^* and Ξ, etc., which can be removed by a spin-dependent mass term as before.

(4) Now Λ and Σ are still degenerate. This degeneracy can be removed by adding a term of the form $\lambda[I(I+1)-\frac{1}{4}Y^2]$ to Eq. (5). Equations (17a) and (17b) are now combined to give the usual octet mass formula,

$$2(M_\Xi+M_N)=3M_\Lambda+M_\Sigma. \tag{17c}$$

We note that Eqs. (10)-(12), (16), and (18) are not changed. [Equation (9) becomes[9] $m_\varphi^{\;2}+\frac{1}{2}(m_\rho^{\;2}+m_\omega^{\;2})=2m_{K^*}^{\;2}$.] The general mass formula can now be written as

$$M=M_0+aY+b[(B\cdot B)_{\mathrm{SU}(4)}-2Q(Q+1)-\tfrac{1}{2}Y^2]$$
$$+\mu J(J+1)+\lambda[I(I+1)-\tfrac{1}{4}Y^2]. \tag{19}$$

So far we have only reproduced some familiar results. Now we proceed to a discussion of the $\underline{70}$ representation $[\underline{70}=(\underline{8},\underline{4})+(\underline{10},\underline{2})+(\underline{8},\underline{2})+(\underline{1},\underline{2})]$. Again we obtain an equation similar to Eq. (8), since

$$\underline{35}\otimes\underline{70}=\underline{20}\oplus\underline{56}\oplus\underline{70}\oplus\underline{70}\oplus\underline{540}\oplus\underline{560}\oplus\underline{1134}. \tag{20}$$

For the spin-$\frac{3}{2}$ baryon resonances we have the familiar octet mass formula.[2] In the case of the spin-$\frac{1}{2}$ resonances we again encounter the mixing problem just as in Eq. (12) where φ^0 and ω^0 get mixed by the symmetry-breaking term. Here the $I=0$, $Y=0$ members of $(\underline{8},\underline{2})$ and $(\underline{1},\underline{2})$ are mixed. Furthermore, the $I=\frac{1}{2}$, $Y=-1$ members of $(\underline{8},\underline{2})$ and $(\underline{10},\underline{2})$ are mixed. So are the $I=1$, $Y=0$ members of $(\underline{8},\underline{2})$ and $(\underline{10},\underline{2})$. The mixing angle is found to be $\theta=45°$ in all three cases. From Eq. (19) there are six mass sum rules among the nine (in general) nondegenerate particles:

$$N_{3/2}'+3\Xi_\pm'=\Omega'+3\Sigma_\pm', \tag{21a}$$

$$\Omega'+N_{3/2}'=\Xi_\mp'+\Sigma_\pm', \tag{21b}$$

$$2(N_{1/2}'+\Xi_\pm')=3\Lambda_\pm'+\Sigma_\mp', \tag{21c}$$

where $N_{3/2,\,1/2}'$ have $I=\frac{3}{2},\frac{1}{2}$, respectively. The subscript + denotes the heavier, and − the lighter of the two particles with the same I and Y. We note that Eq. (21a) takes a form hitherto not discussed. So far very few spin-$\frac{1}{2}$ resonances have been positively identified in the experiments. It is hoped that Eqs. (21) may be helpful in finding spin-$\frac{1}{2}$ resonances in the future.

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Note added in proof. – The transformation properties of the terms $J(J+1)$ and $[I(I+1)-\frac{1}{4}Y^2]$, which are not considered in this Letter, have since been discussed by Bég and Singh.[10] In fact, their Eq. (22) reduces to our Eq. (19) for $b=f=0$.

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