Null Polygons in Conformal Gauge Theory

Enrico Olivucci[1] and Pedro Vieira[1,2]
[1]*Perimeter Institute for Theoretical Physics, Waterloo, Ontario N2L 2Y5, Canada*
[2]*ICTP South American Institute for Fundamental Research, IFT-UNESP, São Paulo, SP Brazil 01440-070*

(Received 18 August 2022; accepted 28 October 2022; published 21 November 2022)

We consider correlation functions of single trace operators approaching the cusps of null polygons in a double-scaling limit where so-called *cusp times* $t_i^2 = g^2 \log x_{i-1,i}^2 \log x_{i,i+1}^2$ are held fixed and the 't Hooft coupling is small. With the help of stampedes, symbols, and educated guesses, we find that any such correlator can be uniquely fixed through a set of coupled lattice PDEs of Toda type with several intriguing novel features. These results hold for most conformal gauge theories with a large number of colors, including planar $\mathcal{N} = 4$ SYM.

DOI: 10.1103/PhysRevLett.129.221601

Introduction.—Correlation functions are the backbone of quantum field theory. In the realm of gauge-string correspondence, correlation functions of n single-trace gauge-invariant operators in the planar limit describe the scattering of n closed strings, whose world-sheet is a punctured sphere. Under special circumstances this sphere simplifies further and factorizes into a product of two disks. (Reminiscent of KLT type relations connecting open and closed strings.) This can happen in at least two ways as depicted in Fig. 1.

One option uses supersymmetry. In $\mathcal{N} = 4$ SYM say, we can take single-trace half-BPS operators made out of *many* fields and use R-symmetry polarizations in order to form a thick frame with many propagators sequentially connected drawing a closed polygon. The frame is SUSY protected from quantum corrections and thus the correlator splits into decoupled dynamics happening inside and outside the polygon [1,2].

An alternative way to decouple the inside and outside of the polygon is to make the polygon's edges approach the light cone. If we send to zero the coupling g^2 as well as the distances between consecutive cusps $x_{i,i+1}^2$ with the so-called *cusp times*

$$t_k^2 \equiv g^2 \log(x_{k-1,k}^2) \log(x_{k,k+1}^2), \tag{1}$$

held fixed, we obtain the decoupling without any requirement on the frame's thickness [3]. This is the regime we will focus on. This decoupling is dynamical: in this limit a fast classical particle travels around the null perimeter decoupling the quantum dynamics that are confined inside and outside the polygon [3–7]. The *null polygon* limit (1) is a double-scaling limit resumming the maximal power of logarithms at each perturbative order. Since it only depends on very universal one-loop features of the conformal gauge theory, most of our results (if not all) should apply to any $4d$ conformal gauge theory with a planar limit and at least one adjoint scalar [8].

The key objects of this paper are these null polygons of disk topology. We think of them as a tree level skeleton which are then dressed by quantum corrections at loop order. The skeleton is described by a set of internal *bridges* which are bundles of propagators connecting nonconsecutive cusps. For squares and pentagons we have a single choice of such bridges (see Figs. 1 and 3) while for hexagons and higher there are multiple different topologies (see for instance Fig. 6).

Belitsky and Korchemsky found a remarkable result for the null square as [9,10] $\texttt{square}_\ell = e^{-s^2}\tau_\ell(s)$, where ℓ is the number of bridges, $s^2 = \sum t_i^2$, and τ_ℓ is a solution of the Toda lattice equation

$$(s\partial_s)^2 \log \tau_\ell(s) = s^2 \tau_{\ell+1}(s)\tau_{\ell-1}(s)/\tau_\ell^2(s). \tag{2}$$

The square with no bridges is simply given by the Sudakov exponential e^{-s^2} so that $\tau_0 = 1$; the exponential can be

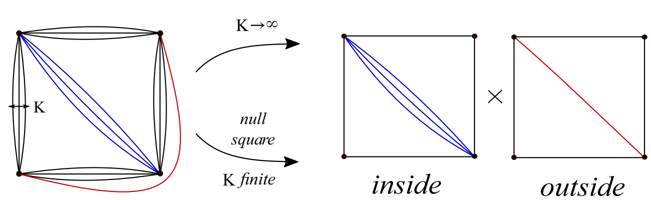

FIG. 1. Correlators with a large BPS frame (left) and doubly scaled correlators with operators approaching the cusps of a null polygon (right) both factorize into products of disk topology correlators (middle).

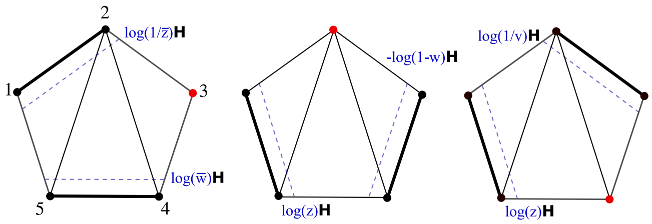

FIG. 2. Three of the five channels for the stampede computation of `pentagon`. (Reflections around the vertical axis yield the two missing channels.) Thick ed ges are null and expanded in `top` and `bot` states, then acted on by the $SL(2)$ Hamiltonian $\mathbf{H}$ which captures the associated leading log. A red dot in j corresponds to subleading cusp time $t_j^2 \to 0$.

thought of as a square null Wilson loop [4,5,9]. The square with one bridge τ_1, *the seed*, is a nontrivial function of s. Together with $\tau_0 = 1$ and the Toda equation it gives us recursively all other squares. Amusingly the seed itself can be fixed by the simple requirement that loop corrections in presence of ℓ bridges kick-in at $(\ell+1)$ loops: $\texttt{square}_\ell = 1 + O(s^{2(\ell+1)})$ uniquely fixes τ_1 to be a modified Bessel function $I_0(2s)$. All other squares then evaluate to simple determinants of Bessel functions.

In this Letter we conjecture an extension of this result and of the Toda system to any null polygon with any bridge configuration. We will encounter a novel hierarchy of Toda equations, determinantal solutions, and generalized Gross-Witten-Wadia matrix integrals [11,12]. More details about our derivations can be found in the supplemental material to this letter [13].

From stampedes to data—We first consider pentagons. If we take two nonconsecutive edges to approach a null limit there is a tool we can use to capture the corresponding leading logarithms: *the stampedes* [3]. There are five different pairs of edges we can take to be null as depicted in Fig. 2.

Consider the choice x_{12} and x_{45} for concreteness; the other choices are treated similarly. A convenient choice of coordinates for this limit is depicted in Fig. 3. In terms of these coordinates the relevant *cusp times* simply read

$$t_1^2 = g^2 \log\left(\frac{1}{\bar z}\right)\log(z), \qquad t_2^2 = g^2 \log\left(\frac{1}{\bar z}\right)\log\left(\frac{1}{v}\right),$$
$$t_4^2 = g^2 \log(\bar w)\log\left(\frac{1}{1-w}\right), \quad t_5^2 = g^2 \log(\bar w)\log(z). \qquad (3)$$

The terms $\log(z)$ and $\log(\frac{1}{1-w})$ arise once we Taylor expand the top and bottom edges in terms of top and bottom states

$$\mathcal{O}_1\mathcal{O}_2 \to \sum_{J=0}^{\infty} \frac{z^J}{J!}|\texttt{top}_J\rangle, \quad \mathcal{O}_4\mathcal{O}_5 \to \sum_{J=0}^{\infty} \frac{w^{-J}}{J!}\langle\texttt{bot}_J|, \qquad (4)$$

where

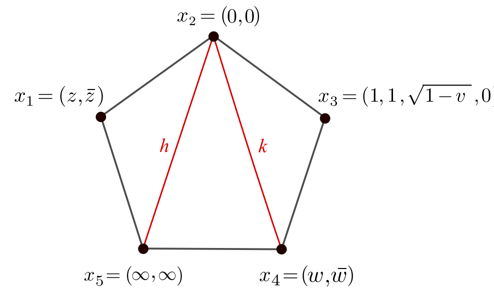

FIG. 3. To parametrize the pentagon we take four points on the plane as indicated in the figure and one point x_3 outside the plane. The five parameters match the five independent cross ratios. In the null limit $\bar z, 1/z, 1/\bar w, 1-w, v$ go to zero.

$$|\texttt{top}_J\rangle = |(\partial_z^J X)\underbrace{X\ldots X}_{h+k+1}\rangle, \quad \langle\texttt{bot}_J| = \langle\underbrace{X\ldots X}_{1+h}\partial_w^J(\underbrace{X\ldots X}_{k+1})|.$$

Here we performed already three major simplifications which we learned from [3]. First, the factorization of the polygon in the limit (1) restricts the analysis only to its interior (or, alternatively, exterior) so that `top` and `bot` states are mapped to vectors of an *open* spin chain where each site is one field that belongs to the inside of the polygon. Second, the one-loop dilation operator that generates quantum corrections is replaced by the $SL(2)$ spin chain hamiltonian $\mathbf{H}$. That is, the leading action of the dilation on the null polygon is generated only by the *hopping* of light-cone derivatives and we replaced every scalar by X. Finally, the thickness K of the frame is irrelevant in this limit where boundary-dependant terms are subleading, hence we set $K = 1$ [19].

The terms $\log(\frac{1}{\bar z})$ and $\log(\bar w)$ in (3) appear through the action of the Hamiltonians that generate the stampedes illustrated in Fig. 4

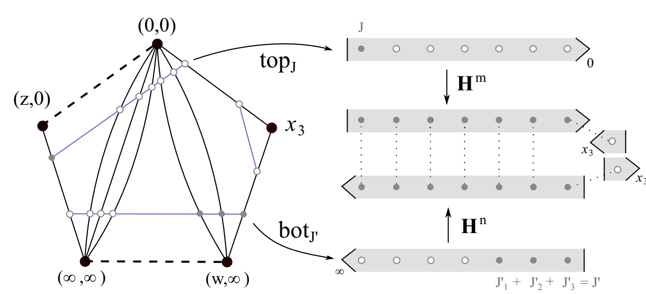

FIG. 4. The Taylor expansion of null edges (dashed) at order J/J' around $0/\infty$ creates $\texttt{top}_J/\texttt{bot}_{J'}$ states, depicted here for $h=3$, $k=2$. Each chain site is a circle: filled ones are those where derivatives are injected. The states `bot` and `top` evolve by the $SL(2)$ action at order $(g^2 \log 1/\bar z)^m$ and $(g^2 \log \bar w)^n$. The resulting states can have derivatives distributed among all sites. Overlap site by site is illustrated by dotted lines.

$$\texttt{stampede}^{h,k}_{J,J'} = \langle \texttt{bot}_{J'}| e^{g^2\log(\bar{w})\mathbf{H}}(\mathbb{I}\otimes|v\rangle\langle v|)e^{g^2\log(1/\bar{z})\mathbf{H}}|\texttt{top}_J\rangle. \quad (5)$$

Dependence on h, k is implicit on the rhs. The stampede is given by bottom and top time evolution together with the overlap with the impurity $|v\rangle\langle v|$ at the right-most site which is the sole effect of the last operator, the off the plane one. In the end we have

$$\mathcal{F}_{h,k} \equiv \sum_{J,J'=0} \frac{z^J}{w^{J'}} \texttt{stampede}^{h,k}_{J,J'}. \quad (6)$$

For example, for two equal bridges of length 1 the pentagon starts at two loops and evaluating the smallest stampedes for the lowest spins $J, J' \leq 1$ yields

$$\begin{aligned}\mathcal{F}_{1,1} &= 1 + \frac{z}{v}(-g^6\log^3\bar{z} + \ldots) + \\ &+ \frac{z}{w}(g^4\log^2\bar{z}/\bar{w} + 4g^6\log^3\bar{z}/\bar{w} + g^6\log^3\bar{z} + \ldots) \\ &+ \frac{1}{w}(-g^4\log^2\bar{w} + 4g^6\log^3\bar{w} + \ldots) + \ldots. \quad (7)\end{aligned}$$

Such expansions are very easy to generate to large orders in z and $1/w$ and $1/v$ (i.e., large spins) and to relatively large order in g^2 (we have gone to five loops, i.e., g^{10}). They are what we call our *data*.

From data to symbols to pentagons—We now need to resum (7) which is an expansion with small $z, 1/v, 1/w$ while we are after the null limit where these three variables are either finite or very large. This is easy. We found that at L loops we can express the full functions of these three variables in terms of iterated Goncharov polylogarithms of transcendentality L which can be nicely coded in terms of symbols [20]. The symbol ansatz is generated as a linear combination of *words* $a_1 \otimes a_2 \otimes \cdots \otimes a_L$ of *letters* a_i picked from the set

$$\{z, w, v, 1-z, 1-w, 1-v, w-z, v-z, v-w\}. \quad (8)$$

At two loops, for instance, the expansion (7) is nothing but the Taylor expansion of

$$\log^2\bar{w}\left[\mathrm{Li}_2(1/w) + \frac{1}{2}\log^2(1-1/w)\right] - \log^2(\bar{w}/\bar{z})\mathrm{Li}_2(z/w),$$

whose symbol is

$$\log^2\bar{w}\left[\frac{1}{1-w}\otimes\frac{1-w}{w}\right] + \log^2(\bar{w}/\bar{z})\left[\frac{z-w}{w}\otimes\frac{z}{w}\right]. \quad (9)$$

Stampedes count movements of bosons (the derivatives) in a spin chain leading to nice rational numbers with combinatorial interpretations. There are no π's or ζ's in expansions like (6) so the symbol misses no information in this case. We simply write an ansatz as in (9) with unknown rational coefficients multiplying each word, which we then fix by matching with expansions such as (6).

We can then expand the resulting functions in the null limits of interest. For example,

$$\lim_{z\to\infty}\lim_{w\to 1}\lim_{v\to 0}\mathcal{F}_{h,k} = \texttt{pentagon}_{h,k}(t_1, t_2, 0, t_4, t_5).$$

Applied to the example $h = k = 1$ we get

$$\begin{aligned}\texttt{pentagon}_{1,1}|_{t_3=0} &= 1 - \frac{1}{4}(2t_1^2t_5^2 + t_1^4 + t_4^4 + t_5^4) \\ &+ \frac{1}{36}(4t_4^6 - 3t_1^2t_2^4 - t_2^6) + \frac{1}{9}(t_1^2 + t_5^2)^3 + \ldots\end{aligned}$$

Considering different channels each time, we can generate a huge amount of null pentagons with a single cusp time set to zero. This allowed us to fix all pentagons up to a *single* constant at five loops when the first completely mixed term $t_1^2t_2^2t_3^2t_4^2t_5^2$ appears. This term can never be detected if we send any of the five cusp times to zero.

A gift from the stampedes—How can we use all these pentagons computed to high perturbative order to unveil an underlying structure leading to their exact expressions? We looked for a set of differential equations for these objects akin to the Toda equation (2) for the square. We made an educated ansatz for such equations and used the stampede data to fix their form. This leads to the main formulas of this Letter:

$$\begin{aligned}(\mathcal{D}_1 + \mathcal{D}_2)(\mathcal{D}_1 + \mathcal{D}_5)\log\mathbb{P}_{h,k} &= 4t_1^2\frac{\mathbb{P}_{h+1,k}\mathbb{P}_{h-1,k}}{\mathbb{P}_{h,k}^2}, \\ (\mathcal{D}_3 + \mathcal{D}_2)(\mathcal{D}_3 + \mathcal{D}_4)\log\mathbb{P}_{h,k} &= 4t_3^2\frac{\mathbb{P}_{h,k+1}\mathbb{P}_{h,k-1}}{\mathbb{P}_{h,k}^2}, \quad (10)\end{aligned}$$

where $\mathcal{D}_j = t_j\partial/\partial t_j$ and

$$\texttt{pentagon}_{h,k} = e^{-t_1^2-\cdots-t_5^2}\times\mathbb{P}_{h,k}(t_j), \quad \mathbb{P}_{0,0} = 1. \quad (11)$$

A second crucial piece of information from the stampede are the reductions of $\texttt{pentagon}_{h,k}$ when only specific cusp times are switched on. In particular, when the times t_1, t_3 are set to zero equations (10) imply factorization. Setting most t_i to zero degenerates the pentagon into squares so we expect these functions to be given by squares. We find precisely such square reductions as

$$\mathbb{P}_{h,k}(0, t_2, 0, t_4, t_5) = \tau_h(t_5)\tau_{h+k}(t_2)\tau_k(t_4), \quad (12)$$

where each τ function has an index given by the total number of bridges emitted by the ith cusp.

Henceforth we take the stampedes gift (10), (12) as our starting point and explain how these equations fix *all* polygons at *any* loop order.

Toda bootstrap.—Under the assumption that $\mathbb{P}_{h,k}$ has a regular Taylor expansion around $t_j^2 = 0$

$$\mathbb{P}_{h,k} = \sum_{n_1 \ldots n_5} \mathbb{c}_{h,k}(n_1, \ldots, n_5) \times t_1^{2n_1} \cdots t_5^{2n_5}, \tag{13}$$

the first of (10) can be used to express any coefficient $\mathbb{c}(n_1, \ldots, n_5)$ as a function of coefficients $\mathbb{c}_{h,k}, \mathbb{c}_{h\pm1,k}$ with lower value of n_1, due to the factor $4t_1^2$ in the r.h.s. of the equation. Likewise, the second equation in (10) can be used to express each $\mathbb{c}_{h,k}$ via coefficients with lower n_3. The iterative application of the two equations eventually reduces the problem to the determination of

$$\mathbb{c}_{h,k}(0, n_2, 0, n_4, n_5), \quad \forall \; h, k \in \mathbb{N}.$$

But these coefficients are precisely those determined by the reduction (12) so we are done.

Pertinent for higher polygon generalizations, an important observation is that $\mathbb{P}_{h,0}$ only depends on times which are either end points of the bridges (t_2, t_5) or sandwiched between two such points (t_1); the two times t_3 and t_4 do not show up. For the first such pentagon we managed to identify its exact resummation as $\mathbb{P}_{1,0} = \phi(t_1, t_2, t_5)$ and $\mathbb{P}_{0,1} = \phi(t_3, t_2, t_4)$, where

$$\phi(x, y, z) = \sum_{n=0}^{\infty} \sum_{m}^{n} \sum_{r}^{n-m} \frac{x^{2(n-m-r)} y^{2m} z^{2r}}{(n-m)!(n-r)!m!r!}. \tag{14}$$

Any solution $\mathbb{P}_{h,k}$ with $h, k \geq 1$ can be cast in the compact form of a nested determinant

$$\mathbb{P}_{h,k} = \frac{|(\mathcal{D}_1 + \mathcal{D}_2)^{i-1}(\mathcal{D}_1 + \mathcal{D}_5)^{j-1}\mathbb{P}_{1,k}|_{i,j\leq h}}{(2t_1)^{h-h^2}\mathbb{P}_{0,k}^{h-1}},$$
$$\mathbb{P}_{1,k} = \frac{|(\mathcal{D}_3 + \mathcal{D}_2)^{i-1}(\mathcal{D}_3 + \mathcal{D}_4)^{j-1}\mathbb{P}_{1,1}|_{i,j\leq k}}{(2t_3)^{k-k^2}\mathbb{P}_{1,0}^{k-1}}. \tag{15}$$

This representation follows iteratively from (10) and from the fact that $\mathbb{P}_{0,k}$ and $\mathbb{P}_{h,0}$ are functions of three cusp times only. Formulas (15) make manifest that everything can be reduced to *seed* functions $\mathbb{P}_{0,1}$ and $\mathbb{P}_{1,1}$. The former is given by (14) while the latter can be bootstrapped by the self-consistency argument presented above.

Higher polygons—Based on the experience collected on `pentagon` functions, we can readily conjecture the form of any null polygon. We express a polygon with n edges as the product of an exponential of cusp times—reminiscent of the Sudakov term in a null polygonal Wilson loop—and a function $\mathbb{H}$ that depends on the configuration of bridges

$$\texttt{polygon} = e^{-\sum_i t_i^2} \times \mathbb{H}(t_1, \ldots, t_n). \tag{16}$$

When there are no bridges, as, for instance, for correlators of short operators, we known from [6] that $\mathbb{H}$ is one.

Let ℓ be the number of bridges stretching between t_{i-1}, t_{i+1}, we denote $\hat{t}_i$ the cusp time comprised between the end points. We focus on this bridge and write $\mathbb{H}_\ell$ leaving other bridge indices implicit. We can state that (i) there is a Toda equation of type (10) in $t_{i-1}, \hat{t}_i, t_{i+1}$ relating $\mathbb{H}_\ell$ to $\mathbb{H}_{\ell\pm1}$. When $\hat{t}_i = 0$ it becomes a factorization condition $\mathcal{D}_{i-1}\mathcal{D}_{i+1}\mathbb{H}_\ell = 0$. (ii) the function $\mathbb{H}$ depends only on cusp times that emit bridges (e.g., t_{i-1}, t_{i+1}) or are sandwiched between the two end points of a bridge (e.g., $\hat{t}_i$). (iii) when $\hat{t}_j = 0$, the function $\mathbb{H}$ factorizes into a product of squares $\tau_n(t_k)$, one for each cusp k emitting a total of n bridges.

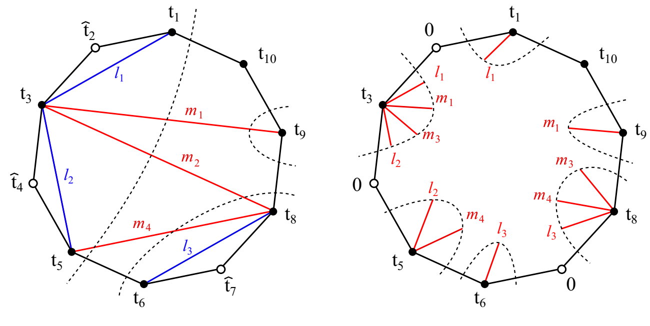

FIG. 5. A decagon example with blue bridges, stretching between next-to-neighboring points, and red bridges. Dashed lines cut the null polygon into factorized contributions, each subject to a set of coupled Toda equations. On the right we depicted the boundary data at $t_j = 0$, which fixes the solution.

This set of statements traces back to `pentagon` observations. For higher polygons the novel feature is the presence of both blue bridges in Fig. 5—connecting next-to-nearest neighbouring cusps $t_{i\pm1}$ and governed by a $2d$ Toda equation—and the red bridges which connect further apart cusps. To complete the conjecture we add (i) The polygon $\mathbb{H}$ is factorized into the product of functions $\mathbb{X}$ of those cusps t_i and $\hat{t}_i$ comprised between the end points of a continuous path of blue bridges. (In Fig. 5, for instance, we have three such blue paths of lengths 2, 1 and 0 and thus the correlator would factor into three factors.) Paths of length 0 correspond to $\text{square}_n(t_j)$ where n is the total number of bridges emitted by the cusp t_j.

Let us apply these conjectures to the general `hexagon`. A planar hexagon can have three different bridge configurations as depicted in Fig. 6. Cases (a) and (b) in the figure feature two blue bridges, connected and disconnected respectively. Hence they are subject to two $2d$ Toda equations as does `pentagon` to which they are closely related. Case (a) is factorized as

$$\mathbb{H}_{h,l,k}^{(a)} = \mathbb{X}_{h,l,k}(t_1, \hat{t}_2, t_3, \hat{t}_4, t_5)\tau_l(t_6).$$

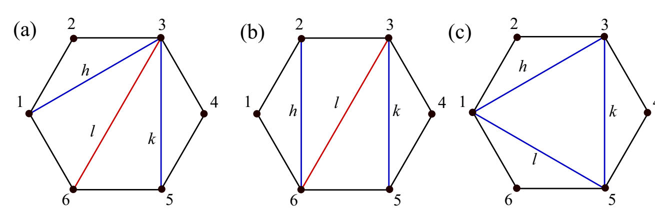

FIG. 6. The three possible configurations of bridges for the `hexagon` in the planar limit. Blue bridges are associated to $2d$ Toda equations of type (10).

The function $\mathbb{X}$ solves two coupled equations in $(t_1, \hat{t}_2, t_3)$ and $(t_3, \hat{t}_4, t_5)$ moving h and k. (Notice that for $l = 0$ the same holds for pentagon, which means $\mathbb{X}_{h,0,k} = \mathbb{P}_{h,k}$.) Case (b) features two disconnected blue bridges, i.e., it solves decoupled equations and is thus factorized into two functions of three cusps and expressed via pentagon functions already determined in this Letter,

$$\mathbb{H}^{(b)}_{h,l,k} = \mathbb{P}_{l,h}(0, t_6, \hat{t}_1, t_2, 0)\mathbb{P}_{l,k}(0, t_3, \hat{t}_4, t_5, 0).$$

Case (c) features three connected blue bridges and thus is solution to three pairwise coupled $2d$ Toda equations in $(t_1, \hat{t}_2, t_3)$, $(t_3, \hat{t}_4, t_5)$ and $(t_5, \hat{t}_6, t_1)$ moving h, k and l, respectively. The boundary data for the three cases are

$$\mathbb{H}^{(a)}_{h,k,l}(t_1, 0, t_3, 0, t_5, t_6) = \tau_h(t_1)\tau_{h+k+l}(t_3)\tau_k(t_5)\tau_l(t_6),$$
$$\mathbb{H}^{(b)}_{h,k,l}(0, t_2, t_3, 0, t_5, t_6) = \tau_h(t_2)\tau_{h+l}(t_6)\tau_{k+l}(t_3)\tau_k(t_5),$$
$$\mathbb{H}^{(c)}_{h,k,l}(t_1, 0, t_3, 0, t_5, 0) = \tau_{h+l}(t_1)\tau_{h+k}(t_3)\tau_{k+l}(t_5).$$

We tested these various statements against stampede data at three loops for a few six-point correlators [21].

Discussion—In this Letter we used the stampede technology developed in [3] to unveil new integrable structures in the null limits of conformal gauge theories. The final picture is quite compact. In the double-scaling limit (1), correlators are characterized by internal bridges which can be blue (if they connect next-to-neighboring cusps) or red (if they connect farther separated vertices). With zero blue bridges the correlators factorize into products of square functions (the boundary data); finite blue lengths can then be constructed recursively by means of Toda equations (10) [22].

Some of the solutions to these PDEs are rather novel. If we take a five point correlator with two nonzero cusp times; for instance, we find

$$\mathbb{P}_{h,k}(t_1, t_2, 0, 0, 0) = \det_{1\le i,j\le h+k} |z_i^{i-j} I_{i-j}(2z_i)|,$$

where I_j are modified Bessel functions of the first kind and $z_i = \sqrt{t_1^2 + t_2^2}$ if $i \le h$ and $z_i = t_2$ for $i > h$. It admits a representation as an exotic matrix integral

$$\int \frac{d\theta_1}{2\pi} \cdots \int \frac{d\theta_\ell}{2\pi} \Delta(Z) e^{\mathrm{Tr}(Z+Z^\dagger)} \prod_k (z_k e^{i\theta_k})^{1-k},$$

where Z is a matrix with eigenvalues $\{z_j e^{i\theta_j}\}$ and Δ is the Vandermonde determinant. When $x_i = y$ this integral reduces to the Gross-Witten-Wadia matrix model [11]. What is the physical origin of these matrix integrals?

It would be fascinating to connect the structures found here to other exact approaches to quantum gauge theories. We can envisage three natural connections.

A first connection would be with conformal bootstrap explorations. In [5–7,23] the conformal bootstrap was applied to the study of null correlation functions. There, no simplifying double-scaling limit was taken. On the other hand the analysis was restricted to correlation functions of the lightest fields of the theory which translates into polygons with no internal bridges. Indeed, if we take the Sudakov and recoil factors bootstrapped there, in the limit (1) we would obtain the universal exponentials

$$e^{-t_1^2-\cdots-t_n^2}, \tag{17}$$

showing up everywhere in our Letter. When the bridges are nonzero, dressing these exponentials we have the very nontrivial solutions to the Toda hierarchy. Would be remarkable to fit them inside a bootstrap analysis. Combining the technology developed here with a bootstrap approach might lead to a more general null limit for any polygon. The physical picture should be one where in the null limit correlators become Wilson loops dressed by further cusp insertions of adjoint fields. The latter then propagate from cusp to cusp interacting with the planar chromodynamic flux tube of the theory.

We computed the hexagon for any bridge topology as depicted in Fig. 6; from an OPE perspective the last one resembles a snowflake while the other two mimic the structure of comb channel decompositions. Despite some encouraging first steps [24,25], there is no analytic result for comb channels for $n > 4$ points. Can one use our hexagon results to improve this state of affairs?

A second connection would be with integrability. In planar $\mathcal{N} = 4$ SYM four-point disk correlators were computed for any kinematics in [1,2] by means of bootstrap plus hexagonalization [26–29] and preliminary higher polygons explorations were started in [30,31]. Would be very interesting to connect our results with these. In the hexagonalization formalism, for instance, the limit (1) is governed by an interesting region where all rapidities and bound-state numbers of the mirror particles are (analytically continued to) zero. The square was identified with an amplitude in the Coulomb branch [32]; are the doubly scaled higher polygons found here higher-point scattering amplitudes in some interesting kinematics?

A last connection could be with supersymmetric localization. Determinants of Bessel functions and Toda equations constantly pop up in such supersymmetric studies, see, e.g., Refs. [33–35]—is this a coincidence?

Finally, can we fix the so-called pentagon at all loops following a bootstrap *à la* [1,3] for the square (called octagon there)? A strategy would be to construct a symbol ansatz for five point functions of large operators. (We already know some of its letters from (8) arising from the stampede analysis here.) Full null polygon limits will not be enough to fix the ansatz but single light-cone limits—which are still fully captured by the stampedes—might suffice once combined with simple additional physical conditions such as Steinmann relations and single valuedness in the physical sheet.

We thank B. Basso, A. V. Belitski, C. Bercini, V. Goncalves, A. Homrich, S. Komatsu, and especially G. P. Korchemsky for discussions and valuable comments on our draft. Research at the Perimeter Institute is supported in part by the Government of Canada through NSERC and by the Province of Ontario through MRI. This work was additionally supported by a grant from the Simons Foundation (Simons Collaboration on the Nonperturbative Bootstrap #488661) and ICTP-SAIFR FAPESP Grant No. 2016/01343-7 and FAPESP Grant No. 2017/03303-1.

[1] F. Coronado, Perturbative four-point functions in planar $\mathcal{N}=4$ SYM from hexagonalization, J. High Energy Phys. 01 (2019) 056.

[2] F. Coronado, Bootstrapping the Simplest Correlator in Planar $\mathcal{N}=4$ Supersymmetric Yang-Mills Theory to All Loops, Phys. Rev. Lett. **124**, 171601 (2020).

[3] E. Olivucci and P. Vieira, Stampedes I: Fishnet OPE and octagon bootstrap with nonzero bridges, J. High Energy Phys. 07 (2022) 017.

[4] L. F. Alday, B. Eden, G. P. Korchemsky, J. Maldacena, and E. Sokatchev, From correlation functions to Wilson loops, J. High Energy Phys. 09 (2011) 123.

[5] L. F. Alday and A. Bissi, Higher-spin correlators, J. High Energy Phys. 10 (2013) 202.

[6] P. Vieira, V. Goncalves, and C. Bercini, Multipoint Bootstrap I: Light-Cone Snowflake OPE and the WL Origin, Phys. Rev. Lett. **126**, 121603 (2021).

[7] C. Bercini, V. Gonçalves, A. Homrich, and P. Vieira, The Wilson loop—Large spin OPE dictionary, J. High Energy Phys. 07 (2022) 079.

[8] It should even hold for non-conformal gauge theories provided the one-loop beta function vanishes. We thank G. P. Korchemsky for pointing this out.

[9] A. V. Belitsky and G. P. Korchemsky, Exact null octagon, J. High Energy Phys. 05 (2020) 070.

[10] A. V. Belitsky and G. P. Korchemsky, Crossing bridges with strong Szego limit theorem, J. High Energy Phys. 04 (2021) 257.

[11] D. J. Gross and E. Witten, Possible third order phase transition in the large N lattice gauge theory, Phys. Rev. D **21**, 446 (1980); S. R. Wadia, $N=$ infinity phase transition in a class of exactly soluble model lattice gauge theories, Phys. Lett. **93B**, 403 (1980).

[12] E. Brezin and D. J. Gross, The external field problem in the large N Limit of QCD, Phys. Lett. **97B**, 120 (1980).

[13] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.129.221601 for an explicit example of five-point correlator (section A); generalization of the spin chain computations (section B);details about symbol ansatz (section C);data up to five-loops for the function $\texttt{pentagon}_{1,1}$ (section D); details about the matrix models mentioned in this letter's introduction (section E). Supplemental material includes additional Refs. [14–18].

[14] N. Beisert, The complete one-loop dilatation operator of $N=4$ super-Yang–Mills theory, Nucl. Phys. **B676**, 3 (2004).

[15] A. Sever, P. Vieira, and T. Wang, From polygon Wilson loops to spin chains and back, J. High Energy Phys. 12 (2012) 065.

[16] D. Gaiotto, J. Maldacena, A. Sever, and P. Vieira, Pulling the straps of polygons, J. High Energy Phys. 12 (2011) 011.

[17] I. M. Gessel and D. Zeilberger, Random walk in a Weyl chamber, Proc. Am. Math. Soc. **115**, 27 (1992).

[18] P. van Moerbeke, Integrable lattices: Random matrices and random permutations, arXiv:0010135.

[19] These three simplifications hold as we take the full null limit of the pentagon. They can easily be relaxed following [3] to describe a more general null limit where only $\bar{w}$ and $1/\bar{z}$ are sent to zero as discussed in the conclusions.

[20] A. B. Goncharov, M. Spradlin, C. Vergu, and A. Volovich, Classical Polylogarithms for Amplitudes and Wilson Loops, Phys. Rev. Lett. **105**, 151605 (2010).

[21] Note that for even sided polygons there is no subleading cusp time in the stampede, as each cusp is extremal of an edge which is expanded in light-like direction; in this sense, the hexagon checks are even cleaner than the pentagon ones.

[22] Taking for instance the equations on the first line of (10), the change of variables $t_j = e^{s_j}$ and $\mathbb{P}_{h+1,k}/\mathbb{P}_{h,k} = \exp(\mathbf{p}_{h,k} - 2s_1 h)$ allows to rewrite them as

$$\partial_x \partial_y \mathbf{p}_{h,k} = \exp(\mathbf{p}_{h+1,k} - \mathbf{p}_{h,k}) - \exp(\mathbf{p}_{h,k} - \mathbf{p}_{h-1,k}),$$

where $x = 2(s_1 + s_2)$ and $y = 2(s_1 + s_5)$ are the Toda times. We thank G. P. Korchemsky for the hint.

[23] A. Antunes, M. S. Costa, V. Goncalves, and J. V. Boas, Lightcone bootstrap at higher points, J. High Energy Phys. 03 (2022) 139.

[24] V. Gonçalves, R. Pereira, and X. Zhou, 20′ five-point function from $AdS_5 \times S^5$ supergravity, J. High Energy Phys. 10 (2019) 247.

[25] V. Rosenhaus, Multipoint conformal blocks in the comb channel, J. High Energy Phys. 02 (2019) 142; J. F. Fortin, W. Ma, and W. Skiba, Higher-point conformal blocks in the comb channel, J. High Energy Phys. 07 (2020) 213; S. Parikh, A multipoint conformal block chain in d dimensions, J. High Energy Phys. 05 (2020) 120; I. Buric, S. Lacroix, J. A. Mann, L. Quintavalle, and V. Schomerus, Gaudin models and multipoint conformal blocks: General theory, J. High Energy Phys. 10 (2021) 139; J. High Energy Phys. 11 (2021) 182; arXiv:2112.10827.

[26] B. Basso, S. Komatsu, and P. Vieira, Structure constants and integrable bootstrap in planar $N=4$ SYM theory, arXiv: 1505.06745.

[27] T. Fleury and S. Komatsu, Hexagonalization of correlation functions, J. High Energy Phys. 01 (2017) 130.

[28] B. Eden and A. Sfondrini, Three-point functions in $\mathcal{N}=4$ SYM: The hexagon proposal at three loops, J. High Energy Phys. 02 (2016) 165.

[29] I. Kostov, V. B. Petkova, and D. Serban, Determinant Formula for the Octagon Form Factor in $N=4$ Supersymmetric Yang-Mills Theory, Phys. Rev. Lett. **122**, 231601 (2019); The octagon as a determinant, J. High Energy Phys. 11 (2019) 178; I. Kostov and V. B. Petkova, Octagon with finite bridge: Free fermions and determinant identities, J. High Energy Phys. 06 (2021) 098.

[30] T. Fleury and S. Komatsu, Hexagonalization of correlation functions II: Two-particle contributions, J. High Energy Phys. 02 (2018) 177.

[31] T. Fleury and V. Goncalves, Decagon at two loops, J. High Energy Phys. 07 (2020) 030.

[32] S. Caron-Huot and F. Coronado, Ten dimensional symmetry of $N = 4$ SYM correlators, J. High Energy Phys. 03 (2022) 151.

[33] S. Giombi and S. Komatsu, Exact correlators on the Wilson loop in $\mathcal{N} = 4$ SYM: Localization, defect CFT, and integrability, J. High Energy Phys. 05 (2018) 109; 11 (2018) 123(E).

[34] N. Gromov and A. Sever, Analytic solution of Bremsstrahlung TBA, J. High Energy Phys. 11 (2012) 075.

[35] E. Gerchkovitz, J. Gomis, N. Ishtiaque, A. Karasik, Z. Komargodski, and S. S. Pufu, Correlation functions of Coulomb branch operators, J. High Energy Phys. 01 (2017) 103.