Exponentially Decreasing Critical Detection Efficiency for Any Bell Inequality

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We address the problem of closing the detection efficiency loophole in Bell experiments, which is crucial for real-world applications. Every Bell inequality has a critical detection efficiency η that must be surpassed to avoid the detection loophole. Here, we propose a general method for reducing the critical detection efficiency of any Bell inequality to arbitrary low values. This is accomplished by entangling two particles in N orthogonal subspaces (e.g., N degrees of freedom) and conducting N Bell tests in parallel. Furthermore, the proposed method is based on the introduction of penalized N-product (PNP) Bell inequalities, for which the so-called simultaneous measurement loophole is closed, and the maximum value for local hidden-variable theories is simply the Nth power of the one of the Bell inequality initially considered. We show that, for the PNP Bell inequalities, the critical detection efficiency decays *exponentially* with N. The strength of our method is illustrated with a detailed study of the PNP Bell inequalities resulting from the Clauser-Horne-Shimony-Holt inequality.

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Introduction.—Quantum correlations arising from local measurements on entangled particles [1] allow for multiple applications, including device-independent randomness expansion [2–5], quantum key distribution [6–10], secret sharing [11,12], self-testing [13,14], and certification of quantum measurements [15–17]. All these tasks require a loophole-free Bell test [18–21] as a necessary condition. The most challenging problem from the applications' perspective is closing the detection loophole [22], since otherwise an adversary can simulate the behavior of entangled particles provided that a sufficient fraction of them remains undetected. Therefore, a fundamental problem is to identify quantum correlations that cannot be simulated with local hidden-variable (LHV) models even when the detection efficiency is relatively low.

The detection efficiency in a Bell inequality test is the ratio between the number of systems detected by the measuring devices and the number of systems emitted by the source. It depends not only on the properties of the detectors, but also on the losses in the channel. Closing the detection loophole requires surpassing a certain threshold detection efficiency, which depends on the quantum correlations chosen. For symmetric Bell tests (i.e., those in which all detectors have the same detection efficiency) and zero background noise, the necessary and sufficient threshold detection efficiency for entangled qubits can be as low as $2/3$ for partially entangled states [23] and 0.828 for maximally entangled states [24]. Massar [25] showed that high-dimensional systems could tolerate a detection efficiency that decreases with the dimension d of the local quantum system. However, this result is of limited practical interest since an improvement over the qubit case occurs only for $d > 1600$. Vértesi, Pironio, and Brunner [26] identified a symmetric Bell inequality for which the efficiency can be lowered down to 0.618 for partially entangled states and 0.77 for maximally entangled states, using four-dimensional systems and assuming perfect visibility, which is still not sufficiently low for practical applications. Other proposals for loophole-free Bell tests with low detection efficiency either combine low-efficient detectors with nearly perfect ones [27–30] or use more than five spatially separated parties [31–33], which is unpractical for real-world applications.

The critical detection efficiency η is not the only important parameter in a loophole-free Bell experiment. Another essential variable is the required visibility v, which quantifies how much noise can be tolerated. The best combinations of parameters (η, v) reported in photonic experiments in distances $\lessapprox 200$ m are (0.774, 0.99) [19], (0.763, 0.99) [5], and (0.8411, 0.9875) [4]. However, these values are very difficult to achieve in longer distances.

In this Letter, we propose a general method to reduce the detection efficiency requirement *exponentially* for any given Bell inequality. This is achieved by violating N Bell inequalities in parallel with a source of N entangled states carried by a single pair of particles. The value of the required detection efficiency then scales like $(C/Q)^N$, where C is the LHV bound, and Q is a quantum value, i.e., the decay is exponential. Moreover, our method reduces the required detection efficiency for a given target visibility or a Bell inequality violation. We analyze in detail the case of parallel violation of N Clauser-Horne-Shimony-Holt (CHSH) Bell inequalities [34]. Another advantage of our approach is that the observed correlations can be directly used for practical applications, since the observed value of N CHSH inequalities can be connected to the violation of an individual CHSH inequality. Hence, there is no need to develop new protocols based on Bell inequalities with more settings [35].

Physical setup.—Consider a Bell experiment in which two spatially separated parties, Alice and Bob, have access to a source of high-dimensional entanglement carried by a single pair of particles. The key examples to keep in mind are hyperentangled states [36], in which two photons are entangled across multiple degrees of freedom, and photon pairs entangled in high-dimensional degrees of freedom [37]. Throughout the text, we consider photons as physical carriers of entanglement; however, similar reasoning can be applied to atoms, ions, etc.

Let us assume now that the carried high-dimensional entangled state is a product of N entangled states, as it is the case for hyperentanglement [36]. We also assume that Alice and Bob can perform joint measurements on their subsystems producing N outcomes each from a single click of their detectors. The main idea of the method is to use N outcomes from each run of the experiment to violate N Bell inequalities *in parallel*. In this way, the probability of detectors' clicks for each of the N inequalities is of the order of the Nth root of the efficiency of the photon detection, i.e., it is effectively increased. We will provide a rigorous analysis that supports this claim.

To the best of our knowledge, the conjecture that the critical detection efficiency could be lowered by integrating several qubit-qubit entangled states in one pair of particles was first made in Ref. [38], without a proof. In Ref. [39], it was shown that the critical detection efficiency could be reduced for the so-called Einstein-Podolsky-Rosen-Bell inequalities that require perfect correlations [40]. Similar ideas have been developed in later works focused on quantum key distribution [41,42] and the P value of a Bell test [43]. Very recently, the idea has been explored for the case of 2-qubit maximally entangled states [44]. In this Letter, we introduce a much more powerful and practical tool: penalized N-product (PNP) Bell inequalities. This tool leads to smaller critical detection efficiencies than those obtained in Ref. [44] and applies to any quantum violation of any Bell inequality, thus opening a new path toward loophole-free Bell tests with longer distances and higher dimensions.

Product Bell inequalities.—Let us consider N Bell inequalities of the same type in parallel. Our first task is to identify a single parameter that quantifies the violation of local realism. One way to do it is to consider the product of the N parameters of all N Bell inequalities. Following this approach, let us start with a Bell inequality of the form

$$\sum_{a,b,x,y} p(a,b|x,y)c_{a,b}^{x,y} \leq C, \tag{1}$$

where $p(a,b|x,y)$ denotes the conditional probability of Alice and Bob to observe outcomes a and b (with $a,b \in [m]$), respectively, given their choice of measurement settings x and y (with $x,y \in [n]$), respectively, and C is the LHV bound. Throughout the text, $[n] = \{0,1,\ldots,n-1\}$. An *N-product Bell inequality* based on Eq. (1) is defined as

$$\sum_{\mathbf{a},\mathbf{b},\mathbf{x},\mathbf{y}} p(\mathbf{a},\mathbf{b}|\mathbf{x},\mathbf{y}) \prod_{i=1}^{N} c_{a_i,b_i}^{x_i,y_i} \leq C_N, \tag{2}$$

where $\mathbf{a} = (a_1,\ldots,a_N)$ is a tuple of Alice's measurement outcomes, with $a_i \in [m]$ for all $i \in \{1,2,\ldots,N\}$, and $\mathbf{b}$, $\mathbf{x}$, $\mathbf{y}$ similarly defined. C_N denotes the maximum value of the N-product Bell inequality attainable by LHV models.

One could expect that $C_N = C^N$. However, this is not the case for arbitrary Bell inequalities of the form given by Eq. (1), including the CHSH inequality [34]. Indeed, for the CHSH inequality, $C = \frac{3}{4}$ but $C_2 = (10/16)$ [38] and $C_3 = (31/64)$ [43]. This fact is also referred to as the *simultaneous measurement loophole* in Bell tests [38]. The problem of determining the closed form for C_N (for the cases when $C_N > C^N$) is closely related to the so-called *parallel repetition theorem* in interactive proof systems [45]. This problem was tackled in Refs. [46–48], where only asymptotic upper-bounds on C_N were reported. Moreover, the authors of Ref. [47] emphasized the difficulty of finding exact values of C_N.

In this Letter, we take a different approach to the problem. Instead of trying to find the values of C_N, we propose a method for modifying the Bell expression in Eq. (2) in a way that $C_N = C^N$ holds for all N. We achieve this by adding a nonlinear "penalty term" to the left-hand side of Eq. (2), which forces a product local strategy (i.e., one in which each outcome a_i depends only on x_i, and similarly for Bob) to be optimal. Given a Bell expression specified by coefficients $c_{a,b}^{x,y}$ and the LHV bound C, we define a *penalized N-product (PNP) Bell inequality* as follows:

$$\sum_{\mathbf{a},\mathbf{b},\mathbf{x},\mathbf{y}} p(\mathbf{a},\mathbf{b}|\mathbf{x},\mathbf{y}) \prod_{i=1}^{N} c_{a_i,b_i}^{x_i,y_i} - \kappa(\mathrm{A}+\mathrm{B}) \leq C^N, \tag{3}$$

where $\kappa \in \mathbb{R}$ is some large positive number and

$$\mathrm{A}=\sum_{i=1}^{N}\sum_{\mathbf{x}}\sum_{\mathbf{x}'\neq\mathbf{x}}^{x_i=x_i'}\sum_{a_i=0}^{m-2}|p(a_i|\mathbf{x})-p(a_i|\mathbf{x}')|, \quad (4a)$$

$$\mathrm{B}=\sum_{i=1}^{N}\sum_{\mathbf{y}}\sum_{\mathbf{y}'\neq\mathbf{y}}^{y_i=y_i'}\sum_{b_i=0}^{m-2}|p(b_i|\mathbf{y})-p(b_i|\mathbf{y}')|. \quad (4b)$$

The sum over $\mathbf{x}'$ is taken such that $\mathbf{x}$ and $\mathbf{x}'$ match on the ith element, but are not the same. The same holds for the sum over $\mathbf{y}'$. $p(a_i|\mathbf{x})$ denotes the marginal probability of outcome a_i of Alice's measurement specified by $\mathbf{x}$. $p(b_i|\mathbf{y})$ is analogously defined for Bob.

The general idea of the method is rather straightforward. By taking large enough κ, we force both quantities A and B in Eq. (4) to be exactly 0. The condition $\mathrm{A}=0$ implies that Alice has to choose her local strategy among nonsignaling ones with respect to her local outcomes a_i and settings x_i. ($\mathrm{B}=0$ implies the same for Bob). Note that this set of strategies is larger than the set of product strategies for which $p(\mathbf{a}|\mathbf{x})=\prod_{i=1}^{N}p(a_i|x_i)$ holds. Nevertheless, it is not difficult to show that the nonsignaling constraints $\mathrm{A}=\mathrm{B}=0$ enforce the bound to be C^N for the ideal case of infinite runs of the experiment [49].

The remaining question is how large should one take κ to be. We answer this question below for the case of $m=2$.

Result 1: Given a Bell inequality specified by the coefficients $c_{a,b}^{x,y}$, with $a,b\in\{0,1\}$, $x,y\in[n]$, it is sufficient to take $\kappa=n^{N-1}(\Sigma_N-C^N)$, such that the LHV bound of the corresponding PNP Bell inequality is C^N, where Σ_N is the algebraic bound of the N-product Bell inequality without the penalty term.

Note that, instead of Σ_N, any known, possibly tighter, bound on C_N can be used [50].

Proof.—For the proof, we use the terminology of probability vectors and local polytopes introduced in Ref. [51]. For the Bell scenario with n settings per party and binary outcomes, the probability vector is defined as $\mathbf{p}=[p(0,0|0,0),\ldots,p(1,1|n-1,n-1)]$; i.e., it is a vector that uniquely specifies the behavior $p(a,b|x,y)$. The local polytope $\mathcal{P}_{\mathrm{LHV}}$ is the region in the space of $\mathbf{p}$, corresponding to LHV models. This polytope is convex and, by the Minkowski-Weyl theorem, it can be described either as a convex hull of its extremal points (in this case determined by local deterministic strategies) or as an intersection of half-spaces (which in this case are tight Bell inequalities and axioms of probabilities). The above concepts generalize straightforwardly to our scenario with multiple inputs and outputs, and we will use $\mathbf{p}$ and $\mathcal{P}_{\mathrm{LHV}}$ to denote these concepts for our case.

For convex polytopes, the maximum of a linear function such as the one in Eq. (1) is attained at one of its extremal points. Although the expression in Eq. (3) is not linear on the whole $\mathcal{P}_{\mathrm{LHV}}$, it is linear in each part of $\mathcal{P}_{\mathrm{LHV}}$ for which every expression inside moduli in Eq. (4) has a definite sign. Hence, the global maximum has to be attained at either one of the extremal points of $\mathcal{P}_{\mathrm{LHV}}$, or at a point resulting from the intersections of the facets of $\mathcal{P}_{\mathrm{LHV}}$ by the hyperplanes $p(a_i|\mathbf{x})-p(a_i|\mathbf{x}')=0$ and $p(b_j|\mathbf{y})-p(b_j|\mathbf{y}')=0$, for some sets of i, j and some pairs $\mathbf{x}\neq\mathbf{x}'$ and $\mathbf{y}\neq\mathbf{y}'$. Let us denote the set of all of such points as $\mathcal{E}=\{\mathbf{p}_e\}_e$.

Among all the points $\mathbf{p}_e$ in $\mathcal{E}$, there are some, let us call them $\mathcal{E}_0$, for which $\mathrm{A}=\mathrm{B}=0$ holds. For points in $\mathcal{E}\backslash\mathcal{E}_0$, the minimal value of $\mathrm{A}+\mathrm{B}$ is $n^{-(N-1)}$ [49]. On the other hand, the value of the expression in Eq. (3) on any of the points $\mathbf{p}_e$ without the penalty term, cannot exceed its algebraic maximum Σ_N. Therefore, taking $\kappa=n^{N-1}(\Sigma_N-C^N)$ ensures that the LHV bound of Eq. (3) cannot exceed the one for strategies compatible with $\mathrm{A}=\mathrm{B}=0$, i.e., the product bound C^N (see the Supplemental Material [49]). ∎

The purpose of the upper bound on the sufficient value of κ is not only theoretical. In practice, even if we use a product quantum strategy, due to experimental errors both A and B will have small yet nonzero values. These errors will be multiplied by κ and could potentially result in large errors in the value of the violation.

Lowering the critical detection efficiency.—Here, we show that having a source of photon pairs carrying N entangled states each alongside with PNP Bell inequalities allows for a significant reduction in the critical detection efficiency requirements for the violation of local realism.

To avoid the *fair sampling assumption* [52], the parties need to either treat "no-click" events as additional outcomes or employ a local assignment strategy [53]. The latter means that whenever one party's detector does not click (when it should), the party draws an outcome according to some local (deterministic) strategy. This allows the parties to use the same Bell inequality without the need to find one with more outcomes.

In this Letter, we consider the local assignment strategy for mitigation of the "no-click" events. Let $\otimes_{i=1}^{N}\rho_{AB}$ be a state carried by photon pair in out setup. Let $\otimes_{i=1}^{N}\mathcal{A}_{a_i}^{x_i}$ and $\otimes_{i=1}^{N}\mathcal{B}_{b_i}^{y_i}$ be the POVM (positive-operator valued measure) elements of Alice and Bob respectively, i.e., they are formed by the POVM elements $\mathcal{A}_a^x$ and $\mathcal{B}_b^y$, that are the same for all i. Evidently, this leads to quantum behavior of the form $p(\mathbf{a},\mathbf{b}|\mathbf{x},\mathbf{y})=\prod_{i=1}^{N}\mathrm{tr}(\mathcal{A}_{a_i}^{x_i}\otimes\mathcal{B}_{b_i}^{y_i}\rho_{AB})$. Let $\alpha\colon[n]\mapsto\{0,1\}$ and $\beta\colon[n]\mapsto\{0,1\}$ be deterministic assignment strategies $a_i=\alpha(x_i)$ and $b_i=\beta(y_i)$, for all i, employed by Alice and Bob respectively in case of a "no-click" event. If, for instance, Bob's detector does not click but Alice's does, the parties' observed behavior is $p(\mathbf{a},\mathbf{b}|\mathbf{x},\mathbf{y})=\prod_{i=1}^{N}\mathrm{tr}(\mathcal{A}_{a_i}^{x_i}\rho_A)\delta_{b_i,\beta(y_i)}$, where ρ_A is Alice's reduced state ρ_{AB} and $\delta_{\cdot,\cdot}$ is the Kronecker delta. Similarly, the parities observe the behavior $p(\mathbf{a},\mathbf{b}|\mathbf{x},\mathbf{y})=\prod_{i=1}^{N}\mathrm{tr}(\mathcal{B}_{b_i}^{y_i}\rho_B)\delta_{a_i,\alpha(x_i)}$ whenever Alice's detector does not click, but the one of Bob does. Finally, for the cases of no clicks on both detectors, the parties observe a local deterministic behavior $p(\mathbf{a},\mathbf{b}|\mathbf{x},\mathbf{y})=\prod_{i=1}^{N}\delta_{a_i,\alpha(x_i)}\delta_{b_i,\beta(y_i)}$.

Let us now take $c_{x,y}^{a,b} \geq 0$ in the considered Bell inequality, which can always be achieved. Assuming the detection efficiency of Alice's and Bob's detectors to be η, the value of the PNP Bell expression is the following:

$$\eta^2 Q^N + \eta(1-\eta)(A^N + B^N) + (1-\eta)^2 C^N, \tag{5}$$

with

$$Q = \sum_{a,b,x,y} c_{a,b}^{x,y} \mathrm{tr}(\mathcal{A}_a^x \otimes \mathcal{B}_b^y \rho_{AB}), \tag{6a}$$

$$A = \sum_{a,b,x,y} c_{a,b}^{x,y} \mathrm{tr}(\mathcal{A}_a^x \rho_A) \delta_{b,\beta(y)}, \tag{6b}$$

$$B = \sum_{a,b,x,y} c_{a,b}^{x,y} \mathrm{tr}(\mathcal{B}_b^y \rho_B) \delta_{a,\alpha(x)}, \tag{6c}$$

where we have assumed that the local strategies α and β reproduce the LHV bound C. Clearly, since all the aforementioned strategies are product, the penalty term is exactly 0. Notice that in Eq. (5), η appears only in its second power, precisely due to the fact that the N-qudit state $\otimes_{i=1}^N \rho_{AB}$ is carried by a single pair of photons. This is what we meant when we said that the effective detection efficiency for each of the N Bell inequality is of the order of $\eta^{(1/N)}$.

To observe a violation of local realism, one needs to ensure that the value of the expression in Eq. (5) is greater than the LHV bound C^N. Solving this inequality with respect to η, we obtain the following value of the required detection efficiency for given Q, A, and B:

$$\eta = \frac{2C^N - A^N - B^N}{Q^N + C^N - A^N - B^N}. \tag{7}$$

This equation has the following interesting implication.

Remark 1: For any given Bell inequality with binary outcomes and a quantum strategy with $Q > C$, it follows from Eq. (7) that the detection efficiency requirement decays exponentially with N.

Indeed, if we take $A = B = \delta C$, then $\eta = 2(C/Q)^N(1-\delta^N) + O((C/Q)^{2N})$. For any Bell inequality, $\delta < 1$ whenever $Q > C$. Hence, the decay of η with $N \to \infty$ is at least exponential with the factor of $\log(C/Q)$. The above remark is in parallel with the results of Massar [25,54].

In order to find the *critical detection efficiency* η_{crit} for a given Bell inequality and its corresponding PNP Bell inequality, one needs to optimize η in Eq. (7) over all possible values of (Q, A, B). In what follows, we solve this optimization problem for the N-product CHSH inequality.

PNP inequality for the CHSH inequality.—The coefficients of the CHSH inequality [34] in its nonlocal game formulation are $c_{a,b}^{x,y} = \frac{1}{4}\delta_{a\oplus b,xy}$, where $a, b, x, y \in \{0, 1\}$ and $\oplus$ denotes addition modulo 2. For this form of the CHSH inequality, we have $C = \frac{3}{4}$, the quantum bound $Q_{\max} = \frac{1}{2} + (1/2\sqrt{2})$, and $\Sigma = 1$. In order to minimize the expression in Eq. (7) over all quantum states and measurements, first we determine the maximal values of A and B attainable for a given value of Q, and then optimize Eq. (7) over Q. In particular, due to the symmetry with respect to A and B in Eq. (7), we are interested in the situation $A = B$. For this case, the optimal relation is the following:

$$A = B = \frac{1}{2} + \frac{1}{4}\sqrt{(1-q)\left(1 + \frac{q}{\sqrt{1+q^2}}\right)}, \tag{8}$$

where $q = \sqrt{(4Q-2)^2 - 1}$. As Q changes from $\frac{3}{4}$ to $\frac{1}{2} + (1/2\sqrt{2})$, q increases from 0 to 1, and, hence, A and B decrease from $\frac{3}{4}$ to $\frac{1}{2}$. For the 2-qubit state ρ_{AB} and qubit measurements $\mathcal{A}_a^x$ and $\mathcal{B}_b^y$ that produce the relation in Eq. (8) see the Supplemental Material [49]. We used the Navascués-Pironio-Acín hierarchy [55] to indicate the dimension-independent optimality of Eq. (8).

Employing the relation in Eq. (8), we optimize η in Eq. (7) over Q in order to obtain the optimal value η_{crit} for a given N. We plot the results in Fig. 1. In the same figure, we show the minimal visibility $v_{0.75}$ for which violation can still be observed with detectors of a given detection efficiency $\eta = 0.75$. As we can see, even though taking 2-, 3-, and 4-product CHSH inequalities does not decrease the value of η_{crit}, one can obtain a significant advantage in terms of visibility for $\eta > \eta_{\mathrm{crit}}$.

In Fig. 2 we plot η_v, the required detection efficiency to observe a violation of the PNP Bell inequality with visibility as low as v. We also account for possible experimental imperfections by taking nonzero values of A + B. Note that the tolerance to the imperfections can be significantly increased if, instead of taking an algebraic

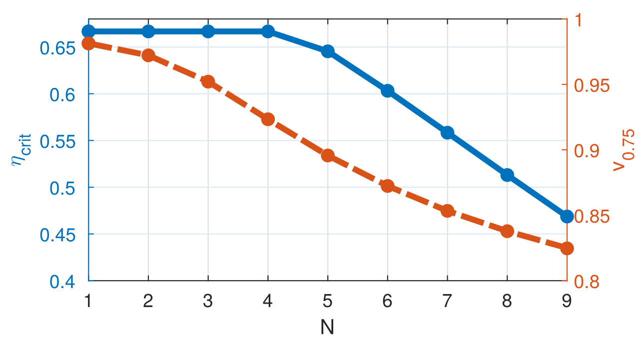

FIG. 1. (Solid line) Critical detection efficiency η_{crit} for the PNP Bell inequality as a function of N. (Dashed line) Visibility (per qubit pair) required for a loophole-free Bell test when $\eta = 0.75$ as a function of N. Perfect statistics is assumed, i.e., the penalty term A + B = 0.

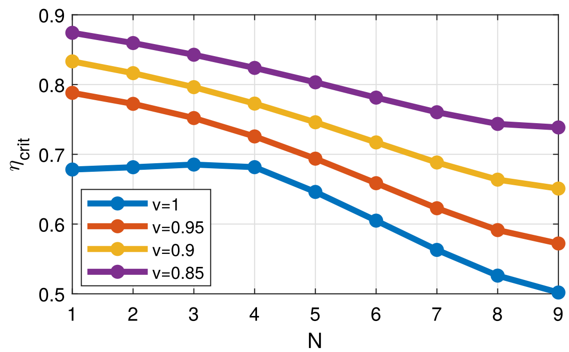

FIG. 2. Required detection efficiency for different values of the visibility v, as a function of N. Lines in the plot are arranged from bottom to top as v changes from 1 to 0.85. The value of the penalty term $\kappa(\mathrm{A}+\mathrm{B})$ in Eq. (4) is taken to be $10^{-5}\kappa$.

maximum $\sum_N$ in Result 1, a known bound on the parallel repetition of the Bell inequality is used [50].

Summary and outlook.—In this Letter, we addressed the problem of reducing the detection efficiency requirements for loophole-free Bell experiments in order to achieve loophole-free Bell tests over longer distances. We presented a method that, when applied to any given Bell inequality, produces a new Bell inequality by taking a penalized product of N copies of it, for which the critical detection efficiency decays exponentially with N. This implies that the critical detection efficiency can be drastically reduced in experiments using photon sources that allow for encoding multiple copies of a qubit-qubit (or qudit-qudit) entangled state on a single pair of particles. Examples of such sources are hyperentanglement sources and sources of high-dimensional entanglement.

We applied our method to several binary Bell inequalities and found that the lowest detection efficiencies occur for the PNP CHSH inequality. The advantage of the CHSH inequality is in terms of both critical detection efficiency and visibility of the violation. Our method can be applied to any Bell inequality with more outcomes, given that Result 1 can be extended to an arbitrary number of outcomes. A natural target for future work is to identify Bell inequalities for which the critical detection efficiencies are low enough for mid-distance photonic loophole-free Bell tests and related applications such as device-independent quantum key distribution.

Other important questions deserve separate investigation. For instance, we believe that the bound on the penalty coefficient κ in Result 1 can be significantly lowered. Another relevant problem is the calculation of P values for PNP Bell inequalities, which would depend on the value of the penalty term. Finally, it is interesting to see whether PNP Bell inequalities can be used for a single-shot Bell test [43].

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