Fractal Fluctuations at Mixed-Order Transitions in Interdependent Networks

Bnaya Gross,[1,*] Ivan Bonamassa,[1,2,†] and Shlomo Havlin[1]
[1]*Department of Physics, Bar Ilan University, Ramat Gan, Israel*
[2]*Department of Network and Data Science, CEU, Quellenstrasse 51, A-1100 Vienna, Austria*

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We study the critical features of the order parameter's fluctuations near the threshold of mixed-order phase transitions in randomly interdependent spatial networks. Remarkably, we find that although the structure of the order parameter is not scale invariant, its fluctuations are fractal up to a well-defined correlation length ξ' that diverges when approaching the mixed-order transition threshold. We characterize the self-similar nature of these critical fluctuations through their effective fractal dimension $d_f' = 3d/4$, and correlation length exponent $\nu' = 2/d$, where d is the dimension of the system. By analyzing percolation and magnetization, we demonstrate that d_f' and ν' are the same for both, i.e., independent of the symmetry of the process for any d of the underlying networks.

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Introduction.—Critical phenomena are fundamental features of phase transitions, showing universal behaviors that emerge in the vicinity of the critical point [1,2]. In second-order transitions, these phenomena are typically reflected in the scaling relations between critical exponents [3,4] as well as in the fractal geometry of the order parameter's structure at criticality [5,6]. In first-order transitions, instead, the lack of a diverging correlation length at the transition threshold leaves the structure of the order parameter to be *compact* [7], i.e., not scale invariant and with the same dimension of the embedding space, further preventing the emergence of scaling laws [8]. Hybrid or mixed-order transitions [9,10] lie in between these two classes, displaying both a discontinuous order parameter with a compact structure and critical scaling at the transition point (see Fig. 1). Because of their mixed nature and their appearance in a broad variety of models [11–14] and real-world systems [15–18], hybrid transitions have attracted much attention, both from a theoretical and an experimental perspective, offering the twofold opportunity of discovering novel universal features and to analyze catastrophic shifts.

In this regard, interdependent networks serve as suitable venues [19,20] for the theoretical and experimental study of mixed-order transitions. They, in fact, typically undergo hybrid structural and/or functional transitions due to cascading failures [21], whose properties can depend on the topological or dynamical features of the interacting systems [22] as well as on the range and fraction of dependency links [23–25]. Random and spatial interdependent networks, in particular, host mixed-order percolation transitions when dependency couplings are random [26,27], serving then as tailored models to analyze the critical properties of these transitions in any spatial dimension.

In this Letter, we study the critical fluctuations of the order parameter $\mathcal{O}$ at the mixed-order transition of randomly interdependent lattices. We find that, although $\mathcal{O}$ is compact at the threshold, its *fluctuations* are *self-similar*

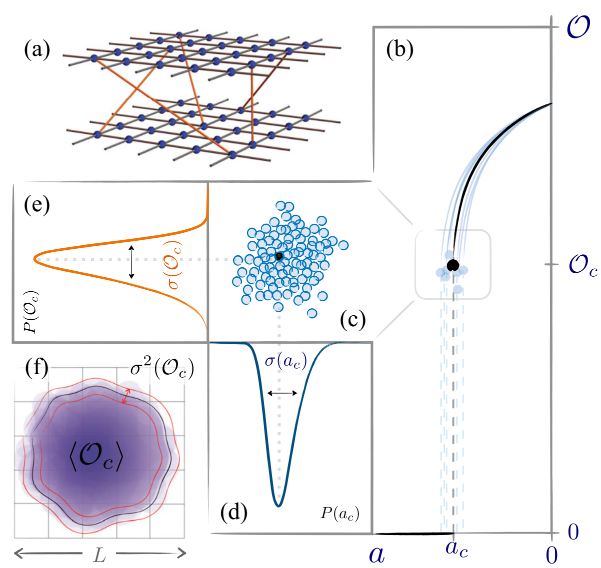

FIG. 1. Fluctuations at a mixed-order transition. (a) Illustration of the model of randomly interdependent d-dimensional networks (here $d = 2$) studied in this Letter, featuring short-range connectivity links (gray links) and long-range dependency links (orange couplings). (b) Each realization of a hybrid transition [see (c) for a close-up of the bounded region] has its own critical threshold a_c and critical mass $\mathcal{O}_c \equiv \mathcal{O}(a_c)$—related to each other via the scaling law [9] $\mathcal{O}(a) \sim \mathcal{O}_c + |a - a_c|^{1/2}$—whose distributions follow a certain profile, as shown in (d) for $P(a_c)$ and in (e) for $P(\mathcal{O}_c)$. (f) Illustration of the fluctuating values of the critical mass, their mean value $\langle \mathcal{O}_c \rangle$, and their statistical variation $\sigma^2(\mathcal{O}_c)$.

and are characterized by an effective fractal dimension d'_f in length scales up to a correlation length, ξ', which diverges close to the mixed-order transition threshold a_c as

$$\xi' \sim |a - a_c|^{-\nu'}, \qquad \nu' = 2/d, \tag{1}$$

where a is a control parameter. We demonstrate this in both interdependent percolation and interdependent magnetization processes in d-dimensional lattices, where we show that (1) the exponents ν' and d'_f are independent of the underlying process, (2) their values are valid for any dimension $d \geq 2$ (i.e., the upper and lower critical dimension is $d_c = 2$) and any number of interdependent layers $M \geq 2$, and (3) they satisfy hyperscaling [28]. Building on the above, we put forward the hypothesis that fractal fluctuations are a universal property of mixed-order transitions and we support this claim by developing and testing a unifying scaling theory for the order parameter's fluctuations in the vicinity of the critical point.

Model and main results.—To present our model, let us consider $M = 2$ randomly interdependent d-dimensional lattices of size $N = L^d$. We insert the dependency links between the layers by *randomly* pairing the functional states of the two lattices' sites [29], thus generating a multilayer system with short-range connectivity and long-range dependency [Fig. 1(a)]. We consider the networks to be fully interdependent, i.e., each node in one layer depends on the state (in what follows, percolation and magnetization) of a randomly chosen node in the other layer. By collecting a large sample [see Table 1 in Supplemental Material (SM) [30]] of independent realizations of the hybrid phase transitions reported in both models [Figs. 1(b) and 1(c)], we study the fluctuations of their critical thresholds $\sigma^2(a_c) = \langle a_c^2 \rangle - \langle a_c \rangle^2$, and of their order parameter's critical mass $\sigma^2(\mathcal{O}_c) = \langle \mathcal{O}_c^2 \rangle - \langle \mathcal{O}_c \rangle^2$. Both quantities are obtained, respectively, from the distributions $P(a_c)$ [Fig. 1(d)] and $P(\mathcal{O}_c)$ [Fig. 1(e)].

Following a method introduced by Levinshtein *et al.* [31] and later discussed by Stauffer [4] (see also [3]), we determine the correlation length critical exponent introduced in Eq. (1) by finite-size scaling as

$$\sigma(a_c) \sim L^{-1/\nu'}, \qquad \nu' = 2/d. \tag{2}$$

A fundamental question then arises: what is the physical role played by the diverging correlation length of *fluctuations*, Eq. (1), at a mixed-order phase transition?

In analogy with continuous transitions, where the order parameter (near criticality) is fractal below the correlation length, we show in what follows that at mixed-order transitions the critical fluctuations of the order parameter's mass [Fig. 1(f)] themselves are self-similar up to ξ', with a well-defined fractal dimension d'_f given by

$$\sigma(\mathcal{O}_c) \sim L^{d'_f}, \qquad d'_f = 3d/4, \tag{3}$$

which we support by extensive simulations and hyperscaling arguments (see Discussion). In light of the above, we advance a scaling theory for the fluctuations of the order parameters' mass close to criticality as follows. At short scales, $L < \xi'$, $\sigma(\mathcal{O})$ follows Eq. (3), while at long scales, $L > \xi'$, the fluctuations are noncritical and satisfy the scaling law $\sigma(\mathcal{O}) \sim \sqrt{N} = L^{d/2}$ [see Fig. S1 (SM) [30]]. Combining the above observation, we obtain the scaling function

$$\frac{\sigma(\mathcal{O})}{L^{d/2}\xi'^{d/4}} = \mathcal{G}(L/\xi'), \tag{4}$$

where $\mathcal{G}(x)$ is constant for $x \gg 1$ and $\mathcal{G}(x) \sim x^{d/4}$ for $x \ll 1$. Remarkably, we find that Eq. (4) is valid for both randomly interdependent percolation (Figs. 2 and 3) and magnetization processes (Fig. 4), both on spatial and on random networks (SM [30], Fig. S5), and it is satisfied for

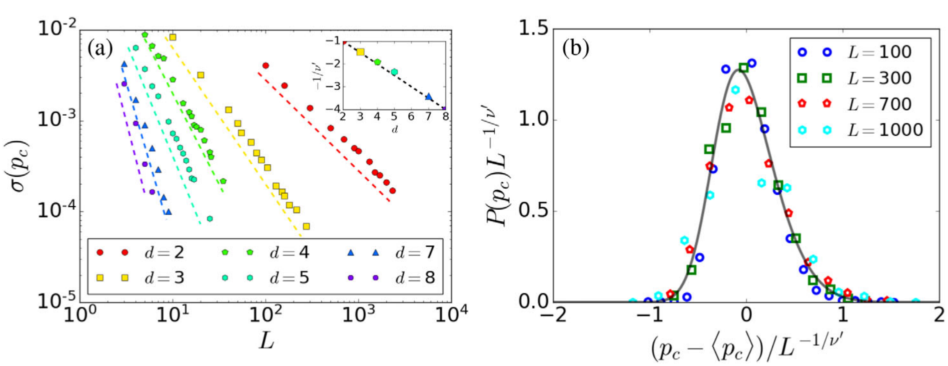

FIG. 2. Correlation length exponents of fluctuations length. (a) Scaling of $\sigma(p_c)$ with L, for all studied dimensions ($d = 2$–8). We find excellent agreement between simulations and the scaling relation in Eq. (2). Inset: the dependence of ν' on the dimension d of the underlying lattices follows the relation $\nu' = 2/d$. (b) The distribution of p_c (here, $d = 2$) fits a skewed Gaussian and follow the scaling in Eq. (5) (black line) with $\gamma_1 \simeq 0.465$ and $\kappa \simeq 0.315$.

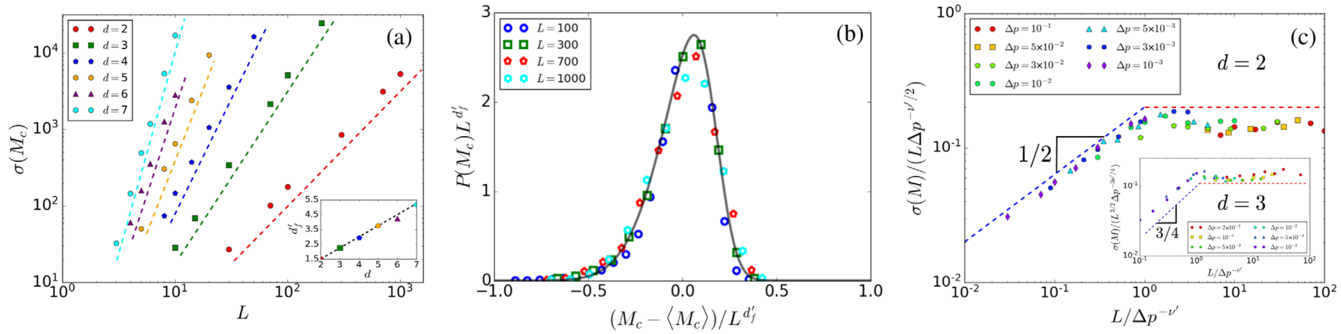

FIG. 3. Fractal fluctuations. (a) Simulations of the scaling of $\sigma(M_c)$ with L show excellent agreement with Eq. (3) where $d'_f = 3d/4$ (see inset). (b) Data collapse of $P(M_c)$ for $d = 2$ under the scaling $P(M_c)L^{d'_f} \sim \mathcal{F}[(M_c - \langle M_c\rangle)/L^{d'_f}]$ (black line), where $\mathcal{F}$ is a skewed Gaussian as in Eq. (5), now with $\gamma_1 \simeq -0.607$ and $\kappa \simeq 0.449$. (c) Close to the hybrid percolation threshold, the MGCC's fluctuations are fractal up to length scales (represented by L) below ξ' [Eq. (1)] and nonfractal otherwise, as described by the universal scaling function in Eq. (4). Results are shown for $d = 2$ and for $d = 3$ in the inset.

any dimension $d \geq 2$ and any number of layers $M \geq 2$ (SM [30], Fig. S6), hinting at its universal nature.

Interdependent percolation.—To percolate our system of randomly interdependent lattices, we remove at random a fraction $1 - p$ of the nodes belonging to one layer and let the cascading of failures to propagate back and forth between the layers. In finite systems, each realization is characterized by a distinct percolation threshold p_c and a critical mass of the mutual giant connected component (MGCC), $M_c = S^c_\infty L^d$, where $S^c_\infty \equiv S_\infty(p_c)$ is the relative fraction of nodes within the MGCC at criticality [Fig. 1(b)]. We find that their best-fitting distribution is a *skewed Gaussian* [Figs. 1(d) and 1(e); and SM [30], Fig. S4] with nonzero skewness [32] $\gamma_1 = \langle x^3\rangle$ and kurtosis $\kappa = \langle x^4\rangle$ [see Figs. 2(b) and 3(b)]. Here, $x = (y - \langle y\rangle)/\sigma(y)$ is the normalized parameter of the distribution and y is the observable of interest (p_c and M_c in our case). The normal form of the two distributions indicate that the *thermodynamic* exponents $\beta', \gamma', \delta', \ldots$ characterizing the critical fluctuations of the order parameter can be cast within the standard ϕ^4-field theory [33] and, as such, they belong to the mean-field Ising universality class (see Discussions). Exponents and scaling relations related to the system's dimensionality d, such as ν' or d'_f, on the other hand, are more delicate since they can be strongly influenced by the presence of dangerous irrelevant variables altering the singular part of the system's free energy [34]. To determine ν' and d'_f at hybrid percolation transitions in randomly interdependent lattices, we follow the method proposed in the above and analyze the finite-size scaling of $\sigma^2(p_c)$ and $\sigma^2(M_c)$. As shown in Fig. 2(a), the scaling of the width of the distribution $P(p_c)$ results in a correlation length exponent ν', whose value explicitly depends on the network's dimension and nicely agrees with the relation $\nu' = 2/d$ [Fig. 2(a), inset]. To further corroborate the expression $\nu' = 2/d$, we perform a data collapse of the distributions $P(p_c)$ obtained for different system sizes which, in light of Eq. (1), can be rescaled [3] as

$$P(p_c)L^{-1/\nu'} \sim \mathcal{F}[(p_c - \langle p_c\rangle)/L^{-1/\nu'}], \tag{5}$$

where $\mathcal{F}(x)$ fits the profile of a skewed Gaussian. As shown in Fig. 2(b) for $d = 2$, the data gathered over different system sizes nicely collapse.

As anticipated in the above, the diverging correlation length ξ' manifests physically the self-similar character of critical fluctuations of the order parameters' mass M_c at the hybrid percolation threshold. Indeed, as displayed in Fig. 3(a), the scaling advanced in Eq. (3) is nicely corroborated by means of extensive simulations and the fractal fluctuation dimension $d'_f = 3d/4$ is observed over several decades in randomly interdependent lattices of dimensions ranging from $d = 2$ up to $d = 7$. Notice that interdependent chains do not undergo any phase transitions due to the absence of ordering already in the isolated $d = 1$ layers themselves. The inset of Fig. 3(a), in particular, highlights the validity of the expression $d'_f = 3d/4$ which we corroborate by performing a data collapse of the distribution $P(M_c)$ [see Fig. 3(b) and details in caption] and further justify by means of hyperscaling arguments in the Discussion. An interesting implication of the above results is that one of the classical properties of continuous phase transitions, i.e., the ratio $\langle M_c\rangle/\sigma(M_c)$ is independent of the system size [35], breaks down at mixed-order transitions. In fact, since the MGCC itself is compact at criticality, then $\langle M_c\rangle \sim L^d$ and the ratio scales with the fluctuations codimension $\Delta' = d - d'_f = d/4$ as $\langle M_c\rangle/\sigma(M_c) \sim L^{d/4}$.

To complete the picture, we analyze the structure of fluctuations near the mixed-order percolation threshold. In light of Eq. (4), we expect that when taking a small displacement $\Delta p = p - p_c$ from the abrupt threshold, the critical fluctuations of the MGCC's mass will be fractal with dimension d'_f up to ξ' and noncritical (i.e., white noise, see Fig. S1 in SM [30]) otherwise. We support this picture in Fig. 3(c) with simulations in $d = 2$ and $d = 3$ [Fig. 3(c), inset] lattices. By rescaling the critical width Δp via Eq. (1), the crossover between the (critical) fractal fluctuations regime and the (noncritical) Gaussian regime is nicely seen. We further verify the presence of this crossover at the hybrid percolation transition in interdependent random graphs [SM [30], Fig. S5(a)] and in $M = 3$ interdependent lattices [SM [30], Figs. S6(a) and S6(b)].

Interdependent magnetization.—To scrutinize the universality of the fractal fluctuations phenomenon at mixed-order transitions, we consider a model of interdependent Ising-spin networks where dependency couplings between layers are realized as thermal interactions [36]. We consider here the particular case of d-dimensional lattices modeled as in Fig. 1(a), where each node is endowed with an Ising spin $\sigma_i = \pm 1$. Dependency couplings between the layers are inserted as local thermal feedback on the level of the flipping probability of spins (see Fig. S2 in SM [30] and discussions therein for details), which intertwine adaptively the stochastic (Metropolis) dynamics of the two layers. In randomly interdependent spins lattices, the average magnetization $M = \sum_i \langle \sigma_i \rangle_\beta$ (where $\langle (\cdots) \rangle_\beta$ is a thermal average) undergoes a spontaneous mixed-order ferromagnetic-to-paramagnetic transition [Fig. 4(a)] at a finite critical temperature $T_c \simeq 1.25d$ [see Fig. 4(a), inset] and for any dimension $d \geq 2$, analogously to the abrupt collapse of the MGCC at p_c in interdependent percolation.

We perform an analysis analogous to the one put forward in the above for interdependent percolation and determine the fluctuation correlation length exponent ν' and its fractal dimension d'_f, respectively, from the scaling of $\sigma(T_c)$ and of $\sigma(M_c)$. Figure 4(b) shows the finite-size scaling of $\sigma(T_c)$, whose behavior nicely follows Eq. (2) with $\nu' = 2/d$ [see Fig. 4(b), inset] and supports the existence of a diverging correlation length, Eq. (1). Moving to the self-similar properties, we display in Fig. 4(c) the scaling of $\sigma(M_c)$ and confirm the validity of the fractal fluctuation dimension $d'_f = 3d/4$ [Fig. 4(c) inset] in agreement with Eq. (3). Both our measurements of ν' and of d'_f are further validated via their corresponding distributional collapse [SM [30], Figs. S3(a), S3(b), and S4(b)]. To close the picture, we show in Fig. 4(d) the behavior of critical fluctuations near the mixed-order magnetization transition in $d = 2$. We find also here excellent agreement with the scaling relation proposed in Eq. (4), supporting further the universality of the crossover at hybrid transitions from the fractal ($L \ll \xi'$) to the nonfractal ($L \gg \xi'$) fluctuation regime. Finally, as with percolation, we support the existence of fractal fluctuations also at the hybrid magnetization transition in interdependent random graphs [SM [30], Fig. S5(b)] and in $M = 3$ randomly interdependent lattices [SM [30], Figs. S6(c) and S6(d)].

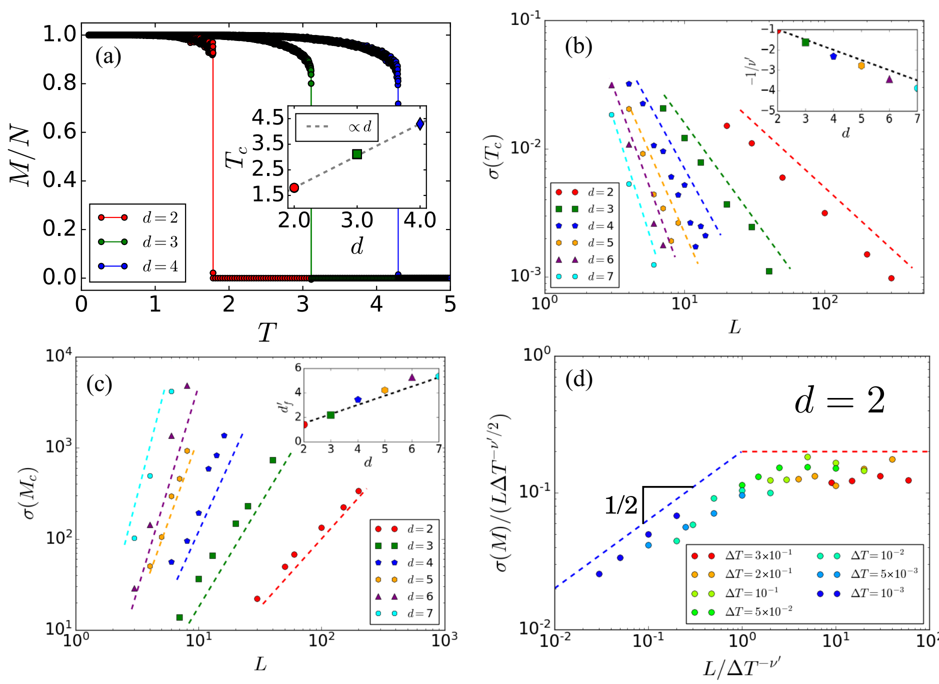

FIG. 4. Fractal fluctuations in interdependent magnetization. (a) Mixed-order transition in randomly interdependent Ising d-dimensional lattices (here $d = 2$, 3, 4) measured via the magnetic density M/N as a function of temperature T. Inset: the critical temperature T_c scales linearly with the lattices' dimension d. (b) Simulations of the scaling of $\sigma(T_c)$ with L show excellent agreement with Eq. (2) where $\nu' = 2/d$ as shown in the inset. (c) At criticality ξ' diverges and the fluctuations of the MGCC are fractal in all length scales and follow Eq. (3) with $d'_f = 3d/4$ as shown in the inset. (d) Fluctuations are fractal up to ξ' [Eq. (1)] and nonfractal above it, confirming the scaling in Eq. (4), here shown with $d = 2$.

Discussion.—In this Letter, we have unveiled the presence of self-similarity and of a diverging length scale, Eq. (1), in the finite-size scaling of critical fluctuations at mixed-order transitions. These critical phenomena are all the more surprising as the order parameter at the transition threshold is always compact, and they suggest that critical fluctuations have a fractal geometry in some appropriate metric space. We characterize these phenomena via an *effective* fractal dimension $d'_f = 3d/4$ and a correlation length exponent $\nu' = 2/d$, which we verify numerically for percolation and magnetization processes both *at*, Eqs. (2), (3), and *close* to, Eq. (4), the transition's threshold. Since the fluctuations are normal-like distributed, a hyperscaling hypothesis can be put forward to justify the exponents (ν', d'_f) in terms of the thermodynamic ones $\beta' = 1/2$, $\gamma' = 1$, etc., characterizing the mean-field Ising universality class [37]. Indeed, one can readily verify that the hyperscaling relations $d\nu' = 2\beta' + \gamma'$ and $d'_f = d - \beta'/\nu'$ are identically satisfied. In this light, our results hold a universal character and we expect the phenomenon of fractal fluctuations to be experimentally observable in other models undergoing mixed-order transitions [38], e.g., in colloidal crystals [16], networks of active gels [15], or interdependent superconductors [39].

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B. G. and I. B. contributed equally to this work.

[*]To whom correspondence should be addressed.
bnaya.gross@gmail.com
[†]To whom correspondence should be addressed.
bonamassai@ceu.edu

[1] C. Domb, *Phase Transitions and Critical Phenomena* (Elsevier, New York, 2000).
[2] H. E. Stanley, *Introduction to Phase Transitions and Critical Phenomena* (Oxford University Press, New York, 1971).
[3] A. Bunde and S. Havlin, *Fractals and Disordered Systems* (Springer-Verlag, New York, 1991).
[4] D. Stauffer, *Introduction to Percolation Theory* (Francis & Taylor, London, 1985).
[5] A. Coniglio, J. Phys. A **15**, 3829 (1982).
[6] A. Coniglio, Phys. Rev. Lett. **62**, 3054 (1989).
[7] F. Family and A. Coniglio, J. Phys. A **13**, L403 (1980).
[8] K. Binder, Rep. Prog. Phys. **50**, 783 (1987).
[9] S. Boccaletti, J. A. Almendral, S. Guan, I. Leyva, Z. Liu, I. Sendiña-Nadal, Z. Wang, and Y. Zou, Phys. Rep. **660**, 1 (2016).
[10] R. M. D'Souza, J. Gómez-Gardenes, J. Nagler, and A. Arenas, Adv. Phys. **68**, 123 (2019).
[11] G. Parisi and T. Rizzo, Phys. Rev. E **78**, 022101 (2008).
[12] S. N. Dorogovtsev, A. V. Goltsev, and J. F. F. Mendes, Rev. Mod. Phys. **80**, 1275 (2008).
[13] A. Bar and D. Mukamel, Phys. Rev. Lett. **112**, 015701 (2014).
[14] A. de Candia, A. Fierro, and A. Coniglio, Sci. Rep. **6**, 26481 (2016).
[15] M. Sheinman, A. Sharma, J. Alvarado, G. H. Koenderink, and F. C. MacKintosh, Phys. Rev. Lett. **114**, 098104 (2015).
[16] R. Alert, P. Tierno, and J. Casademunt, Proc. Natl. Acad. Sci. U.S.A. **114**, 12906 (2017).
[17] J. C. Rocha, G. Peterson, Ö. Bodin, and S. Levin, Science **362**, 1379 (2018).
[18] G. F. de Arruda, G. Petri, P. M. Rodriguez, and Y. Moreno, arXiv:2112.04273.
[19] S. V. Buldyrev, R. Parshani, G. Paul, H. E. Stanley, and S. Havlin, Nature (London) **464**, 1025 (2010).
[20] J. Gao, A. Bashan, L. Shekhtman, and S. Havlin, *Introduction to Networks of Networks* (IOP Publishing, 2022).
[21] M. M. Danziger, I. Bonamassa, S. Boccaletti, and S. Havlin, Nat. Phys. **15**, 178 (2019).
[22] V. Nicosia, G. Bianconi, V. Latora, and M. Barthelemy, Phys. Rev. Lett. **111**, 058701 (2013).
[23] R. Parshani, S. V. Buldyrev, and S. Havlin, Phys. Rev. Lett. **105**, 048701 (2010).
[24] R. Parshani, S. V. Buldyrev, and S. Havlin, Proc. Natl. Acad. Sci. U.S.A. **108**, 1007 (2011).
[25] M. M. Danziger, L. M. Shekhtman, Y. Berezin, and S. Havlin, Europhys. Lett. **115**, 36002 (2016).
[26] W. Li, A. Bashan, S. V. Buldyrev, H. E. Stanley, and S. Havlin, Phys. Rev. Lett. **108**, 228702 (2012).
[27] A. Bashan, Y. Berezin, S. V. Buldyrev, and S. Havlin, Nat. Phys. **9**, 667 (2013).
[28] S.-K. Ma, *Modern Theory of Critical Phenomena* (Routledge, New York, 2018).
[29] While connectivity links allow physical processes to hop from one node to another within layers, dependency links make those processes to depend on each other while preventing their hopping between layers.
[30] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.129.268301 for more details.
[31] M. E. Levinshtein, B. I. Shklovskii, M. S. Shur, and A. L. Efros, Zh. Exp. Theor. Fis. **69**, 386 (1975) Sov. Phys., JETP **42**, 197 (1976)].
[32] Critical fluctuations are biased toward the low stable state with larger deviations from the mean compared to the metastable side.
[33] G. Parisi, *Statistical Field Theory* (Addison-Wesley, Redwood, 1988), Vol. 66.
[34] M. E. Fisher, Lect. Theor. Phys. **VIIC**, 1 (1983).
[35] S. T. Bramwell, J. Y. Fortin, P. C. W. Holdsworth, S. Peysson, J. F. Pinton, B. Portelli, and M. Sellitto, Phys. Rev. E **63**, 041106 (2001).
[36] I. Bonamassa, B. Gross, and S. Havlin, arXiv:2110.08907.
[37] D. Lee, S. Choi, M. Stippinger, J. Kertész, and B. Kahng, Phys. Rev. E **93**, 042109 (2016).
[38] T. R. Kirkpatrick and D. Thirumalai, Rev. Mod. Phys. **87**, 183 (2015).
[39] I. Bonamassa, B. Gross, M. Laav, I. Volotsenko, A. Frydman, and S. Havlin, arXiv:2207.01669.