

Constant-Cost Implementations of Clifford Operations and Multiply-Controlled Gates Using Global Interactions

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We consider quantum circuits composed of single-qubit operations and global entangling gates generated by Ising-type Hamiltonians. It is shown that such circuits can implement a large class of unitary operators commonly used in quantum algorithms at a very low cost—using a constant or effectively constant number of global entangling gates. Specifically, we report constant-cost implementations of Clifford operations with and without ancillae, constant-cost implementation of the multiply-controlled gates with linearly many ancillae, and an $O(\log^*(n))$ cost implementation of the n -controlled single-target gates using logarithmically many ancillae. This shows a significant asymptotic advantage of circuits enabled by the global entangling gates.

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Executing a quantum computer program requires its compilation to native gates directly supported by the controlling apparatus. While most of the existing quantum computing hardware supports arbitrary single-qubit gates as their native physical-level instruction, the choice of entangling multiqubit operations may vary strongly across different platforms. Here we focus on leveraging parallel global instructions available in some quantum computer architectures [1–3]. Such global operations draw their power from the ability to apply a layer of pairwise commuting 2-qubit gates simultaneously. A notable example is the Molmer-Sorensen gate available in the trapped-ion architecture [2,4] that evolves a system of qubits under the Ising-type Hamiltonian. In this Letter, we report a streamlined implementation of two common quantum computing subroutines enabled by global entangling gates: elements of the Clifford group and multiply-controlled gates. Clifford operations play a central role in quantum error correction, certain simulation algorithms [5], and a variety of applications based on pseudorandom unitary operators [6–8]. Multiply-controlled gates are ubiquitous in applications that rely on arithmetic circuits or the quantum singular value transformation [9,10].

We consider computational primitives that implement a selectable set of commuting two-body Ising interactions in one go, which we call global tunable gates. We furthermore treat single-qubit gates as a free resource, which is justified since the implementation of entangling operators on the physical level tends to be more difficult and error prone compared with the single-qubit gates. Such a selection of the costing metric implies that the particular nature of the Ising interaction (e.g., XX, YY, ZZ) used is

irrelevant—indeed, the results hold uniformly for all possible choices of the interaction since the changes between them are accomplished by the single-qubit gates (basis change) and can be merged into the single-qubit gate layers. For the sake of simplicity of mathematical expressions, we chose to work with the CZ^a interaction defined as $CZ^a = e^{i\pi a|1\rangle\langle 1|}$. Here $a \in [0, 1]$ is a tunable parameter. In other words, CZ^0 is equivalent to the identity, whereas $CZ^1 = CZ$ is single-qubit equivalent to the CNOT gate. We define an n -qubit global tunable (GT) gate as a diagonal unitary operator

$$\prod_{1 \leq i < j \leq n} CZ_{i,j}^{a(i,j)},$$

where i and j indicate pairs of qubits acted upon by each CZ^a gate, and we allow the parameter $a = a(i, j)$ to be tuned individually for each pair of qubits i and j . We assume all-to-all qubit connectivity.

Multiple versions of GT gates have been experimentally demonstrated [1,2,11,12]. In particular, Ref. [2] showed that the pulse complexity required for a GT gate is not much more than that required for a single 2-qubit gate, i.e., $3n - 1$ degrees of freedom used for a GT instruction and $2n + 1$ for a 2-qubit gate, for an n -qubit quantum computer. This is understood by considering the pulse design space, where some number of degrees of freedom must first be spent to decouple the computational states from the mutually shared information bus at the end of the entangling operation ($2n$) and then using additional degrees of freedom to induce the desired degree of entanglement between qubit pairs of

choice. Since there are at most $O(n^2)$ pairs to entangle, $O(n)$ additional degrees of freedom introduced to each of n qubits suffice, making the hardware-level requirement for a GT gate and that for a 2-qubit gate not too different.

Our main contributions are as follows. First, we show that any n -qubit Clifford operation can be implemented using a constant number of GT gates (at most 21) using no ancilla. The number of GT gates is at most four if n ancillae are available. The latter construction meets the information-theoretic lower bound that applies to the no-ancilla case. Second, we show how to construct a multiply-controlled Toffoli (Toffoli- n) gate using $O(\log^*(n))$ [13] GT gates with $O(\log(n))$ ancillae or using 4 GT gates with $O(n)$ ancillae. We also consider *adaptive* quantum circuits that include intermediate measurements and classical feed forward. We show that Toffoli- n can be implemented by an adaptive circuit with only two GT gates and $O(n)$ ancillae. Our results achieve a substantial improvement over the state of the art.

Previous work.—The use of the GT gates to efficiently implement an n -qubit Clifford operation has been studied in the literature. An implementation with $12n + O(1)$ GT gates requiring no ancilla was developed in Ref. [14] and superseded by Ref. [15], which improved it to $6n + O(1)$ GT gates, still using no ancilla. Subsequently in Ref. [16] the complexity was improved to $6\log(n) + O(1)$ GT gates, however, with $n/2$ ancillae. Recently in Ref. [3], an ancilla-free, $(n, 2n]$ -GT method was reported. Finally, an ancilla-free implementation with $2\log(n) + O(1)$ GT gates follows from Ref. [17] by noticing that $R_A^{-1}W01$ in formula (4) therein is a single GT gate, and nonoverlapping GT gates can be implemented in parallel. In all these approaches, one employs a decomposition of an n -qubit Clifford into stages of 1- and 2-qubit gate layers, to then implement the 2-qubit gate layers efficiently using GT gates. Here, we too employ this decomposition; see, e.g., Lemma 8 of Ref. [18], where any n -qubit Clifford U can be realized by a layered circuit

$$U = -L-CX-CZ-L-CZ-L-. \quad (1)$$

Here $-L-$ stands for a layer of single-qubit gates, $-CX-$ is a linear reversible circuit, and $-CZ-$ is a layer of CZ gates. In contrast to previous results, we show how to implement arbitrary Clifford operation using $O(1)$ GT gates either with or without ancilla (the constant is smaller with ancilla). We note that a single GT gate suffices if the input state of a Clifford circuit is a fixed basis vector, such as $|0^n\rangle$. Indeed, in this case, the output state $U|0^n\rangle$ can be prepared by a layered circuit $-L-CZ-L-$, as follows from the well-known local equivalence between stabilizer states and graph states [19]. In contrast, our implementation with $O(1)$ GT gates works for an arbitrary input state.

GT gates were also used to efficiently implement a Toffoli- n , implying the same asymptotic for arbitrary multicontrolled unitaries. Briefly, a 3-GT construction

of Toffoli-3 was reported in Ref. [20] and a 7-GT construction of Toffoli-4 was reported in Ref. [21]. Toffoli-4 was improved to rely on 3 GTs in Ref. [14], with one clean ancilla, along with $3n + O(1)$ GT implementation of a multiply-controlled Toffoli using $n/2 + O(1)$ ancillae. An ancilla-free implementation (of, technically speaking, multicontrolled iX) that takes $2n$ GT gates was reported in Ref. [22]. In all of these references, the GT gates used were the single-angle, all-pair (drawn from a selectable subset of qubits) kind. Indeed, when using a more general GT, the complexity can be improved to $O(\log(n))$ by nesting Toffoli- m , $m \leq n$, in a depth-optimal fashion with $O(n)$ ancillae, or $3n/2$ GTs with no more than seven ancillae [16].

Clifford operations.—We focus on implementing n -qubit Clifford operations by quantum circuits composed of GT gates and “free” single-qubit gates. Here, we use only Clifford GT gates that apply CZ to a subset of qubits. Following Ref. [15] we will refer to such gates as Global CZ (GCZ). Our main result is the following.

Theorem 1: Any n -qubit Clifford operator can be implemented using 4 GCZs with n ancillae or using 26 GCZs with no ancilla.

To prove the theorem we first show how to realize a special class of Clifford operators using 3 GCZs. Let $GL(n)$ be the group of binary $n \times n$ invertible matrices. Here and below multiplication and inversion for binary matrices are defined over the binary field \mathbb{F}_2 .

Lemma 1: For any matrix $A \in GL(n)$ there exist $2n$ -qubit Clifford circuits $C_3(A)$ and $C'_3(A)$, each with the GCZ cost of 3, such that

$$C_3(A)|x, y\rangle = |A^{-1}y, Ax\rangle \quad \text{and} \quad (2)$$

$$C'_3(A)|x, y\rangle = |y \oplus Ax, A^{-1}y\rangle \quad (3)$$

for all $x, y \in \mathbb{F}_2^n$. These circuits require no ancilla.

Proof.—Given a binary $n \times n$ matrix M and a pair of n -qubit registers R_1 and R_2 , let $CX_{1,2}(M)$ be a $2n$ -qubit unitary operator that EXORs the i th qubit of R_1 to the j th qubit of R_2 for each pair $i, j \in \{1, 2, \dots, n\}$ such that $M_{i,j} = 1$. Here and below EXOR stands for the Exclusive OR logical gate. By definition we have $CX_{1,2}(M)|x, y\rangle = |x, y \oplus Mx\rangle$ and $CX_{2,1}(M)|x, y\rangle = |x \oplus My, y\rangle$ for all $x, y \in \mathbb{F}_2^n$. These operators can be implemented with a single GCZ by conjugating each qubit in the register R_2 and R_1 respectively by the Hadamard gate (which converts all CNOTs to CZs) and noticing that any layer of CZs is a single GCZ gate. Now the desired transformations in Eqs. (2) and (3) can be implemented as

$$C_3(A) = CX_{1,2}(A)CX_{2,1}(A^{-1})CX_{1,2}(A)$$

and

$$C'_3(A) = \text{CX}_{1,2}(I \oplus A^{-1})\text{CX}_{2,1}(I)\text{CX}_{1,2}(I \oplus A).$$

We are now ready to prove Theorem 1.

Proof.—According to Eq. (1), it takes two -CZ- layers and one -CX- layer to implement arbitrary n -qubit Clifford operation U . This implementation uses no ancilla. Each -CZ- layer is, trivially, a single GCZ gate. Thus below we focus on the -CX- layer. Let V be the circuit implementation of the -CX- layer in Eq. (1). We have $V|x\rangle = |Ax\rangle$ for all $x \in \mathbb{F}_2^n$ and some matrix $A \in \text{GL}(n)$. Using Lemma 1 with $y = 0^n$ one concludes that the map $|x, 0^n\rangle \mapsto |Ax, 0^n\rangle$ can be implemented with 3 GCZs; see Eq. (3). Equivalently, V admits a 3-GCZ implementation using n ancillae initialized to and returned in the state $|0\rangle$. This shows that U admits a 5-GCZ implementation using n ancilla. Moreover, the left -CZ- layer in Eq. (1) can be merged with the rightmost GCZ gate in the implementation of V . Indeed, this -CZ- layer acts nontrivially only on the data (top) n qubits, and thus it commutes with the layer of Hadamards in the implementation of V which acts nontrivially only on the ancillary (bottom) n qubits. Thus U admits a 4-GCZ implementation using n ancilla. This proves the first part of the theorem.

To prove the second part, we develop an ancilla-free implementation with at most 25 GCZs when $3|n$ (3 divides n) and 26 GCZs otherwise. Based on Eq. (1), it suffices to show that any n -qubit linear reversible circuit V can be implemented using at most 23 GCZs with zero ancilla (when $3|n$).

Let us first describe a 12-GCZ implementation of a $3k$ -qubit linear reversible circuit $W(A)$ that uses $2k$ dirty ancillae (that is, ancillae that store an unknown data and must be returned in the initial state) to implement a k -qubit linear reversible transformation $|x\rangle \mapsto |Ax\rangle$ up to a register swap. Specifically, suppose the initial state of k data qubits and $2k$ ancillae is $|x, y, z\rangle$. We would like to implement the map

$$W(A)|x, y, z\rangle = |z, Ax, y\rangle. \quad (4)$$

We will use the following fact proved in Ref. [23].

Fact: Suppose $n \geq 3$. Then any matrix $A \in \text{GL}(n)$ can be expressed as a group commutator $A = D^{-1}B^{-1}DB$ for some $B, D \in \text{GL}(n)$.

Apply the circuit $C_3(A)$ constructed in Lemma 1 four times with the matrix A replaced by B (first application), D (second application), B^{-1} (third application), and D^{-1} (fourth application). This gives the sequence of transformations as

$$\begin{aligned} |x, y, z\rangle &\xrightarrow{C_3(B)} |B^{-1}y, Bx, z\rangle \xrightarrow{C_3(D)} |B^{-1}y, D^{-1}z, DBx\rangle \\ &\xrightarrow{C_3(B^{-1})} |B^{-1}DBx, D^{-1}z, y\rangle \\ &\xrightarrow{C_3(D^{-1})} |z, D^{-1}B^{-1}DBx, y\rangle = |z, Ax, y\rangle. \end{aligned}$$

In other words, we implemented the k -qubit transformation $|x\rangle \mapsto |Ax\rangle$ using $2k$ dirty ancilla up to qubit register permutation using four applications of the C_3 gate, which costs $12(= 4 \cdot 3)$ GCZs.

Assume that $3|n$. We want to implement the map $|v\rangle \mapsto |Av\rangle$ with a given $A \in \text{GL}(n)$ using constantly many GCZ gates and no ancilla. Specifically, write A as

$$A = \begin{bmatrix} A_{00} & A_{01} & A_{02} \\ A_{10} & A_{11} & A_{12} \\ A_{20} & A_{21} & A_{22} \end{bmatrix}. \quad (5)$$

Here each block $A_{i,j}$ has size k , where $k = n/3$, which is well defined given $3|n$. First, we use two GCZ gates to transform A into a form where A_{00} and $\begin{pmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{pmatrix}$ are invertible.

To accomplish this, label the individual matrix rows as $x_1, x_2, \dots, x_k, y_1, y_2, \dots, y_k, z_1, z_2, \dots, z_k$ top to bottom. Consider the first k elements of these rows. We need to make x_1, x_2, \dots, x_k restricted to the first k bits be linearly independent. This can be done by adding a row from the y_* or z_* sets onto some of x_* rows. This is accomplished in parallel by a depth-1 CNOT circuit made with $\text{CNOT}(y_i, x_j)$ or $\text{CNOT}(z_i, x_j)$ gates. The x_* rows are now linearly independent as prefix rows of length k , and as a result also as prefix rows of length $2k$. Thus, to make the first $2k$ rows linearly independent as length- $2k$ prefix rows, we can add some $\text{CNOT}(z_i, y_i)$ in the CNOT depth 1. Composing both stages we obtain the transformation of A into the desired form at the GCZ cost of 2, since each CNOT layer can be implemented as a GCZ cost-1 operation.

Next, we use 3 GCZ gates to transform A into the block-diagonal form such that the only nonzero blocks are A_{00} , A_{11} , and A_{22} . First, apply the transformation

$$A \leftarrow \begin{bmatrix} I & 0 & 0 \\ A_{10}A_{00}^{-1} & I & 0 \\ A_{20}A_{00}^{-1} & 0 & I \end{bmatrix} \cdot A. \quad (6)$$

It sets to zero the blocks A_{10} and A_{20} (and, possibly, modifies the blocks A_{11} , A_{12} , A_{21} , and A_{22}). This transformation can be implemented by the CNOT gates with controls on the top third of qubits and targets on the bottom two thirds and thus it costs one GCZ gate (and some extra Hadamard gates). Since we assumed that $\begin{pmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{pmatrix}$ is invertible, after the transformation [Eq. (6)] the block A_{11} is invertible. Thus the same argument applies to the remaining off-diagonal blocks A_{01} and A_{21} , and then again for the pair of blocks A_{02} and A_{12} . Transforming A to the block-diagonal form

$$A \leftarrow \begin{bmatrix} A_{00} & 0 & 0 \\ 0 & A_{11} & 0 \\ 0 & 0 & A_{22} \end{bmatrix} \quad (7)$$

thus costs 5 GCZs. Note that each diagonal block describes a linear reversible circuit acting nontrivially on exactly $n/3$ qubits. To implement all three, we can combine C_3 from Lemma 1 twice (first time applied to first and third multiregisters, second time applied to first and second multiregisters) with the 12-GCZ circuit implementing the transformation in Eq. (4) for $k = n/3$ as follows:

$$\begin{aligned} |x, y, z\rangle &\xrightarrow{C_3(A_{00})} |A_{00}^{-1}z, y, A_{00}x\rangle \\ &\xrightarrow{C_3(A_{22}A_{00})} |(A_{22}A_{00})^{-1}y, A_{22}z, A_{00}x\rangle \\ &\xrightarrow{W(A_{11}A_{22}A_{00})} |A_{00}x, A_{11}y, A_{22}z\rangle. \end{aligned}$$

The overall GCZ cost of A is $5 + 3 \cdot 2 + 12 = 23$, using zero ancilla. Thus the overall cost of the Clifford U when $3|n$ is 25 GCZs.

Finally, consider the case $3 \nmid n$. Suppose $n = 3k + 2$ ($n = 3k + 1$ is handled similarly). Then, in Eq. (5) the qubits are broken into the sets of k , $k + 1$, and $k + 1$. To employ the circuit $W(A)$ in Eq. (4), we have one less qubit than we need. To compensate for this lack of a qubit, an intermediate step is used after the reduction to block diagonal form, Eq. (7), where, at the cost of one GCZ gate, we disentangle (diagonalize) one qubit each in the two qubit sets of the size $k + 1$. Then we have a block diagonal decomposition with five blocks, of which three have sizes $k \times k$, and the remaining two have sizes 1×1 . The circuit $W(A)$ can now be implemented since there is enough ancillary space to accommodate it. The overall cost of the Clifford U for arbitrary n is thus at most 26 GCZs.

In the Supplemental Material [24] we describe local optimizations reducing the cost further to 20 or 21 GCZs, depending on whether $3|n$. ■

Multiply-controlled Toffoli.—We show two different methods to implement an n -qubit Toffoli gate. The first uses $O(\log(n))$ ancillae and $O(\log^*(n))$ GT gates. The second uses $O(n)$ ancillae and $O(1)$ GT gates. Below we assume $n \geq 3$. Let $\text{OR}_n: \{0, 1\}^n \rightarrow \{0, 1\}$ be the n -bit Boolean OR function. Define an n -qubit unitary operator as

$$\text{OR}_n|x\rangle = (-1)^{\text{OR}_n(x)}|x\rangle.$$

It flips the phase of the n -qubit register $|x\rangle$ if and only if at least one bit of x is nonzero. Clearly, OR_n coincides with the multiply-controlled Z gate up to relabeling of basis states 0 and 1 (which, in turn, coincides with the n -qubit Toffoli up to a conjugation by the Hadamard gate on the target qubit).

Here we show how to implement OR_n by a quantum circuit composed of single-qubit gates and fractional CNOT gates, that are then mapped into a small number of GT gates.

Suppose $q \geq 0$ is an integer. Let X_q be some fixed 2^q th root of Pauli X , such that $(X_q)^{2^q} = X$. One can choose the primary root,

$$X_q = H \begin{bmatrix} 1 & 0 \\ 0 & \exp[i\pi/2^q] \end{bmatrix} H, \quad (8)$$

where H is the Hadamard gate. For example, $X_0 = X$ is Pauli- X . Let the fractional CNOT gate be

$$CX_q = |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes X_q.$$

Our construction closely follows Refs. [25,26]. Choose an integer $p := \lceil \log_2(n+1) \rceil \approx \log(n)$. Consider a bit string $x \in \{0, 1\}^n$ with the Hamming weight $w(x) = x_1 + x_2 + \dots + x_n$. Note that $\text{OR}_n(x) = 1$ if and only if

$$(X_q)^{w(x)}|0\rangle = |1\rangle \quad \text{for some } q = 0, 1, \dots, p-1.$$

If $w(x)$ is odd then $(X_0)^{w(x)}|0\rangle = |1\rangle$. Otherwise $w(x)$ is even, and thus $w(x)/2$ is an integer. If $w(x)/2$ is odd then $(X_1)^{w(x)}|0\rangle = (X_0)^{w(x)/2}|0\rangle = |1\rangle$. Otherwise $w(x) \equiv 0 \pmod{4}$ and thus $w(x)/4$ is an integer. If $w(x)/4$ is odd then $(X_2)^{w(x)}|0\rangle = (X_0)^{w(x)/4}|0\rangle = |1\rangle$, etc. Thus we have the identity

$$\begin{aligned} (-1)^{\text{OR}_n(x)} &= \langle 0^p | (X_0^{w(x)} \otimes X_1^{w(x)} \otimes \dots \otimes X_{p-1}^{w(x)})^\dagger \text{OR}_p \\ &\quad \cdot (X_0^{w(x)} \otimes X_1^{w(x)} \otimes \dots \otimes X_{p-1}^{w(x)}) |0^p\rangle. \end{aligned} \quad (9)$$

Consider a register with $n + p$ qubits where the last p qubits are ancillae initialized in $|0\rangle$. One can implement the gates $X_q^{w(x)}$ controlled by the Hamming weight $w(x)$ of the first n qubits and target on the last p qubits using n fractional CNOT gates CX_q . In fact, all np fractional CNOT gates can be implemented using one GT gate, since all controls are on the first n qubits and all targets are on the last p qubits; hence all commute (note that adding left and right layers of Hadamards on the ancillae transforms each CX_q gate to CZ^a with $a = 1/2^q$). Next, we need to implement OR_p on the ancilla register of size p , which is nothing but an exponentially reduced version of the original problem of implementing OR_n . One can thus recursively apply the reductions until only two qubits are left, after which OR_2 can trivially be implemented using a single 2-qubit gate. This results in the $2\log^*(n) - 1$ GT construction of an n -qubit Toffoli using $\log(n) + O(\log(\log(n)))$ ancillae.

To obtain an $O(1)$ GT circuit, we avoid the recursion. Instead, as soon as the problem is reduced to that of

implementing the OR_p , we invoke two-GT construction to obtain OR_p , which uses $2^p - p - 1$ ancillae. This is a GT-enabled modification of a well-known method [27] to implement a multicontrolled Z gate that relies on computing all possible EXORs of p Boolean inputs (which takes one GT gate)—note the EXORs with a single literal do not need to be computed on the ancillae, since they are the input. Once all possible EXORs are available, apply the single-qubit gate $Z^{1/2^{p-1}}$ to all EXORs to obtain OR_p (to obtain multicontrolled Z odd and even weight EXORs experience the application of phases with opposite angles). This costs no GT gates. Finally, uncompute the EXORs using one GCZ gate. The overall cost of implementing n -fold OR is thus $4 [= 1 + (1 + 1) + 1]$ GT gates using $2^{\lceil \log(n) \rceil} - 1 < 2n$ ancillae. In the Supplemental Material [24] we show how to halve the gate counts in the above implementation using adaptive quantum circuits. We note that the gate count $2\log^*(n) - 1$, is in fact limited by the constant 9, since $\log^*(n)$ cannot exceed the value of 5 due to not enough elementary particles in the known universe to constitute qubits. Thus, in effect both reported constructions carry a constant cost.

Discussion.—Our results significantly improve over state of the art, and for the most part settle asymptotic complexities. We show that GT gates are a very powerful primitive for compiling short quantum circuits, since a serial approach would require $O(n^2/\log(n))$ [28] and $O(n)$ gates respectively, for the Clifford and multiply-controlled Toffoli gates. Our results demonstrate the power of parallel single instruction multiple data operations in quantum computer settings, not unlike its classical, conventional computer counterparts.

Our result also implies that GT gates enable more efficient generation of pseudorandom unitary operators known as unitary t designs [29]. In particular, n -qubit approximate unitary t designs can be realized with $O(1)$ GT gates for any constant $t = O(1)$ since such designs can be realized using single-qubit gates and at most $\tilde{O}(t^4)$ Clifford layers [7].

This study sheds more light on space-time tradeoffs in quantum circuits. In particular, it turned out that the no-ancilla implementation of a Clifford unitary is possible without increasing the asymptotic complexity compared with a much simpler ancilla-enabled implementation. An analogy can be drawn to the multiply-controlled Toffoli gate implemented using single-qubit and 2-qubit gates, where it too turned out that a no-ancilla implementation is possible [30] at the same asymptotic cost as a significantly more straightforward and practical ancilla-enabled implementation [27]. Our study left the door open to discover a constant GT gate count implementation of the multiply-controlled Toffoli gate using few or no ancilla, as well as implementation of other commonly used unitary operations, such as the quantum Fourier transform.

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