Wrapping Corrections for Long-Range Spin Chains

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The long-range spin chains play an important role in the gauge-string duality. The aim of this Letter is to generalize the recently introduced transfer matrices of integrable medium-range spin chains to long-range models. These transfer matrices define a large set of conserved charges for every length of the spin chain. These charges agree with the original definition of long-range spin chains for infinite length. However, our construction works for every length, providing the definition of integrable finite-size long-range spin chains whose spectrum already contains the wrapping corrections.

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Introduction.—In the early studies of the planar limit of the $\mathcal{N}=4$ super-Yang-Mills theory, it turned out that the anomalous dimensions of single-trace operators can be obtained from the spectrum of an integrable Hamiltonian with long-range interaction. At one loop, the dilatation operator corresponds to an integrable nearest-neighbor interacting model [1]. For higher loops, the interaction range increases; more precisely, the interaction range is $\ell+1$ at ℓ loops.

In the region where the spin chain length J is bigger than the loop order (the asymptotic region), the Hamiltonian can be written as a sum of local densities. For these local operators, the integrability condition can be generalized, and it was shown that the Hamiltonian of the $SU(2)$ sector preserves integrability for higher loops [2]. These local Hamiltonians can be diagonalized with the asymptotic Bethe ansatz [3], and the result can be generalized to the full $\mathfrak{psu}(2,2|4)$ spectrum [4]. However, this result is correct only in the asymptotic region. In the region where the spin chain length J is smaller than the loop order (the wrapping region), wrapping corrections appear [5]. So far, it was not clear whether good spin chain toy models, which mimic the wrapping corrections, could be found—i.e., even if an asymptotic Hamiltonian were given, we could not define the corresponding finite-size Hamiltonian.

The solution for the wrapping corrections came from holographic duality. In the string theory side, the scaling dimensions correspond to the energy spectrum of strings which can be described as a $1+1$-dimensional integrable field theory [6]. In field theory, if we know the dispersion relation and the scattering matrix at infinite volume, then we can calculate the finite-volume spectrum as well (at least in principle). The finite-size corrections can be obtained from the thermodynamic Bethe ansatz [7–10], and it has been shown that they agree with the wrapping corrections [11–13].

Since the asymptotic data of the string theory (dispersion relation, scattering matrix) completely defines the finite-size corrections, a natural conclusion is that the asymptotic data on the spin chain side should also define the wrapping corrections. In other words, there must be a procedure that gives the finite-size Hamiltonians from the asymptotic ones. The aim of this Letter is to present such a method.

Recently, an algebraic framework was developed for integrable medium-range spin chains (with an interaction range bigger than 2, but finite) [14]. This method gives a recipe for how to define transfer matrices which are the generating functions of the conserved quantities, including the Hamiltonians. An interesting observation is that this transfer matrix is well defined even when the length of the spin chain is smaller than the interaction range; therefore, generalizing this method to long-range spin chains, we obtain transfer matrices which define the finite-length Hamiltonians even for the lengths where the wrapping corrections appear.

Preliminaries.—In this section, we summarize the definition of the long-range spin chain following Refs. [3,15] and specify our goals.

An integrable long-range spin chain has a tower of coupling constant λ-dependent commuting charges $\mathcal{Q}_r(\lambda)\equiv\mathcal{Q}_r$ [16] which have the following series expansions:

$$\mathcal{Q}_r=Q_r^{(0)}+\sum_{j=1}^{\infty}\lambda^jQ_r^{(j)},\qquad(1)$$

where $r \geq 2$, and the λ-independent operators $Q_r^{(\ell)}$ are sums of local operators with range $r+\ell$:

$$Q_r^{(\ell)} = \sum_{j=-\infty}^{\infty} q_j^{r,\ell} = \sum_{j=-\infty}^{\infty} q_{j,j+1,\ldots,j+r+\ell-1}^{r,\ell}, \quad (2)$$

where the local densities $q_j^{r,\ell} \equiv q_{j,j+1,\ldots,j+r+\ell-1}^{r,\ell}$ act on the sites $j, j+1, \ldots, j+r+\ell-1$. The Hamiltonian is the charge $\mathcal{Q}_2$.

It turns out that, for a fixed nearest-neighbor model $\mathcal{Q}_k(\lambda=0)$, a large class of integrable deformations exists. The moduli space is given by four sets of parameters: $\alpha_r(\lambda)$, $\beta_{r,s}(\lambda)$, $\gamma_{r,s}(\lambda)$, and $\epsilon_k(\lambda)$. The last two sets are unphysical parameters, and they correspond to the linear combinations of the charges $\mathcal{Q}_r \to \sum \gamma_{r,s}(\lambda)\mathcal{Q}_s$ and the similarity transformations

$$\mathcal{Q}_r \to e^{\mathcal{X}} \mathcal{Q}_r e^{-\mathcal{X}}, \qquad \mathcal{X} = \sum_{j=-\infty}^{\infty} \sum_k \epsilon_k(\lambda) X_j^k, \quad (3)$$

where $X_j^k \equiv X_{j,\ldots,j+\ell_k-1}^k$'s are local operators with range ℓ_k. The remaining parameters are the physical ones. The α_r and $\beta_{r,s}$ parameters appear in the rapidity map and the scattering phase [15].

It is clear that the operators $Q_k^{(\ell)}$ can also be defined on a finite length J for $J \geq \ell + k$. More concretely, the Hamiltonian $\mathcal{H}$ on size J is defined up to order λ^{J-2} (asymptotic region). Our goal is to find an integrability-preserving method which defines the finite-volume version of the asymptotic Hamiltonians even for higher orders than λ^{J-2} (wrapping region).

Medium range to long range.—In this section, we generalize the construction of Ref. [14] (the basics appeared first in Ref. [17]) to obtain transfer matrices for perturbative long-range spin chains [3]. In Ref. [14], an algebraic framework was introduced for integrable spin chains with the interaction range $\ell+2$, which is defined by the Hamiltonian

$$H^{(\ell)} = \sum_{j=1}^{J} h_{j,j+1,\ldots,j+\ell+1} = \sum_j h_j^{(\ell)}, \quad (4)$$

where $h_j^{(\ell)}$ is the Hamiltonian density which acts on the sites $j, j+1, \ldots, j+\ell+1$. We use a periodic boundary condition. The construction of Ref. [14] is based on the existence of the Lax and the R operators:

$$\check{L}_j^{(\ell)}(u) = \check{L}_{j,j+1,\ldots,j+\ell+1}^{(\ell)}(u) = 1 + u h_j^{(\ell)} + \mathcal{O}(u^2), \quad (5)$$

$$\check{R}_j^{(\ell)}(u,v) = \check{R}_{j,j+1,\ldots,j+2\ell+1}^{(\ell)}(u,v), \quad (6)$$

which satisfy the RLL relation

$$\check{R}_2^{(\ell)}(u,v)\check{L}_1^{(\ell)}(u)\check{L}_{\ell+2}^{(\ell)}(v) = \check{L}_1^{(\ell)}(v)\check{L}_{\ell+2}^{(\ell)}(u)\check{R}_1^{(\ell)}(u,v). \quad (7)$$

In this Letter, we choose to write these operators in the "checked" form (the R matrix is multiplied by a permutation), which might be less familiar to some readers [18], although it has the advantage that the Lax operator has a simpler expansion in the spectral parameter [Eq. (5)]. In the alternative "unchecked" convention, the quantum and auxiliary spaces are separated. Figure 1 shows graphical presentations of Lax operators and RLL relations, and the colored legs denote the auxiliary spaces of the "unchecked" convention.

The consequence the RLL relation is that the transfer matrix

$$\check{T}^{(\ell)}(u) = \widehat{\mathrm{Tr}}_{J,\ell+1}\left(\check{L}_J^{(\ell)}(u)\ldots\check{L}_1^{(\ell)}(u)\right) \quad (8)$$

defines commuting quantities: $[\check{T}^{(\ell)}(u), \check{T}^{(\ell)}(v)] = 0$ [18]. In Eq. (8), we define the twisted trace operator $\widehat{\mathrm{Tr}}_{J,\ell}$, which acts on an operator X as

$$\widehat{\mathrm{Tr}}_{J,\ell}(X) = \mathrm{Tr}_{J+1,\ldots,J+\ell}(X P_{\ell,J+\ell} P_{\ell-1,J+\ell-1} \ldots P_{1,J+1}), \quad (9)$$

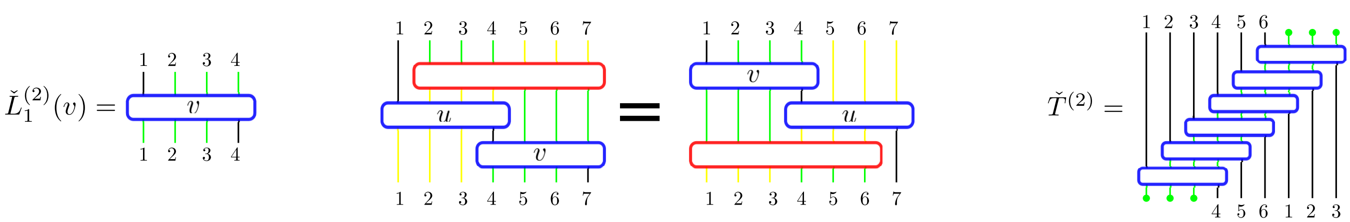

FIG. 1. Graphical illustration of the Lax operator, RLL relation, and transfer matrix for $\ell = 2$. The left graph shows the Lax operator $\check{L}_1^{(2)}(v) = \check{L}_{1,2,3,4}^{(2)}(v)$. In the middle, we can see the RLL relation, where the red box is the R matrix $\check{R}_1^{(2)}(u,v) = \check{R}_{1,2,3,4,5,6}^{(2)}(u,v)$. The right graph shows the transfer matrix for $J = 6$, where the dots on the incoming and outgoing legs denote the summations for the auxiliary spaces. The green and yellow lines mark the auxiliary spaces.

where $P_{j,k}$ is the permutation operator and $\mathrm{Tr}_{J+1,\ldots,J+\ell}$ is the usual trace on the sites $J+1,\ldots,J+\ell$. The transfer matrix generates the local conserved charges

$$Q_{k+1}^{(\ell)} = \frac{\partial^k}{\partial u^k} \log \check{T}^{(\ell)}(u)\bigg|_{u=0}. \tag{10}$$

The interaction range of $Q_k^{(\ell)}$ is $(k-1)\ell + k$, and $H^{(\ell)} = Q_2^{(\ell)}$.

Let us turn to the long-range spin chains. At first, we have to introduce the coupling constant λ-dependent truncated operators $\check{\mathcal{L}}_1^{(\ell)}(u,\lambda) \equiv \check{\mathcal{L}}_1^{(\ell)}(u)$ and $\check{\mathcal{R}}_1^{(\ell)}(u,v,\lambda) \equiv \check{\mathcal{R}}_1^{(\ell)}(u,v)$ with range $\ell+2$ and $2\ell+2$ as

$$\check{\mathcal{L}}_1^{(\ell)}(u) = \check{L}_1^{(0)}(u) + \sum_{j=1}^{\ell} \lambda^j \check{L}_1^{(j)}(u), \tag{11}$$

$$\check{\mathcal{R}}_1^{(\ell)}(u,v) = \check{R}_1^{(\ell,0)}(u,v) + \sum_{j=1}^{\ell} \lambda^j \check{R}_1^{(\ell,j)}(u,v), \tag{12}$$

which satisfy the RLL relation up to order $\mathcal{O}(\lambda^{\ell+1})$:

$$\check{\mathcal{R}}_2^{(\ell)} \check{\mathcal{L}}_1^{(\ell)}(u) \check{\mathcal{L}}_{\ell+2}^{(\ell)}(v) = \check{\mathcal{L}}_1^{(\ell)}(v) \check{\mathcal{L}}_{\ell+2}^{(\ell)}(u) \check{\mathcal{R}}_1^{(\ell)} + \mathcal{O}(\lambda^{\ell+1}), \tag{13}$$

where we use the shorthand notation $\check{\mathcal{R}}_j^{(\ell)} \coloneqq \check{\mathcal{R}}_j^{(\ell)}(u,v)$. We also require that

$$\check{\mathcal{L}}_1^{(\ell)}(u) = 1 + u\mathfrak{h}_1^{(\ell)} + \mathcal{O}(u^2), \quad \mathfrak{h}_1^{(\ell)} = \sum_{j=0}^{\ell} \lambda^j h_1^{(j)}, \tag{14}$$

where $h_1^{(j)}$ are λ- and u-independent operators with interaction range $j+2$.

At first sight, we might think that the truncated RLL relation [Eq. (13)] and the matrices $\check{R}_1^{(\ell,j)}$ are completely independent for every order ℓ, but this is not true. It turns out that Eq. (13) up to order $\mathcal{O}(\lambda^\ell)$ is equivalent with the truncated RLL relation for $\ell-1$. We can show that

$$\check{\mathcal{R}}_1^{(\ell)} + \mathcal{O}(\lambda^\ell) = \check{\mathcal{L}}_{\ell+1}^{(\ell-1)}(u) \check{\mathcal{R}}_1^{(\ell-1)} \check{\mathcal{J}}_{\ell+1}^{(\ell-1)}(v), \tag{15}$$

where we define the perturbative inverse $\check{\mathcal{J}}_1^{(\ell-1)}(u)$ as $\check{\mathcal{J}}_1^{(\ell-1)}(u)\check{\mathcal{L}}_1^{(\ell-1)}(u) = 1 + \mathcal{O}(\lambda^\ell)$ [18].

The consequence of Eq. (15) is that the matrices $\check{R}_1^{(\ell,j)}$ are determined by $\check{R}_1^{(\ell-1,j)}$ for $j = 1,\ldots\ell-1$; therefore, the full truncated R matrix $\check{\mathcal{R}}_1^{(\ell)}$ is completely determined by the matrices $\check{R}_1^{(j,j)}$ for $j = 1,\ldots,\ell$. By fixing the leading-order $\check{L}_{1,2}^{(0)}, \check{R}_{1,2}^{(0,0)}$ to already known L and R matrices of a nearest-neighbor interacting model, we can calculate the matrices $\check{L}_1^{(\ell)}(u)$, $\check{R}_1^{(\ell,\ell)}(u,v)$ order by order from the highest order λ^ℓ of the truncated RLL relation [Eq. (13)].

As in the medium-range case, the transfer matrix

$$\check{\mathcal{T}}^{(\ell)}(u) = \widehat{\mathrm{Tr}}_{J,\ell+1}\left(\check{\mathcal{L}}_J^{(\ell)}(u)\ldots\check{\mathcal{L}}_1^{(\ell)}(u)\right) + \mathcal{O}(\lambda^{\ell+1}) \tag{16}$$

defines commuting quantities up to order $\lambda^{\ell+1}$: $[\check{\mathcal{T}}^{(\ell)}(u), \check{\mathcal{T}}^{(\ell)}(v)] = \mathcal{O}(\lambda^{\ell+1})$. The transfer matrix generates the conserved charges up to order $\mathcal{O}(\lambda^{\ell+1})$

$$\mathcal{Q}_{k+1}^{(\ell)} = \frac{\partial^k}{\partial u^k} \log \check{\mathcal{T}}^{(\ell)}(u)\bigg|_{u=0} + \mathcal{O}(\lambda^{\ell+1}), \tag{17}$$

where

$$\mathcal{Q}_k^{(\ell)} = Q_k^{(0)} + \sum_{j=1}^{\ell} \lambda^j Q_k^{(j)}(u) + \mathcal{O}(\lambda^{\ell+1}). \tag{18}$$

It turns out that the charge $Q_k^{(\ell)}$ has the interaction range $\ell+k$.

Since the Lax operators have the property in Eq. (14), the Hamiltonian reads as

$$\mathcal{Q}_2^{(\ell)} = \sum_{j=1}^{J} \widehat{\mathrm{Tr}}_{J,\ell+1}\left(\mathfrak{h}_j^{(\ell)}\right) + \mathcal{O}(\lambda^{\ell+1}). \tag{19}$$

Since $\mathfrak{h}_j^{(\ell)} = \sum_{k=0}^{\ell} \lambda^k h_j^{(k)}$, we obtain that

$$\mathcal{Q}_2^{(\ell)} = \sum_{j=1}^{J} \widehat{\mathrm{Tr}}_{J,\ell+1}\left(h_j^{(\ell)}\right). \tag{20}$$

For the asymptotic region (i.e., $J > \ell+1$), we have the identity $\widehat{\mathrm{Tr}}_{J,\ell+1}(h_1^{(\ell)}) = h_1^{(\ell)}$; therefore, this charge has the usual form $\mathcal{Q}_2^{(\ell)} = \sum_{j=1}^{J} h_j^{(\ell)}$.

Above, we have shown that the solutions of the RLL relations [Eq. (13)] define long-range charges [Eq. (1)] in the asymptotic limit. An important question is whether the reverse statement is also true: i.e., do there exist Lax operators for every integrable long-range charge $\mathcal{Q}_2$? At this point, we do not know the answer. However, I investigated the long-range $\mathfrak{gl}(N)$ spin chains of Ref. [3] up to order $\mathcal{O}(\lambda^3)$. After fixing the unphysical parameters $\gamma(\lambda)$, $\epsilon(\lambda)$, I found the matrices $\check{\mathcal{L}}_1^{(2)}(u)$ which give the $\mathcal{Q}_2^{(2)}$ values for every set of physical parameters $\alpha(\lambda)$, $\beta(\lambda)$ [18].

Long-range spin chains at the wrapping region.—The main advantage of the algebraic construction of the previous section is that the transfer matrix is well defined and satisfies $[\check{\mathcal{T}}^{(\ell)}(u), \check{\mathcal{T}}^{(\ell)}(v)] = \mathcal{O}(\lambda^{\ell+1})$ [20] even for $J < \ell+2$—i.e., the wrapping region. So far, it has not been clear how to define the Hamiltonian in the wrapping

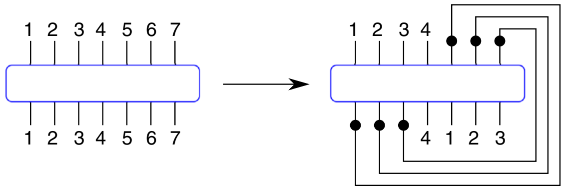

FIG. 2. To the left is a graph for the seven-site operator $h_1^{(5)} = h_{1,2,\ldots,7}$. On the right, we can see the wrapped operator $\tilde{h}_1^{(5)}$ for $J = 4$. The contracted dots denote the summations—e.g., we have to trace out the first incoming and the fifth outgoing legs of the operator $h_{1,2,\ldots,7}$.

region in an integrability-preserving way, but our transfer matrix gives a recipe. We emphasis that the Lax operator [Eq. (11)] is an asymptotic, density-like quantity (since it is defined on an infinite chain and contains the asymptotic Hamiltonian density); therefore, it describes the elementary physical interaction. The transfer matrix shows a consistent possibility how to define this interaction for finite sizes in a translation-invariant and integrability-preserving way.

To obtain the integrable Hamiltonian for the wrapping region ($J \leq \ell + 1$), we only have to use the definition in Eq. (17). We can repeat the previous calculation up to Eq. (20); i.e., the Hamiltonian reads as $Q_2^{(\ell)} = \sum_{j=1}^{J} \tilde{h}_j^{(J,\ell)}$, where we introduce a "wrapped" Hamiltonian density (see Fig. 2)

$$\tilde{h}_1^{(J,\ell)} \equiv \tilde{h}_{12\ldots J}^{(J,\ell)} \coloneqq \widehat{\mathrm{Tr}}_{J,\ell+2-J}\left(h_1^{(\ell)}\right), \tag{21}$$

and the periodic boundary condition is prescribed—i.e., $\tilde{h}_j^{(J,\ell)} = \tilde{h}_{j,j+1,\ldots,J,1,2,\ldots j-1}^{(J,\ell)}$. We see that the twisted trace acts as an identity for the asymptotic region, but for the wrapping region it defines a new operator which "'fits" with the length of the chain.

Let us summarize what we have learned from this analysis. Let us take an asymptotic integrable long-range Hamiltonian

$$\mathcal{H}^\infty = \sum_{j=-\infty}^{\infty} \mathfrak{h}_j^\infty = \sum_{j=-\infty}^{\infty} \sum_{\ell=0}^{\infty} \mathcal{X}_j^{(\ell)}, \tag{22}$$

where $\mathcal{X}_j^{(\ell)} \equiv \mathcal{X}_{j,\ldots,j+\ell+1}^{(\ell)}$ is a coupling-constant-dependent operator with interaction range $\ell + 2$. We see that our method defines a unique integrability-preserving Hamiltonian for every finite length J as

$$\mathcal{H}^J = \sum_{j=1}^{J} \sum_{\ell=0}^{\infty} \tilde{\mathcal{X}}_j^{(J,\ell)},$$
$$\tilde{\mathcal{X}}_1^{(J,\ell)} = \begin{cases} \mathcal{X}_1^{(\ell)}, & \ell + 2 \leq J, \\ \widehat{\mathrm{Tr}}_{J,\ell+2-J}(\mathcal{X}_1^{(\ell)}), & \ell + 2 > J. \end{cases} \tag{23}$$

Inozemtsev's spin chain.—In this section, we demonstrate that our finite-volume Hamiltonian is consistent with a naive physical argument. Let us take a long-range interaction and assume that we already know its manifestation for every length J; i.e., for every length J we have the Hamiltonians $\mathcal{H}^J$ which correspond to the same physical interaction. Since we know the Hamiltonians for every length, we can obtain the asymptotic model by the limit

$$\mathcal{H}^\infty = \lim_{J\to\infty} \mathcal{H}^J. \tag{24}$$

A natural requirement is that our procedure in Eq. (23) should return to the original models $\mathcal{H}^J$.

In the following, we validate this requirement on Inozemtsev's spin chain [21], for which the finite-volume Hamiltonian reads as

$$\mathcal{H}^J = \sum_{1\leq j,k\leq J} \left(\wp(k-j) + \frac{2}{\omega}\zeta\left(\frac{\omega}{2}\right)\right) P_{j,k}, \tag{25}$$

where $\wp(z)$, $\zeta(z)$ are the Weierstrass functions defined on the torus $\mathbb{C}/\mathbb{Z}J + \mathbb{Z}\omega$, $\omega = i(\pi/\kappa)$, and the local Hilbert spaces are $\mathbb{C}^N$. The J is length of the spin chain, and $\kappa \in \mathbb{R}$ is the coupling. In the asymptotic limit, we obtain the hyperbolic Inozemtsev spin chain [21]:

$$\mathcal{H}^\infty = \sum_{-\infty<j<k<\infty} V(k-j) P_{j,k}, \qquad V(j) = \left(\frac{\kappa}{\sinh(j\kappa)}\right)^2. \tag{26}$$

After a renormalization of the coupling constant $\kappa(\lambda)$, this Hamiltonian is compatible with the perturbative long-range description [22]. Let us rewrite the Hamiltonian as

$$\mathcal{H}^\infty = \sum_{-\infty<j<\infty} \mathfrak{h}_j^\infty = \sum_{\ell=0}^{\infty} \sum_{-\infty<j<\infty} \mathcal{X}_j^{(\ell)},$$
$$\mathcal{X}_1^{(\ell)} = V(\ell+1) P_{1,\ell+2}. \tag{27}$$

Now, let us apply Eq. (23) on $\mathcal{X}_1^{(\ell)}$. At first, let us wrap the permutations

$$\tilde{P}_{1,k} = \begin{cases} N, & k \equiv 1 \bmod J \\ P_{1,k_J}, & k \equiv 2, \ldots, J \bmod J, \end{cases} \tag{28}$$

where $1 < k_J \leq J$ and $k_J \equiv k \bmod J$. Now, we can wrap the full $\mathfrak{h}_1^\infty = \sum_{\ell=0}^{\infty} \mathcal{X}_1^{(\ell)}$ as

$$\mathfrak{h}_1^J \coloneqq \sum_{\ell=0}^{\infty} \tilde{\mathcal{X}}_1^{(J,\ell)} = \sum_{1<k\leq J} P_{1,k} \sum_{0\leq l<\infty} V(k + lJ - 1) + c, \tag{29}$$

where $c = N\sum_{1\leq l<\infty} V(lJ)$. The full finite-volume Hamiltonian is

$$\mathcal{H}^J = \sum_{1\leq j\leq J} \mathfrak{h}_j^J = \sum_{1\leq j<k\leq J} P_{j,k} \sum_{-\infty<l<\infty} V(k-j+lJ) + Jc. \tag{30}$$

The infinite sum can be written in the following closed form [Eq. (23.8.3) in Ref. [23]]:

$$\sum_{-\infty<l<\infty} V(k+lJ) = \wp(k) + \frac{2}{\omega}\zeta\left(\frac{\omega}{2}\right). \tag{31}$$

Substituting back and dropping the irrelevant identity operator, we simply obtain the original Inozemtsev Hamiltonian [Eq. (25)]. We can see that our wrapping method gives the finite-volume Inozemtsev spin chain from the infinite-volume hyperbolic Inozemtsev spin chain, which is an expectation for a consistent wrapping procedure.

Wrapping corrections in AdS/CFT.—In this section, we summarize some properties of the wrapping corrections in the planar $\mathcal{N}=4$ super Yang-Mills theory (SYM). We show that our finite-volume Hamiltonians are compatible with these requirements.

Argument 1: On the string theory side (1 + 1-dimensional field theory description), we know that the asymptotic data (dispersion relation and scattering matrix) uniquely define the wrapping corrections. This fact is in agreement with our method, which uniquely defines finite-size Hamiltonians [Eq. (23)] for a given asymptotic Hamiltonian [Eq. (22)].

Argument 2: In the dilatation operator of the $\mathcal{N}=4$ SYM, there are unfixed parameters coming from the free choice of the renormalization scheme [24,25]. These are unphysical parameters which disappear from the spectrum. On the asymptotic level, these parameters correspond to $\epsilon_k(\lambda)$—i.e., the global rotations in Eq. (3); therefore, it is clear that they have no effect on the spectrum. The disappearance on finite volume is a nontrivial condition for the physical finite-size Hamiltonians. It turns out that the spectrum of our finite-volume Hamiltonians is free from $\epsilon_k(\lambda)$ as well [18].

Argument 3: In the asymptotic limit, the spectrum of the closed sectors is completely independent from the full theory. To be more concrete, let us consider three asymptotic Hamiltonians $\mathcal{H}^\infty_{\mathcal{N}=4}$, $\mathcal{H}^\infty_{SU(N)}$, and $\mathcal{H}^\infty_{SU(2)}$, which correspond to the $\mathcal{N}=4$ SYM, one of the $SU(N)$ and the $SU(2)$ long-range models for which the restriction to the $SU(2)$ sector is the same—i.e.,

$$\mathcal{H}^\infty_{\mathcal{N}=4}\Big|_{SU(2)} = \mathcal{H}^\infty_{SU(N)}\Big|_{SU(2)} = \mathcal{H}^\infty_{SU(2)}.$$

Clearly, the spectrum of a closed sector does not know about the full model in which it is embedded. However, we know that for proper wrapping corrections, we have to consider contributions from the full spectrum (for Lüsher corrections, we have to sum for all virtual particles of the mirror model [11]); therefore,

$$\mathcal{H}^J_{\mathcal{N}=4}\Big|_{SU(2)} \neq \mathcal{H}^J_{SU(2)}.$$

This is an important requirement for the definition of the finite-size long-range Hamiltonians. Let us take our definition in Eq. (23). We can see that the wrapped operator $\tilde{\mathcal{X}}_1^{(J,\ell)}$ contains a sum for a tensor product of the *full* local Hilbert spaces. Therefore, these wrapped operators, even in the closed subsectors, explicitly depend on the *full asymptotic Hamiltonian*; therefore, our definition satisfies

$$\mathcal{H}^J_{SU(N)}\Big|_{SU(2)} \neq \mathcal{H}^J_{SU(2)}.$$

Argument 4: We also know that, in the wrapping corrections, extra transcendental numbers appear. For example, let us consider the Konishi operator [a length-4 operator in the $SU(2)$ sector]. At four loops, the asymptotic dilatation operator of the $SU(2)$ sector contains only one transcendental number, $\zeta(3)$ [24,25]. However, the length-4 Hamiltonian at four loops [24,25] contains an extra $\zeta(5)$ compared to the asymptotic Hamiltonian. We already mentioned that our finite-volume Hamiltonian includes a sum for the full one-site Hilbert space in the wrapping region; therefore, extra transcendental numbers can appear in the finite-size Hamiltonians if the one-site Hilbert space is infinite-dimensional, which is the case for $\mathcal{N}=4$ SYM. We note that transcendental numbers appear in the spectrum of higher chargers of nearest-neighbor spin chains with infinite-dimensional local Hilbert spaces [26].

Conclusions.—In this Letter, we generalized the algebraic framework of medium-range spin chains [14] to perturbative long-range spin chains [3]. Using this method, we were able to define finite-volume Hamiltonians [Eq. (23)] for every asymptotic long-range model. We demonstrated that this definition is physically relevant by showing that our definition is in agreement with several physical requirements coming from Inozemtsev's spin chain and AdS/CFT.

We saw that our wrapping procedure [Eq. (23)] leads to wrapping corrections with similar properties to what we expect from the $\mathcal{N}=4$ SYM. This is an important result, because previously, the finite-size corrections under simpler conditions could be studied using integrable field theories. From now on, the wrapping corrections can be also tested on spin chains, which can be simpler in many ways.

I believe that this result could open up a number of new research directions. One possible direction is to generalize the integrable boundary states [27,28] for long-range spin chains as well. Combining this with the method of this Letter, we could investigate the wrapping corrections of the overlaps between boundary and Bethe states which describe certain one- and three-point functions in $\mathcal{N}=4$ SYM [29–35] and ABJM theories [36,37].

It would be interesting to apply the algebraic Bethe ansatz, although it is not clear how this should be done due to the increasing number of auxiliary spaces. However, there are other ways to diagonalize the transfer matrices—e.g., functional techniques [38] (quantum spectral curve [39] for simpler long-range models?) and the separation of variables [40–42].

Another interesting direction would be to give some nonperturbative definitions of the quantities appearing in this Letter (Lax operators, transfer matrix), the derivation of the Yangian symmetry [43] from our framework, and connection for the $T\bar{T}$ deformations of spin chains [44].

Finally, we need to address a major shortcoming of our method. The spin chain which appears in the perturbation theory of $\mathcal{N} = 4$ SYM is dynamical, which means that the Hamiltonian can change the length of the spin chain. Our method in its present form is not suitable for describing such models. In the future, we plan to extend the process to dynamic spin chains, but in the meantime, the nondynamical Hamiltonians like Eq. (23) can serve as good toy models of wrapping effects.

It is worth mentioning that parallel research has also been started on the topic of long-range spin chains [45].

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