Nonperturbative Double Copy in Flatland

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We derive a nonperturbative, Lagrangian-level formulation of the double copy in two spacetime dimensions. Our results elucidate the field theoretic underpinnings of the double copy in a broad class of scalar theories which can include masses and higher-dimension operators. An immediate corollary is the amplitudes-level double copy at all orders in perturbation theory. Applied to certain integrable models, the double copy defines an isomorphism between Lax connections, Wilson lines, and infinite towers of conserved currents. We also implement the double copy at the level of nonperturbative classical solutions, both analytically and numerically, and present a generalization of the double copy map that includes a fixed tower of higher-dimension corrections given by the Moyal algebra.

Introduction.—Recent breakthroughs in scattering theory have unveiled an extraordinary hidden structure lying dormant within the fundamental laws of nature. The so-called "double copy" [1–4] is a mathematical formula, only proven at tree level, that very simply relates *perturbative scattering amplitudes* of gravitons in Einstein's general relativity (GR) to those of gluons in Yang-Mills theory (YM).

At a purely practical level, the double copy is an immensely efficient tool for recycling past results in gauge theory to derive new ones for gravity. This approach has made feasible many previously intractable calculations, for example, those relevant to the finiteness of supergravity theories [5–12] and more recently, post-Minkowskian computations for black hole binary dynamics [13–16] which are directly relevant to the LIGO experimental program [17,18] and are, within the last three years, competing with the state of the art.

At the conceptual level, the double copy remains deeply mysterious. Its structure transcends gauge theory and gravity and applies to a broad web of theories [3]. For example, the exact same double copy *also relates* all tree-level amplitudes of pions in the chiral limit to those of certain hypothetical scalars known as Galileons, which have been studied independently as viable theories of cosmology and modified gravity.

In broad strokes, the double copy maps gauge theory to gravity by first expressing every gauge theory amplitude as a sum over cubic graphs,

$$A_n = \sum_{\text{cubic}} \frac{c_i n_i}{d_i}, \qquad (1)$$

where the c_i are color factors (structure constants), the n_i are kinematic numerators, and the d_i are propagators [1–3,19,20]. *Color-kinematics duality* states that there exists a rearrangement of terms such that the kinematic numerators obey the same Jacobi identities as the color factors. Gravity—as the *square* or *double copy* of gauge theory—is simply obtained by replacing each color factor with the associated kinematic numerator, $c_i \to n_i$.

The double copy is an established fact about flat space, perturbative scattering amplitudes, but its generality is far from understood. To what extent does it apply off-shell [21–27]? To curved geometries [28–38]? Nonperturbatively? Finding answers to these questions could provide a nonperturbative, background independent mapping between gravity and far simpler quantum field theories.

In this Letter, we present a nonperturbative double copy in two spacetime dimensions. This is the first off-shell Lagrangian level formulation of the double copy for interacting theories. [39]. Extending the proof of the double copy from tree level to all loop orders has implications for the understanding of all double copy constructions. Our approach is inspired by a remarkable isomorphism between the algebras of unitary transformations and diffeomorphisms [40],

$$\lim_{N \to \infty} U(N) \sim \text{Diff}_{S^1 \times S^1}, \qquad (2)$$

and applies to an enormous class of scalar theories, including masses and higher-dimension operators.

We apply this construction successively to map bi-adjoint scalar (BAS) theory to Zakharov-Mikhailov (ZM) theory [41] to the special Galileon (SG) [42–44], thus

deriving the corresponding and more familiar amplitudes-level double copy at all orders in perturbation theory. [45] Since ZM theory is classically integrable, it furnishes a Lax connection whose Wilson lines define an infinite tower of conserved currents, all of which are shown to double copy into corresponding objects in the SG. An extension of the double copy based on the Moyal algebra is presented where N, the rank of the gauge group, parameterizes an infinite tower of higher-dimension operators [46]. Note that at the classical level, ZM theory is very closely related to self-dual Yang-Mills (SDYM) theory [21,22], which exhibits identical integrable and Moyal structures [49].

Implementing the double copy on nonperturbative, large-field configurations, we show analytically that every classical solution of the SG theory is isomorphic to corresponding dual solutions in ZM and BAS theory. As a highly nontrivial check, we compute an explicit, large-field, numerical solution for soliton scattering in the SG theory, map it to a corresponding configuration in ZM theory for $U(N)$ at large N, and verify that it satisfies the ZM equations of motion to high precision.

Color algebra.—A field in the adjoint of $U(N)$ is a Hermitian matrix, $\boldsymbol{V} = V^a \boldsymbol{T}_a$, where $[\boldsymbol{T}_b]_a{}^c = if_{ab}{}^c$ and $[\boldsymbol{T}_a, \boldsymbol{T}_b] = if_{ab}{}^c \boldsymbol{T}_c$. For odd N there exists a basis of generators $\boldsymbol{T}_p$ labeled by a two-vector, $p \in \mathbb{Z}_N \times \mathbb{Z}_N$ [40]. In this basis, $\boldsymbol{V} = V^p \boldsymbol{T}_p$ where $V^{p*} = V^{-p}$, and

$$[\boldsymbol{T}_{p_i}, \boldsymbol{T}_{p_j}] = if_{p_ip_j}{}^{p_k} \boldsymbol{T}_{p_k}, \tag{3}$$

with the corresponding color structure constant [50],

$$f_{p_ip_j}{}^{p_k} = -\frac{N}{2\pi} \sin\left(\frac{2\pi}{N} \langle ij \rangle\right) \delta_{p_i+p_j,p_k} \overset{N\to\infty}{=} -\langle ij \rangle \delta_{p_i+p_j,p_k}. \tag{4}$$

Hence, the $N \to \infty$ limit literally defines the algebra of volume-preserving diffeomorphisms on the torus [40], or equivalently, the Poisson algebra. The toroidal geometry arises because the generator labels p are defined mod N.

Kinematic algebra.—Equation (2) implies that fields in the adjoint of $U(N)$ at large N are isomorphic to field-dependent diffeomorphisms,

$$\boldsymbol{V} = \epsilon^{\mu\nu} \partial_\mu V \partial_\nu = \partial_\mu V \tilde{\partial}^\mu = -\tilde{\partial}^\mu V \partial_\mu, \tag{5}$$

which are volume-preserving because $\partial_\mu \tilde{\partial}^\mu V = 0$. This algebra is closed since the commutator of diffeomorphisms yields another diffeomorphism via

$$\boldsymbol{Z} = [\boldsymbol{V}, \boldsymbol{W}] = [\partial_\mu V \tilde{\partial}^\mu, \partial_\nu W \tilde{\partial}^\nu] = \partial_\mu Z \tilde{\partial}^\mu, \tag{6}$$

where $Z = \partial_\mu V \tilde{\partial}^\mu W$. Motivated by these structures, we propose a color-kinematic duality replacement,

$$
\begin{aligned}
V^a &\to V \\
f_{ab}{}^c V^a W^b &\to \partial_\mu V \tilde{\partial}^\mu W \\
g_{ab} V^a W^b &\to \int VW.
\end{aligned}
\tag{7}
$$

The first line simply maps any color-adjoint field to a corresponding singlet field. The second line maps the color structure constant to a kinematic structure constant whose momentum space representation is

$$f_{p_ip_j}{}^{p_k} = -\langle ij \rangle \delta^2(p_i + p_j - p_k). \tag{8}$$

This is literally the continuum limit of Eq. (4), in accordance with the algebra isomorphism in Eq. (2). The third line is obtained from the Killing form of $U(N)$ at large N, which effectively defines a Killing form for the diffeomorphism algebra [40].

Lagrangian double copy.—The color-kinematic replacement rules in Eq. (7) can be applied directly at the level of the Lagrangian, thus giving an off-shell, nonperturbative definition of the double copy.

BAS theory: The Lagrangian for BAS theory is

$$\mathcal{L}_{\mathrm{BAS}} = \frac{1}{2} \partial_\mu \phi_{a\bar{a}} \partial^\mu \phi^{a\bar{a}} + \frac{1}{6} f_{abc} f_{\bar{a}\bar{b}\bar{c}} \phi^{a\bar{a}} \phi^{b\bar{b}} \phi^{c\bar{c}}, \tag{9}$$

while the corresponding equation of motion is

$$\Box \phi^{c\bar{c}} - \frac{1}{2} f_{ab}{}^c f_{\bar{a}\bar{b}}{}^{\bar{c}} \phi^{a\bar{a}} \phi^{b\bar{b}} = 0. \tag{10}$$

The tree-level four-point off-shell BAS amplitude is

$$-A_{\mathrm{BAS}} = \frac{c_s \bar{c}_s}{s} + \frac{c_t \bar{c}_t}{t} + \frac{c_u \bar{c}_u}{u}, \tag{11}$$

where $s = (p_1 + p_2)^2$, $t = (p_2 + p_3)^2$, $u = (p_3 + p_1)^2$, and the color factors are $c_s = f_{a_1a_2}{}^b f_{ba_3a_4}$, $c_t = f_{a_2a_3}{}^b f_{ba_1a_4}$, $c_u = f_{a_3a_1}{}^b f_{ba_2a_4}$, and likewise for barred color.

Massless *on-shell* kinematics in two dimensions is famously plagued by infrared singularities since all asymptotic states are either left or right movers. For example, for the case of four-point scattering with color-ordered external states, the external momenta exhibit kinematic configurations which we classify as "split," where $p_1 + p_2 = p_3 + p_4 = 0$ or $p_2 + p_3 = p_1 + p_4 = 0$, versus "alternating," where $p_3 + p_1 = p_2 + p_4$. Since either s, t, or u is always zero, there is a vanishing Gram determinant, $stu = 0$, and propagator exchanges generically exhibit collinear singularities.

The precise method of infrared regulation—be it going off-shell, introducing a physical mass term to the theory, or analytically continuing away from two dimensions—can yield different answers for nominally classical equivalent

theories, and special care must be taken [51]. Nevertheless, the claim of the present Letter is that *assuming* a particular infrared regulator, our construction can be applied to map any given infrared-regulated theory to a corresponding infrared-regulated double copy theory.

ZM theory: Applying the replacement rules in Eq. (7) to the *Lagrangian* of BAS theory in Eq. (9), we obtain the *action* of ZM theory, whose Lagrangian is [41,51–53]

$$\mathcal{L}_{\mathrm{ZM}} = \frac{1}{2}\partial_\mu\phi_a\partial^\mu\phi^a + \frac{1}{6}f_{abc}\phi^a\partial_\mu\phi^b\tilde{\partial}^\mu\phi^c. \quad (12)$$

The resulting equation of motion is

$$\Box\phi^c - \frac{1}{2}f_{ab}{}^c\partial_\mu\phi^a\tilde{\partial}^\mu\phi^b = 0, \quad (13)$$

which can alternatively be obtained from Eq. (10) via Eq. (7). Note that Eq. (13) also encodes the dynamics of SDYM theory [21,22,49].

As is well-known [41,51,53], ZM theory is classically equivalent to the principal chiral model (PCM), otherwise known as the nonlinear sigma model (NLSM) in two dimensions. In general dimensions, the NLSM is classically defined by

$$\partial_{[\mu}j^c_{\nu]} + f_{ab}{}^c j^a_\mu j^b_\nu = 0 \quad \text{and} \quad \partial^\mu j^a_\mu = 0, \quad (14)$$

where the former is a pure gauge condition implying that $j^a_\mu T_a \sim g^{-1}\partial_\mu g$ and the latter is the NLSM equation of motion. By defining $j^a_\mu = \epsilon_{\mu\nu}\partial^\nu\phi^a$, we trivially enforce the latter, while the former is equivalent to Eq. (13).

The three-point Feynman vertex defined by Eq. (12) is

$$= -if_{abc}\langle 12\rangle, \quad (15)$$

which is fully antisymmetric because off-shell two-dimensional kinematics implies that $\langle 12\rangle = \langle 23\rangle = \langle 31\rangle$.

The tree-level four-point off-shell ZM amplitude is

$$-A_{\mathrm{ZM}} = \frac{c_s n_s}{s} + \frac{c_t n_t}{t} + \frac{c_u n_u}{u}, \quad (16)$$

where the kinematic numerators,

$$n_s = \langle 12\rangle\langle 34\rangle, \quad n_t = \langle 23\rangle\langle 14\rangle, \quad n_u = \langle 31\rangle\langle 24\rangle, \quad (17)$$

satisfy the off-shell kinematic Jacobi identity, $n_s + n_t + n_u = 0$, on account of the Schouten identity. Applying the standard color decomposition [54], the color-ordered ZM amplitude is $A_{\mathrm{ZM}}[1234] = (n_s/s) - (n_t/t)$.

For the alternating configuration described previously, $u = s + t = 0$, which implies that $A_{\mathrm{ZM}}[1234]$ is free of collinear singularities. In this case $n_s = -n_t = \langle 12\rangle^2$, so $A_{\mathrm{ZM}}[1234] = 0$, in accordance with the phenomenon of no-particle production described in [55]. For the split configurations, $A_{\mathrm{ZM}}[1234]$ is nonzero but must be evaluated with some choice of infrared regulator [51].

At loop level, integrands at arbitrary order are mechanically calculated using the Feynman vertex in Eq. (15). By construction, all loop-level kinematic Jacobi identities are automatically satisfied, even off-shell. While enforcing "global color-kinematics constraints" is a well-known difficulty in gauge theory starting at two loops [56], we learn here that there is no obstacle to this for ZM theory at all orders in perturbation theory.

SG theory: Equation (7) maps the ZM Lagrangian in Eq. (12) to the action of the SG theory, whose Lagrangian is

$$\mathcal{L}_{\mathrm{SG}} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{6}\phi\partial_\mu\partial_\nu\phi\tilde{\partial}^\mu\tilde{\partial}^\nu\phi, \quad (18)$$

and whose equation of motion is

$$\Box\phi - \frac{1}{2}\partial_\mu\partial_\nu\phi\tilde{\partial}^\mu\tilde{\partial}^\nu\phi = 0. \quad (19)$$

The three-point Feynman vertex is then

$$= i\langle 12\rangle^2, \quad (20)$$

which is fully permutation invariant. Applying either an off-shell or mass regulator for infrared singularities, the on-shell amplitude is

$$-A_{\mathrm{SG}} = \frac{n_s^2}{s} + \frac{n_t^2}{t} + \frac{n_u^2}{u} \sim stu = 0, \quad (21)$$

which is proportional to the Gram determinant and thus vanishes in two dimensions. This reflects the fact that the SG is field-redefinition equivalent to a two-dimensional free theory [57,58].

Masses and higher-dimension operators.—Thus far we have only considered those theories which have historically appeared in the amplitudes-level double copy [3]. Our construction extends far more broadly, however. In particular, the color-kinematic replacement rules in Eq. (7) can be applied to *any operator* that does not have (i) a closed loop of color structure constants, nor (ii) multiple color traces. In both cases, the third line of Eq. (7) will induce ill-defined or nonlocal integrals over the volume of spacetime which enter into the Lagrangian. The mildness of the restrictions (i) and (ii) means that a very large class of operators manifestly obey color-kinematics duality, in sharp contrast to the typical intuition that almost all operators will fail the duality.

By these rules, mass terms are perfectly fine and trivially double copy via the same color-kinematic replacements as the kinetic terms. These mass terms serve only to change the propagator denominators.

Equation (7) can also be implemented for an infinite class of higher-dimension operators. For example, consider the higher-dimension operator in BAS theory, $\mathcal{O}_{\mathrm{BAS}} = f_{abe}f_{cde}f_{\bar{a}\,\bar{b}\,\bar{e}}f_{\bar{c}\,\bar{d}\,\bar{e}}\phi^{a\bar{a}}\phi^{b\bar{b}}\phi^{c\bar{c}}\phi^{d\bar{d}}$, where both the color and dual color structures are single trace. Applying Eq. (7) to dual color, we obtain the spacetime integral of $\mathcal{O}_{\mathrm{ZM}} = f_{abe}f_{cde}\partial_\mu\phi^a\tilde{\partial}^\mu\phi^b\partial_\nu\phi^c\tilde{\partial}^\nu\phi^d$, which is the color-kinematic dual operator in ZM theory. Then applying Eq. (7) to the remaining color structures, we obtain the spacetime integral of $\mathcal{O}_{\mathrm{SG}} = \partial_\mu\partial_\nu\phi\tilde{\partial}^\mu\tilde{\partial}^\nu\phi\partial_\rho\partial_\sigma\phi\tilde{\partial}^\rho\tilde{\partial}^\sigma\phi$, which is the color-kinematic dual operator in the SG theory.

Now consider $\mathcal{O}'_{\mathrm{BAS}} = g_{ac}g_{bd}f_{\bar{a}\,\bar{b}\,\bar{e}}f_{\bar{c}\,\bar{d}\,\bar{e}}\phi^{a\bar{a}}\phi^{b\bar{b}}\phi^{c\bar{c}}\phi^{d\bar{d}}$, which is double trace in color and single trace in dual color. Applying Eq. (7) to the latter, we obtain $\mathcal{O}'_{\mathrm{ZM}} = \partial_\mu\phi_a\tilde{\partial}^\mu\phi_b\partial_\nu\phi^a\tilde{\partial}^\nu\phi^b$. Since the resulting operator is double-trace in color, it cannot be further double copied via Eq. (7) without generating an additional integral over all of spacetime.

Fundamental BCJ relation.—Our Lagrangian-level formulation of the double copy does not preserve the fundamental Bern-Carrasco-Johansson (BCJ) relation [1,2] nor the so-called minimal rank condition [59]. Ultimately, this is not so surprising because the fundamental BCJ relation is literally equivalent to the conservation equation for the kinematic current in theories with purely cubic interactions [24,28], and our generalized double copy construction allows for quartic and higher interactions.

Crucially, failure of the minimal rank condition implies that our framework should be interpreted as a generalization of the color-kinematic dual formulation of the double copy [1,2], which is built upon the kinematic Jacobi identities, rather than the Kawai-Lewellen-Tye (KLT) formulation [4], which relies on relations amongst color-ordered amplitudes.

As an example, consider BAS theory deformed by a mass and the higher-dimension operator defined earlier,

$$\mathcal{L} = \mathcal{L}_{\mathrm{BAS}} - \frac{m^2}{2}\phi_{a\bar{a}}\phi^{a\bar{a}} + \frac{\tau}{16}\mathcal{O}_{\mathrm{BAS}}. \qquad (22)$$

For the moment, let us work in general dimensions, where infrared divergences are absent. The matrix of doubly color-ordered amplitudes is

$$H(m,\tau) = \begin{pmatrix} A[1234|1234] & A[1234|1324] \\ A[1324|1234] & A[1324|1324] \end{pmatrix}$$

$$= -\begin{pmatrix} \frac{1}{s-m^2}+\frac{1}{t-m^2}-\tau & -\frac{1}{t-m^2}+\frac{\tau}{2} \\ -\frac{1}{t-m^2}+\frac{\tau}{2} & \frac{1}{t-m^2}+\frac{1}{u-m^2}-\tau \end{pmatrix}. \qquad (23)$$

The minimal rank condition holds for pure BAS theory in general dimensions since $\det H(0,0) = 0$ on-shell.

However, it fails in the presence of masses [60,61] and higher-dimension operators [59],

$$\det H(m,0) = \frac{m^2}{(s-m^2)(t-m^2)(u-m^2)}$$

$$\det H(0,\tau) = -\tau\left(\frac{1}{s}+\frac{1}{t}+\frac{1}{u}\right)+\frac{3\tau^2}{4}. \qquad (24)$$

Evaluating these expressions for two-dimensional kinematics, we encounter the usual annoyances of infrared divergences, but irrespective of choice of regulator, the above determinants are still nonzero.

Integrable models.—Since ZM theory is classically equivalent to the PCM, it is similarly integrable [41,52,62]. Moreover, ZM theory maps to the SG under the double copy, so we will see that the latter is also integrable. Note that mapping the integrability of one theory to another would not be possible with the standard amplitudes-level double copy because the integrability conditions are expressed in terms of currents and off-shell fields.

As a brief review, integrability is achieved by casting the equations of motion into the form of the *Lax equation*, $\dot{L} = [M,L]$, where the operators L and M constitute a *Lax pair* [63–67]. By virtue of this form of the equations of motion, the eigenvalues of L are conserved charges. A familiar Lax pair is the Hamiltonian together with an observable in the Heisenberg picture. In two dimensions, integrability requires infinitely many charges where the infinitude of Lax pairs is parameterized by a *spectral parameter* λ. The Lax pair comes from a Wilson line and a gauge field, the *Lax connection*, where flatness of the gauge connection yields the Lax equation.

Integrability of ZM theory: Let us review the integrability properties of ZM theory [41,52,62]. To begin, we define the Lax connection [41,62,68,69],

$$A_\mu = \frac{1}{1-\lambda^2}(\tilde{\partial}_\mu\boldsymbol{\phi} + \lambda\partial_\mu\boldsymbol{\phi}), \qquad (25)$$

whose corresponding field strength,

$$F_{\mu\nu} = \partial_{[\mu}A_{\nu]} + [A_\mu, A_\nu] = 0, \qquad (26)$$

vanishes for all values of the parameter λ due to the ZM equation of motion in Eq. (13). Since the Lax connection is pure gauge, we can construct the Wilson line,

$$W(x) = P\exp\left[-\int^x dx'^\mu A_\mu(x')\right]$$

$$= 1 - \int^x dx' A(x') + \int^x dx' A(x')$$

$$\times \int^{x'} dx'' A(x'') + \cdots, \qquad (27)$$

which is path independent and satisfies $D_\mu W = \partial_\mu W + A_\mu W = 0$. Next, we define the Lax current [41,52,62]

$$\boldsymbol{J}_\mu = \tilde{\partial}_\mu \boldsymbol{W} = \sum_{k=0}^{\infty} \lambda^{-k} \boldsymbol{J}_\mu^{(k)}, \qquad (28)$$

which furnishes an infinite tower of currents, including

$$\boldsymbol{J}_\mu^{(1)} = \tilde{\partial}_\mu \boldsymbol{\phi}, \qquad \boldsymbol{J}_\mu^{(2)} = \partial_\mu \boldsymbol{\phi} + \tilde{\partial}_\mu \boldsymbol{\phi}\boldsymbol{\phi}$$

$$\boldsymbol{J}_\mu^{(3)} = \tilde{\partial}_\mu \boldsymbol{\phi} + \partial_\mu \boldsymbol{\phi}\boldsymbol{\phi} + \tilde{\partial}_\mu \boldsymbol{\phi} \int^x dx' \tilde{\partial}\boldsymbol{\phi} + \tilde{\partial}_\mu \boldsymbol{\phi} \int^x dx' \partial\boldsymbol{\phi}\boldsymbol{\phi}, \qquad (29)$$

which become increasingly nonlocal at higher orders. On the support of the equations of motion in Eq. (13), these currents are conserved, so $\partial^\mu \boldsymbol{J}_\mu = \partial^\mu \boldsymbol{J}_\mu^{(k)} = 0$.

Integrability of SG theory: Applying the color-kinematics replacement in Eq. (7) to Eqs. (25) and (26) we obtain the Lax connection for SG theory,

$$A_\mu = \frac{1}{1 - \lambda^2} (\tilde{\partial}_\mu \phi + \lambda \partial_\mu \phi), \qquad (30)$$

whose corresponding field strength is also vanishing,

$$F_{\mu\nu} = \partial_{[\mu} A_{\nu]} + \partial_\rho A_\mu \tilde{\partial}^\rho A_\nu = 0. \qquad (31)$$

Meanwhile, the Wilson line maps from a color matrix to a diffeomorphism via

$$W(x) = P \exp\left[\int^x dx'^\mu \tilde{\partial}^\nu A_\mu \partial_\nu \right]$$

$$= 1 + \int^x dx'^\mu \tilde{\partial}^\nu A_\mu \partial_\nu + \int^x dx'^\mu \tilde{\partial}^\nu A_\mu \tilde{\partial}^\rho A_\nu \partial_\rho + \cdots$$

$$= \int^x dx'^\mu K_\mu^\nu \partial_\nu \quad \text{where } K_\mu^\nu = (\delta_\mu^\nu - \tilde{\partial}^\nu A_\mu)^{-1}. \qquad (32)$$

As per Eq. (28), the Lax current for the SG theory is

$$J_\mu = \epsilon_{\mu\nu} \tilde{\partial}^\rho A^\nu K_\rho^\sigma \partial_\sigma, \qquad (33)$$

which is conserved since

$$\partial^\mu J_\mu = -\tilde{\partial}^\nu (\tilde{\partial}^\rho A_\nu K_\rho^\sigma) \partial_\sigma = -\tilde{\partial}^\nu (K_\nu^\rho \tilde{\partial}^\sigma A_\rho) \partial_\sigma = 0, \qquad (34)$$

where $K_\nu^\rho \partial_\mu A_\rho = \partial_\nu A_\mu$ follows directly from Eq. (31). The series expansion of Eq. (33) yields an infinite tower of conserved currents in the SG theory which include

$$J_\mu^{(1)} = \tilde{\partial}_\mu \partial_\nu \phi \tilde{\partial}^\nu, \qquad J_\mu^{(2)} = \partial_\mu \partial_\nu \phi \tilde{\partial}^\nu + \tilde{\partial}_\mu \partial_\nu \phi \tilde{\partial}^\nu \partial_\rho \phi \tilde{\partial}^\rho, \qquad (35)$$

and can also be obtained trivially from the currents of ZM theory in Eq. (29) by applying the color-kinematic replacement rules in Eq. (7).

Nonperturbative solutions.—Equation (2) implies a *nonperturbative* map between the classical solutions of the equations of motion of BAS, ZM, and SG theory.

Since the SG theory is field-redefinition equivalent to a two-dimensional free theory [57,58,70,71], any arbitrary configuration of left- and right-moving wave packets will pass through each other unscathed even though the collision itself will be highly nonlinear and nonperturbative. Thus if we restrict to scattering on a spatial circle of circumference 2π, then the time evolution will be similarly 2π periodic. Since every classical solution of the SG theory effectively resides on a spacetime torus, it can be expressed as a double discrete Fourier transform,

$$\phi(x) = \sum_{p\in\mathbb{Z}\times\mathbb{Z}} e^{ipx} \tilde{\phi}(p) = \sum_{p\in\mathbb{Z}_N\times\mathbb{Z}_N} e^{ipx} \tilde{\phi}(p) + \mathcal{O}\left(\frac{1}{N}\right), \qquad (36)$$

where the corrections to the right-hand side are negligible as long as the field does not vary on distances shorter than $1/N$, which is always true for sufficiently large N.

We now construct a dual field configuration in ZM theory defined for $U(N)$,

$$\phi^a(x) \boldsymbol{T}_a = \sum_{p\in\mathbb{Z}_N\times\mathbb{Z}_N} e^{ipx} \tilde{\phi}(p) \boldsymbol{T}_p, \qquad (37)$$

which is literally the SG solution under the replacement $e^{ipx} \rightarrow e^{ipx} T_p$. It is straightforward to see that Eq. (37) *automatically satisfies* the ZM equations of motion in Eq. (13) up to $1/N$ corrections, since the commutator in Eq. (3) and Eq. (4) yields a color structure constant that exactly transforms the interaction vertex of ZM theory into that of SG theory. Repeating this procedure, we obtain

$$\phi^{a\bar{a}}(x) \boldsymbol{T}_a \otimes \boldsymbol{T}_{\bar{a}} = \sum_{p\in\mathbb{Z}_N\times\mathbb{Z}_N} e^{ipx} \tilde{\phi}(p) \boldsymbol{T}_p \otimes \boldsymbol{T}_p, \qquad (38)$$

which is a classical solution of BAS theory.

Remarkably, the above analytic construction can be verified *numerically*, as described in Fig. 1. Using the double copy replacement, we map a numerical solution of SG theory onto a corresponding matrix-valued field configuration of ZM theory, which is then shown to satisfy the ZM equations of motion to high precision.

Note that *every* solution of the SG theory maps to a dual solution in ZM theory but the converse is not true. This is not actually surprising given what is known from scattering: *every* gravity amplitude maps to a gauge theory amplitude with very specific color structures which are precisely chosen to be certain kinematic numerators. On the other hand, a generic gauge theory amplitude with arbitrary color structures will not have any interpretation as a gravity amplitude.

That the SG is secretly free certainly detracts from the miracle of a nonperturbative mapping in this context.
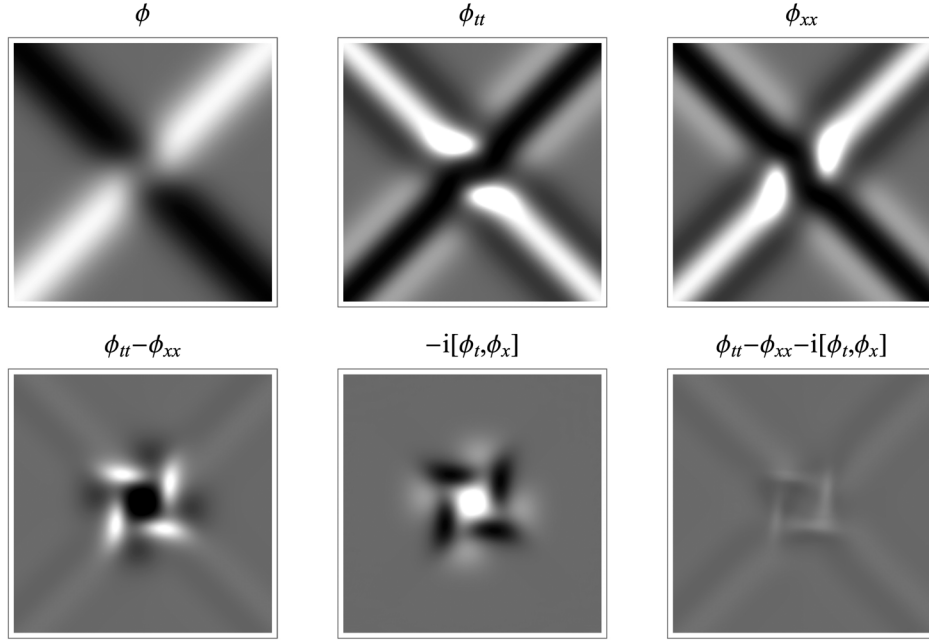
FIG. 1. We numerically solve the SG equations of motion in Eq. (19) for a pair of colliding Gaussian wave packets. The discrete Fourier transform of this solution, defined in Eq. (36), is inserted into Eq. (37) to obtain a putative matrix-valued solution of ZM theory. The above density plots characterize this ZM configuration, where the horizontal (vertical) axes denote space (time) and lighter (darker) colors denote positive (negative) field values. Each panel depicts a different matrix-valued, spacetime-dependent operator, $\mathcal{O} = \phi, \phi_{tt}, \phi_{xx}$, etc., where the subscripts denote derivatives. Each operator is visualized by plotting its projection onto a single component, $\mathrm{tr}(\mathcal{O}\boldsymbol{T}_0)$, where $\boldsymbol{T}_0 = \sum_p \boldsymbol{T}_p$. Each term in the ZM equations of motion in Eq. (13) is nonzero, but they nevertheless cancel to high precision in the final panel. These results were obtained for $U(N)$ with $N = 499$. See Ref. [72] for an animation of this scattering process.

However, recall that very general deformations of BAS and ZM theory—for example including masses or higher-dimension operators—also double copy mechanically into analogous deformations of the SG theory. Nonperturbative solutions of this much larger class of nonfree theories will also exhibit the nonperturbative double copy defined in Eqs. (36)–(38).

Generalization using the Moyal algebra.—We observed in Eq. (2) that the $N \to \infty$ limit of $U(N)$ yields the diffeomorphism algebra. What about finite N? In this case the continuum version of Eq. (4) is the Moyal algebra [73],

$$f_{p_1 p_2}{}^{p_3} = -\frac{1}{\alpha'}\sin(\alpha'\langle 12 \rangle)\delta^2(p_i + p_j - p_k), \quad (39)$$

which is the unique deformation of the Poisson algebra [74] encoding an infinite tower of higher-dimension corrections to the original kinematic structure constant in Eq. (8). Here we have defined a new coupling constant $\alpha' \sim (2\pi/N)$. At the level of fields, the generalized color-kinematic replacement rule is

$$f_{ab}{}^c V^a W^b \to \frac{1}{\alpha'}\sin(\alpha' \partial_V \tilde{\partial}_W)VW, \quad (40)$$

where the subscripts denote which fields the derivatives act upon. Under this substitution, BAS theory maps to

$$\mathcal{L}_{\mathrm{ZM},\alpha'} = \frac{1}{2}\partial_\mu \phi^a \partial^\mu \phi^a + \frac{1}{6\alpha'} f_{abc}\phi^a \sin(\alpha' \partial_{\phi^b} \tilde{\partial}_{\phi^c})\phi^b \phi^c, \quad (41)$$

a Moyal-deformed variation of ZM theory which has also appeared in the context of SDYM theory [49].

The corresponding three-point Feynman vertex is

$$\phi^a(p_1) \!\!\!\!\!\!\!\!\!\!\!\!\!\!\! \overset{\phi^b(p_2)}{\underset{\phi^c(p_3)}{<}} \quad = \quad -\frac{i}{\alpha'} f_{abc} \sin(\alpha'\langle 12 \rangle), \quad (42)$$

which is totally antisymmetric because of two-dimensional kinematics. The resulting four-point amplitude is given by Eq. (16) with the numerators

$$n_s = \frac{1}{\alpha'^2}\sin(\alpha'\langle 12 \rangle)\sin(\alpha'\langle 34 \rangle)$$

$$n_t = \frac{1}{\alpha'^2}\sin(\alpha'\langle 23 \rangle)\sin(\alpha'\langle 14 \rangle)$$

$$n_u = \frac{1}{\alpha'^2}\sin(\alpha'\langle 31 \rangle)\sin(\alpha'\langle 24 \rangle). \quad (43)$$

Remarkably, these satisfy the kinematic Jacobi identity for any value of α', so for example

$$\sin\langle 12\rangle \sin\langle 34\rangle + \sin\langle 23\rangle \sin\langle 14\rangle + \sin\langle 31\rangle \sin\langle 24\rangle = 0, \tag{44}$$

for arbitrary off-shell two-dimensional kinematics.

The generalized replacement rule in Eq. (40) can be reapplied to ZM theory to generate a deformation of SG theory that includes a fixed tower of higher-dimension corrections, analogous to the infinite tower of corrections to self-dual gravity in Ref. [49].

Future directions.—The double copy is an extremely potent computational tool but it is fundamentally unclear *why* it works. Our results mark a radical departure from the status quo of the double copy in several ways. Typical theories that admit color-kinematics duality have a single coupling constant, massless particles, square in any space-time dimension, only double copy on-shell, and all of this is only provable at tree level [3]. On the other hand we have presented an enormous class of scalar theories with arbitrary Wilson coefficients and masses that square off-shell (to all orders in perturbation theory) in two dimensions. A Lagrangian formulation coupled with an understanding of the algebra mapping also broadens the scope of the double copy to include Wilson lines, currents, and nonperturbative (non-Abelian) classical solutions.

The present Letter leaves several avenues for further inquiry. In general dimensions, the kinematic algebra for the NLSM is that of volume-preserving diffeomorphisms [24]. Generalizing this tree-level observation to the full loop-level action is an open problem. The two-dimensional results presented here suggest that this generalization may be possible, at least in principle. While we have found an enormous class of operators that double copy, it may be possible to enlarge the space even further by overcoming the restrictions (i) and (ii) given above.

Finally, it would also be interesting to apply our approach to gauge theory and gravity in two dimensions and beyond. The kinematic algebra for gauge theory [24], even at tree level, is not as well understood as for the NLSM. However, the self-dual sector of Yang-Mills theory has a simple kinematic algebra so it may be possible to systematically perturb away from the self-dual sector [21,22].

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