# Associativity of One-Loop Corrections to the Celestial Operator Product Expansion

Kevin Costello*
*Perimeter Institute for Theoretical Physics, Waterloo, Ontario N2L 2Y5, Canada*

Natalie M. Paquette†
*Department of Physics, University of Washington, Seattle, Washington 98195, USA*

(Received 27 April 2022; accepted 15 November 2022; published 1 December 2022)

There has been recent interest in the question of whether QCD collinear singularities can be viewed as the operator product expansion of a two-dimensional conformal field theory. We analyze a version of this question for the self-dual limit of pure gauge theory (incorporating states of both helicities). We show that the known one-loop collinear singularities do not form an associative chiral algebra. The failure of associativity can be traced to a novel gauge anomaly on twistor space. We find that associativity can be restored for certain gauge groups if we introduce an unusual axion, which cancels the twistor space anomaly by a Green-Schwarz mechanism. Alternatively, associativity can be restored for some gauge groups with carefully chosen matter.

DOI: 10.1103/PhysRevLett.129.231604

*Introduction.*—The celestial holography program (see the recent reviews [1–3] and references therein) suggests, among other things, that collinear singularities in the scattering amplitudes of gauge theory and gravity are controlled by a conformal field theory (CFT). This is known to be true at tree level [4], and in the beautiful paper [5] it was shown to persist to one-loop level in one formulation of self-dual gravity and gauge theory.

In this Letter we analyze a different formulation of this question for self-dual gauge theory, and find a different result. In our Letter, we define self-dual gauge theory to include states of both helicities, but with the Lagrangian $\int BF(A)_-$. This is in contrast to [5], where only states of positive helicity are considered; see also [6–9] for earlier studies of self-dual QCD. With our definition, which is common in twistor studies [10], self-dual gauge theory can be deformed to QCD by adding the operator $\frac{1}{2}\mathrm{tr}(B^2)$. Throughout this Letter, we will use "QCD" to mean pure Yang-Mills theory (unless otherwise specified).

We analyze collinear singularities that appear not just in amplitudes, but in *form factors*. Form factors are scattering amplitudes in the presence of a local operator.

We say a collinear singularity is *universal* if it appears in the same way in all form factors. Universal collinear singularities in self-dual gauge theory capture certain collinear singularities in QCD. This is because certain form factors of self-dual gauge theory for the operator $\mathrm{tr}(B^2)$ are the same as certain QCD amplitudes (e.g., at one loop they compute QCD amplitudes with one negative helicity gluon).

We ask the question: do universal collinear singularities in self-dual gauge theory form a CFT? We find that the answer is *no*: associativity of the operator product expansion (OPE) fails at one loop [11].

We can trace the failure of associativity to an anomaly on twistor space. In [12] we studied universal collinear singularities of self-dual gauge theory, using a twistor space analysis. The twistor uplift of self-dual gauge theory is holomorphic BF theory [13]. It was shown in [14] that holomorphic BF theory has a one-loop gauge anomaly on twistor space, that can be canceled by the introduction of an additional field [15]. The cancellation takes the form of a Green-Schwarz mechanism, and holds if the gauge group is SU(2), SU(3), SO(8) or an exceptional group. See Sec. 1.6 of [12] for the explicit cancellation of 4D amplitudes.

On space-time this additional field becomes an axionlike field (which we simply refer to as an axion hereafter), with a fourth-order kinetic term [17]:

$$\int \mathrm{tr}[BF(A)_-] - \frac{1}{2}\int (\Box\rho)^2 - \frac{2\sqrt{5}\mathsf{h}^\vee}{\sqrt{2(\dim\mathfrak{g}+2)}8\pi\sqrt{3}} \times \int \rho\, \mathrm{tr}[F(A)\wedge F(A)], \tag{1}$$

where $\mathsf{h}^\vee$ is the dual Coxeter number. The gauge field $A$ has helicity $+1$, while $B$ has helicity $-1$. We emphasize that the 4D theory, including the axion, is a conformal field theory.

The twistor origin of the theory implies that it has vanishing scattering amplitudes, but form factors are nonvanishing.

In [12] we showed on abstract grounds that, when we cancel the twistor space anomaly, the universal collinear singularities have the structure of a chiral algebra.

We find (by an explicit computation) that associativity of QCD collinear singularities is *restored* when we add the axion. This is a Green-Schwarz mechanism for associativity of the collinear singularities.

Conversely, once we introduce the axion, associativity of the OPE forces the collinear singularities in the gauge sector to have certain one-loop corrections. These include the standard one-loop QCD correction. In this way, we find a purely chiral-algebraic computation of the standard one-loop collinear singularities.

Finally, we leverage associativity to find a remarkably simple formula for the one-loop amplitudes of QCD with an axion (or of QCD without an axion, but with carefully chosen matter and gauge group).

*The chiral algebra.*—We will start by reviewing the tree-level chiral algebra encoding collinear singularities in self-dual gauge theory. Our analysis uses analytically continued momenta, and so works in any signature. As is standard, in the spinor-helicity formalism states are expressed in terms of spinors $\lambda_\alpha, \tilde{\lambda}^{\dot{\alpha}}$. The chiral algebra lives on a copy of $\mathbb{CP}^1$ with homogeneous coordinates $(\lambda_1 : \lambda_2)$. We will use a coordinate $z$ corresponding to $(\lambda_1 : \lambda_2) = (1 : z)$.

The chiral algebra is generated by two towers of states $J_a[r,s](z)$, $\tilde{J}_a[r,s](z)$, corresponding to particles of positive and negative helicity, respectively. We can arrange these into generating functions

$$J_a[\tilde{\lambda}](z) = \sum \omega^{r+s} \frac{1}{r!s!} (\tilde{\lambda}^{\dot{1}})^r (\tilde{\lambda}^{\dot{2}})^s J_a[r,s](z),$$
$$\tilde{J}[\tilde{\lambda}](z) = \sum \omega^{r+s} \frac{1}{r!s!} (\tilde{\lambda}^{\dot{1}})^r (\tilde{\lambda}^{\dot{2}})^s \tilde{J}_a[r,s](z). \qquad (2)$$

These generating functions correspond to gauge theory states of positive and negative helicity, with momenta encoded in the spinors $\tilde{\lambda}$ and $\lambda = (1, z)$. Because we are expanding in powers of the energy $\omega$, these chiral algebra states should be thought of as soft modes. Precisely, we have $\mathcal{O}_a^-(z) = \omega \tilde{J}_a[\tilde{\lambda}](z), \mathcal{O}_a^+(z) = (1/\omega) J_a[\tilde{\lambda}](z)$, where $\mathcal{O}^\pm$ are the positive and negative helicity hard gluon operators dual to momentum eigenstates of energy $\omega$.

In expressing the OPE, we write

$$\langle ij \rangle = 2\pi i (z_i - z_j), \qquad [ij] = -2\epsilon_{\dot{\alpha}\dot{\beta}} \tilde{\lambda}_i^{\dot{\alpha}} \tilde{\lambda}_j^{\dot{\beta}}. \qquad (3)$$

In Supplemental Material [22] we provide more details on these conventions. The normalization of $[ij]$ is in order to match standard conventions where $\langle ij \rangle [ij] = 2 p_i \cdot p_j$.

The tree-level OPE was derived in [12] using twistor space methods, but also matches the standard [23] tree-level splitting amplitudes. The tree-level OPE is

$$J_a[\tilde{\lambda}_1](z_1) J_b[\tilde{\lambda}_2](z_2) \sim f_{ab}^c \frac{1}{\langle 12 \rangle} J_c[\tilde{\lambda}_1 + \tilde{\lambda}_2](z_1),$$
$$J_a[\tilde{\lambda}_1](z_1) \tilde{J}_b[\tilde{\lambda}_2](z_2) \sim f_{ab}^c \frac{1}{\langle 12 \rangle} \tilde{J}_c[\tilde{\lambda}_1 + \tilde{\lambda}_2](z_1). \qquad (4)$$

These two OPEs correspond to splitting amplitudes $+ \mapsto ++$ and $- \mapsto +-$. In QCD, there are also the parity-conjugate tree level splitting amplitudes of the form $- \mapsto --$ and $+ \mapsto -+$. These do not appear in our self-dual gauge theory, which only has a $++-$ vertex.

*One loop corrections.*—One loop QCD splitting amplitudes have been analyzed in [24–26]. There are two new processes at one loop, namely the $- \mapsto ++$ amplitude and its parity conjugate. In the normalization of [26] this splitting amplitude is

$$\mathrm{split}_+^{[1]}(a^+, b^+) = -\frac{N_c}{96\pi^2} \frac{[ab]}{\langle ab \rangle^2}. \qquad (5)$$

This is the only one-loop amplitude from the analysis of [26] that contributes to self-dual gauge theory. Indeed, all one-loop diagrams in self-dual gauge theory, when all particles are viewed as incoming, have positive helicity external lines. Therefore they can only contribute to a $- \mapsto ++$ splitting amplitude. The remaining one-loop splitting amplitudes of QCD are either multiplicative corrections to the tree-level splitting amplitudes, or the parity-conjugate $+ \mapsto --$ splitting amplitude. Neither of these can appear from a one-loop diagram in self-dual gauge theory.

Perhaps surprisingly, our analysis shows that there are two other terms in the OPE of the form $-+ \mapsto ++$ and $-- \mapsto -+$. These contributions, corresponding to two states becoming a normal-ordered product of two states, seem not to have appeared in other work on the topic; perhaps they are not visible in standard QCD amplitudes, but only in form factors. It would be fascinating to understand the 4D interpretation of these terms.

Let us implement the splitting amplitude (5) in the chiral algebra. Looking at the definition of $[ab]$ and $\langle ab \rangle$ in the chiral algebra, we see that the only natural way is to add on a term in the OPE of the form

$$J_a[\tilde{\lambda}_1](z_1) J_b[\tilde{\lambda}_2](z_2) \sim -\frac{N_c}{96\pi^2} \frac{[12]}{\langle 12 \rangle^2} f_{ab}^c \tilde{J}_c[\tilde{\lambda}_1 + \tilde{\lambda}_2] \left( \frac{1}{2} z \right). \qquad (6)$$

On the right hand side, the operator is evaluated at $\frac{1}{2} z \coloneqq (z_1 + z_2/2)$; this is forced by symmetry. We can rewrite this OPE as

$$J_a[\tilde{\lambda}_1](z_1) J_b[\tilde{\lambda}_2](z_2) \sim -\frac{N_c}{96\pi^2} \frac{[12]}{\langle 12 \rangle^2} f_{ab}^c \tilde{J}_c[\tilde{\lambda}_1 + \tilde{\lambda}_2](z_1) + \frac{N_c}{192\pi^2} \frac{[12]}{\langle 12 \rangle} f_{ab}^c \frac{1}{2\pi i} \partial_z \tilde{J}_c[\tilde{\lambda}_1 + \tilde{\lambda}_2](z_1). \qquad (7)$$

Finally, specializing the generating function $J[\tilde{\lambda}]$ to the terms linear in $\tilde{\lambda}$, we find the OPE

$$J_a[1,0](z_1)J_b[0,1](z_2) \sim \frac{N_c}{48\pi^2}\frac{1}{\langle 12\rangle^2}f^c_{ab}\tilde{J}_c[0,0](z_1) - \frac{N_c}{96\pi^2}\frac{1}{\langle 12\rangle}f^c_{ab}\frac{1}{2\pi i}\partial_z\tilde{J}_c[0,0](z_1). \tag{8}$$

*Failure of associativity of the corrected OPE.*—Now we can ask if this new OPE satisfies the associativity relation of a chiral algebra. The associativity of the OPE is encoded in equalities between contour integrals of OPEs involving three operators, coming from the change of contour. For instance, one of the associativity equations is the identity

$$\begin{aligned} &K^{ab}\oint_{|z|=1,|w|=2} J_a[1,0](0)J_b[0,1](z)J_c[0,0](w)w\,dw\,dz \\ &= K^{ab}\oint_{|z|=2,|w|=1}(J_a[1,0](0)J_c[0,0](w)w\,dw)J_b[0,1](z)\,dz \\ &\quad + K^{ab}\oint_{|z|=2,|w|=1} J_a[1,0](0) \\ &\quad \times (J_b[0,1](z)J_c[0,0](w+z)(w+z)dw)dz, \end{aligned} \tag{9}$$

where $K^{ab}$ is the Killing form. We find that this identity fails to hold with our quantum corrected OPE [27].

We first note that the left hand side vanishes. If we take the $z$ contour integral first, the terms with a first order pole are antisymmetric in the $a$ and $b$ indices. Similarly, the first term on the right hand side vanishes because there is no second order pole in $w$.

For the second term on the right hand, we can perform the $w$ contour integral first. This yields

$$-\oint_z J_a[1,0](0)f^d_{bc}J_d[0,1](z)z\,dz. \tag{10}$$

This is nonzero, because the one-loop correction to the OPE introduces a second-order pole. The result of this contour integral is

$$-\frac{1}{2\pi i}\frac{N_c}{48\pi^2}\tilde{J}_e[0,0]K^{ab}f^e_{ad}f^d_{bc}. \tag{11}$$

Since $K^{ab}f^e_{ad}f^d_{bc}$ is the action of the quadratic Casimir in the adjoint representation, it is proportional to $\delta^e_c$. We conclude that the one-loop corrected OPE is not associative.

Alternatively, if we build a chiral algebra in which associativity is forced to hold, we find that all states of negative helicity become zero.

*Twistor space anomalies and chiral algebra associativity.*—This failure of associativity is connected to the twistor space anomaly we have already mentioned. As explained in [12], from any local, anomaly-free theory on twistor space we can build a chiral algebra living on the twistor $\mathbb{CP}^1$.

If we do this for the twistor uplift of self-dual gauge theory, then, at tree level, this matches the chiral algebra describing the tree-level collinear singularities in self-dual gauge theory. However, the twistor uplift is anomalous at loop level. To cancel this anomaly, we need to introduce a new field on twistor space, corresponding to the axion in Eq. (1).

We know on general grounds [12] that the theory including the axion corresponds to a consistent chiral algebra. Here, we will determine that this chiral algebra contains a one-loop correction to the classical OPE which matches the one-loop splitting amplitude.

For associativity to hold, the axion field is essential. This tells us that the failure of associativity of the quantum-corrected OPE is a reflection of the twistor space anomaly, and is solved by the same Green-Schwarz mechanism.

To perform the calculation, we need to describe the extra elements in the chiral algebra coming from the axion field, and their OPEs. Let us now do this.

*Chiral algebra including the axion.*—The chiral algebra has four towers of states, each living in an infinite sum of finite-dimensional representations of SU(2). They are enumerated in Table I. In the chiral algebra presentation, we write the Lorentz group as $\mathrm{SU}(2)\times SL_2(\mathbb{R})$, where $SL_2(\mathbb{R})$ rotates the chiral algebra plane. Each state with label $m$, $n$ transforms in a representation of SU(2) of highest weight $\frac{1}{2}(m+n)$ and is a weight vector of weight $\frac{1}{2}(m-n)$. Each tower of generators arises from on-shell background field configurations of the 6d theory (which reduce to on-shell configurations in 4D) which are localized at points on the twistor sphere $\mathbb{CP}^1$. The quartic kinetic term of the axion gives rise to two towers of generators, $E$, $F$, which arise geometrically from the two-dimensional basis of closed two-forms on twistor space. Since the 4D theory is scale invariant, the generators also transform with a specified weight, which we denote by dimension in Table I, under homogeneous scaling of the 4D coordinates; these quantum numbers can be deduced most easily from the couplings in Sec. 7 of [12]. The OPEs involving the $E$, $F$ towers are

TABLE I. The generators of our 2D chiral algebra and their quantum numbers. Dimension refers to the charge under scaling of $\mathbb{R}^4$.

| Generator | Spin | Field | Dimension |
|---|---|---|---|
| $J[m,n]$, $m$, $n\geq 0$ | $1-(m+n)/2$ | $A$ | $-m-n$ |
| $\tilde{J}[m,n]$, $m$, $n\geq 0$ | $-1-(m+n)/2$ | $B$ | $-m-n-2$ |
| $E[m,n]$, $m+n>0$ | $-(m+n)/2$ | $\rho$ | $-m-n$ |
| $F[m,n]$, $m$, $n\geq 0$ | $-(m+n)/2$ | $\rho$ | $-m-n-2$ |

$$J^a[r,s](0)E[t,u](z) \sim \frac{1}{2\pi i z}\frac{(ts-ur)}{t+u}\hat{\lambda}_{\mathfrak{g}}\tilde{J}^a[t+r-1,s+u-1](0), \quad (12)$$

$$J^a[r,s](0)F[t,u](z) \sim -\hat{\lambda}_{\mathfrak{g}}\frac{1}{2\pi i z}\partial_z\tilde{J}^a[r+t,s+u](0) - \hat{\lambda}_{\mathfrak{g}}\frac{1}{2\pi i z^2}\left(1+\frac{r+s}{t+u+2}\right)\tilde{J}^a[r+t,s+u](0), \quad (13)$$

$$J^a[r,s](0)J^b[t,u](z) \sim \hat{\lambda}_{\mathfrak{g}}\frac{1}{2\pi i z}K^{ab}(ru-st)F[r+t-1,s+u-1](0) - \hat{\lambda}_{\mathfrak{g}}\frac{1}{2\pi i z}K^{ab}(t+u)\partial_z E[r+t,s+u](0) - \hat{\lambda}_{\mathfrak{g}}\frac{1}{2\pi i z^2}K^{ab}(r+s+t+u)E[r+t,s+u](0). \quad (14)$$

Let us explain the constant $\hat{\lambda}_{\mathfrak{g}}$. First, we define $\lambda_{\mathfrak{g}}$ so that, for $X \in \mathfrak{g}$, $\mathrm{Tr}(X^4) = \lambda_{\mathfrak{g}}^2 \mathrm{tr}(X^2)^2$, where Tr means the trace in the adjoint and tr means the minimal trace [i.e., the fundamental for $\mathrm{SU}(N)$]. This tensor identity is necessary for the Green-Schwarz mechanism to hold. According to [28], we have

$$\lambda_{\mathfrak{g}}^2 = \frac{10(\mathsf{h}^\vee)^2}{\dim\,\mathfrak{g}+2}, \quad (15)$$

where $\mathsf{h}^\vee$ is the dual Coxeter number, equal to $N_c$ for $\mathrm{SU}(N_c)$.

To account for the normalization of the interaction between the gauge field on twistor space and the field corresponding to the axion, we let

$$\hat{\lambda}_{\mathfrak{g}} = \frac{\lambda_{\mathfrak{g}}}{(2\pi i)^{3/2}\sqrt{12}}. \quad (16)$$

*One loop corrections.*—Let us now consider the possible one-loop corrections to the chiral algebra. We will then normalize them using associativity. (In principle, one can compute these using the method of [12], where the OPE coefficients are computed by an analysis of Feynman diagrams on twistor space. This was done in a similar situation for $5d$ gauge theories in [29], and a related analysis appears in the forthcoming work of [30]. However, we find that fixing the OPE coefficients using associativity is significantly easier.)

Our chiral algebra has a symmetry called dimension in the Table I. This comes from scaling on $\mathbb{R}^4$. It is a symmetry that persists to the quantum level; therefore any OPE must respect this symmetry.

One-loop corrections to the OPE cannot involve the axion, as axion exchanges are already a one-loop effect. One-loop corrections are determined by the OPEs involving $J[0,0]$, $\tilde{J}[0,0]$, $J[1,0]$, and $J[0,1]$. This is because these generate the algebra. Further, one-loop corrections must increase the number of $\tilde{J}$'s in an expression by one.

The most general correction to the OPE between $J[1,0]$, $J[0,1]$ currents is

$$J_a[1,0](0)J_b[0,1](z) = -\frac{1}{2\pi i z}CK^{fe}(f^c_{ae}f^d_{bf}+f^d_{ae}f^c_{bf}):J_c[0,0]\tilde{J}_d[0,0]:(0) + \frac{1}{2\pi i z}\frac{1}{2}Df^c_{ab}\partial_z\tilde{J}_c(0) + \frac{1}{2\pi i z^2}Df^c_{ab}\tilde{J}_c(0), \quad (17)$$

where $C$, $D$ are constants to be determined. The terms whose coefficient is $D$ correspond to the known one-loop splitting amplitudes of self-dual Yang-Mills theory.

OPEs involving $J_a[0,0]$ cannot get deformed. For instance, consider the OPE between $J_a[0,0]$ and $J_b[0,0]$. This must result in a state in the chiral algebra of dimension 0 with exactly one copy of $\tilde{J}$. A glance at Table I tells us it is not possible. A similar remark tells us that the OPE of $J_a[0,0]$ with any of the other states we are considering does not change.

In addition to this OPE, associativity of the chiral algebra also forces a correction to the $J-\tilde{J}$ OPE, which is

$$J_a[1,0](0)\tilde{J}_b[0,1](z) \simeq C\frac{1}{z}K^{fe}f^c_{ae}f^d_{bf}:\tilde{J}_c[0,0]\tilde{J}_d[0,0]:$$
$$J_a[0,1](0)\tilde{J}_b[1,0](z) \simeq -C\frac{1}{z}K^{fe}f^c_{ae}f^d_{bf}:\tilde{J}_c[0,0]\tilde{J}_d[0,0]: \quad (18)$$

for the same value of $C$ as above.

*Normalizing the quantum corrections using associativity.*—The constants $C$ and $D$ in the quantum-corrected OPE will be chosen so that the following identity holds:

$$\oint_{|z|=1,|w|=2} J_a[1,0](0)J_b[0,1](z)J_c[0,0](w)w\,dw\,dz = \oint_{|z|=2,|w|=1}(J_a[1,0](0)J_c[0,0](w)w\,dw)J_b[0,1](z)dz + \oint_{|z|=2,|z-w|=1}J_a[1,0](0)(J_b[0,1](z)J_c[0,0](w)w\,dw)dz. \quad (19)$$

This identity is, of course, a consequence of the associativity of the OPE.

*Proposition 1.*—The OPE associativity identity (19) holds if and only if we have the trace identity

$$\mathrm{Tr}(X^4) = \lambda_{\mathfrak{g}}^2\mathrm{tr}(X^2)^2 \quad (20)$$

and we take the constants $C$, $D$ to be

$$C = \frac{3}{2(2\pi i)^2 12}, \qquad D = -\frac{\mathsf{h}^\vee}{(2\pi i)^3 12}. \tag{21}$$

The proof of this is provided in Supplemental Material [22]. It is an entirely routine argument: we simply compute both sides of Eq. (19) and compare them.

*Matching coefficients.*—To compare to the known results about one-loop splitting amplitudes, we should recall that $\langle ij\rangle$ comes with a factor of $2\pi i$ and $[ij]$ with a factor of $-2$. This brings the coefficient of the one-loop $++ \mapsto -$ term in the OPE to

$$\frac{[ij]}{\langle ij\rangle^2}\frac{\mathsf{h}^\vee}{(2\pi i)^2 24} = -\frac{\mathsf{h}^\vee}{96\pi^2}\frac{[ij]}{\langle ij\rangle^2}. \tag{22}$$

For the groups $SU(N_c)$, $\mathsf{h}^\vee = N_c$. This matches the coefficient in [26].

We should emphasize that the chiral algebra, as an abstract chiral algebra, does not know about this constant. Multiplying the generators $\tilde{J}, E, F$ of the chiral algebra by a constant changes the coefficient of the one-loop OPE. In our analysis, however, we have taken care that our chiral algebra generators match states in the gauge theory exactly, without a prefactor. When we do this, we do find the correct coefficient, providing a chiral-algebraic derivation of the one-loop splitting amplitude.

*Amplitudes.*—Associativity of the chiral algebra in the presence of the axion gives us a remarkably simple formula for the one-loop form factors of self-dual Yang-Mills theory in the presence of the operator $\frac{1}{2}\mathrm{tr}(B^2)$ inserted at a fixed position (the origin). This form factor has the same functional form (without the momentum conserving delta function; see Ref. [12]) as the one loop amplitude for QCD with an axion, with one particle of negative helicity (the form factor with all-positive helicities vanishes). We find that the form factor is

$$\begin{aligned}&\left\langle \frac{1}{2}\mathrm{tr}(B^2)\middle| 1^- 2^+ \ldots n^+\right\rangle \\ &= -\frac{1}{192\pi^2}\sum_{2\le i<j\le n}\frac{[ij]\langle 1i\rangle^2\langle 1j\rangle^2}{\langle ij\rangle\langle 12\rangle\langle 23\rangle, \ldots, \langle n1\rangle}\mathrm{Tr}_{\mathfrak{g}}(\mathrm{t}_1, \ldots, \mathrm{t}_n) \\ &\quad + \text{permutations in } S_{n-1},\end{aligned} \tag{23}$$

where we sum over permutations of the labels $2, \ldots, n$, and the trace is in the adjoint representation. We note that this expression has *both* the first order pole we found when the indices $i$, $j$ are not adjacent in the trace, as well as the second order pole when they are adjacent in the trace. We derive this result in Sec. III of Supplemental Material [22].

We remark that since our chiral algebra is nonunitary, there are many ways to define correlation functions consistent with its OPE. In Sec. 8 of [12] we proved that a local operator insertion in 4D corresponds to a choice of conformal block, which gives a prescription for computing a correlation function, in the chiral algebra. The diversity of local operators, and form factors, in 4D maps to a plethora of 2D correlation functions. See Ref. [12] for additional form factor computations and more details on this correspondence.

The expression (23) is valid also when the anomaly is cancelled by carefully chosen matter, instead of an axion, e.g., if the gauge group is $SU(2)$ and $N_f = 8$ (meaning 8 fundamental and 8 antifundamental fermions). In that case we must take the trace in the adjoint minus the matter representation. In the case of $\mathcal{N} = 1$ supersymmetric Yang-Mills theory, i.e., an adjoint matter fermion, the anomaly is always cancelled and Eq. (23) vanishes.

This formula is a great deal simpler than the formulas [24,31] for the corresponding QCD amplitudes without an axion, bolstering our contention that the presence of the axion simplifies amplitudes greatly.

We thank R. Bittleston, A. Sharma, and A. Strominger for helpful comments on a draft of this manuscript. K. C. is supported by the NSERC Discovery Grant program and by the Perimeter Institute for Theoretical Physics. Research at Perimeter Institute is supported by the Government of Canada through Industry Canada and by the Province of Ontario through the Ministry of Research and Innovation. N. P. acknowledges support from the University of Washington, past support from DOE Award No. DE-SC0022347, and current support from the DOE Early Career Research Program under Award No. DE-SC0022924.

*kcostello@perimeterinstitute.ca
†npaquett@uw.edu

[1] A. Strominger, arXiv:1703.05448.
[2] A.-M. Raclariu, arXiv:2107.02075.
[3] S. Pasterski, Eur. Phys. J. C **81**, 1062 (2021).
[4] A. Guevara, E. Himwich, M. Pate, and A. Strominger, J. High Energy Phys. 11 (2021) 152.
[5] A. Ball, S. Narayanan, J. Salzer, and A. Strominger, J. High Energy Phys. 01 (2022) 114.
[6] A. N. Leznov, Theor. Math. Phys. **73**, 302 (1988), https://www.osti.gov/biblio/5974023.
[7] A. Leznov and M. Mukhtarov, J. Math. Phys. (N.Y.) **28**, 2574 (1987).
[8] A. Parkes, Phys. Lett. B **286**, 265 (1992).
[9] G. Chalmers and W. Siegel, Phys. Rev. D **54**, 7628 (1996).
[10] L. J. Mason, J. High Energy Phys. 10 (2005) 009.
[11] We emphasize that there is no contradiction with the work of [5], because of the different ways in which we define self-dual gauge theory, and because of our use of form factors in addition to amplitudes.
[12] K. Costello and N. M. Paquette, J. High Energy Phys. 10 (2022) 193.
[13] R. S. Ward, Phys. Lett. **61A**, 81 (1977).

[14] K. J. Costello, arXiv:2111.08879.
[15] It would be fascinating to connect this anomaly with the anomaly discussed in, e.g., [16] as the origin of all-+ scattering in QCD.
[16] W. A. Bardeen, Prog. Theor. Phys. Suppl. **123**, 1 (1996).
[17] It is interesting to note that a scalar dilaton field with a fourth-order kinetic term has been used to cancel conformal anomalies in four dimensions [18–21]. We will continue to call our field the axion because it couples to a topological term in the gauge field.
[18] E. S. Fradkin and A. A. Tseytlin, Nucl. Phys. **B203**, 157 (1982).
[19] E. S. Fradkin and A. A. Tseytlin, Phys. Lett. **134B**, 187 (1984).
[20] R. J. Riegert, Phys. Lett. **134B**, 56 (1984).
[21] Z. Komargodski and A. Schwimmer, J. High Energy Phys. 12 (2011) 099.
[22] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.129.231604 for a proof of Prop. 21, the derivation of Eq. 23, and details on our conventions.
[23] M. L. Mangano and S. J. Parke, Nucl. Phys. **B299**, 673 (1988).
[24] Z. Bern, L. J. Dixon, D. C. Dunbar, and D. A. Kosower, Nucl. Phys. **B425**, 217 (1994).
[25] D. A. Kosower and P. Uwer, Nucl. Phys. **B563**, 477 (1999).
[26] Z. Bern, L. J. Dixon, and D. A. Kosower, Phys. Rev. D **71**, 105013 (2005).
[27] In this computation, as we will show shortly, there are no quantum corrections to OPEs involving $J[0,0]$ and $\tilde{J}[0,0]$; indeed, no such corrections are possible.
[28] S. Okubo, J. Math. Phys. (N.Y.) **20**, 586 (1979).
[29] K. Costello, arXiv:1705.02500.
[30] R. Lorgat, F. Moosavian, and Y. Zhou (2022).
[31] G. Mahlon, Phys. Rev. D **49**, 2197 (1994).