Critical Probability Distributions of the Order Parameter from the Functional Renormalization Group

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We show that the functional renormalization group (FRG) allows for the calculation of the probability distribution function of the sum of strongly correlated random variables. On the example of the three-dimensional Ising model at criticality and using the simplest implementation of the FRG, we compute the probability distribution functions of the order parameter or, equivalently, its logarithm, called the rate functions in large deviation theory. We compute the entire family of universal scaling functions, obtained in the limit where the system size L and the correlation length of the infinite system ξ_∞ diverge, with the ratio $\zeta = L/\xi_\infty$ held fixed. It compares very accurately with numerical simulations.

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In many different fields of research, mathematicians, physicists, and even specialists of quantitative finance have paid considerable attention to the probability distribution of the sums of random variables. Here the central limit theorem (CLT) plays a crucial role [1,2]. It asserts that, given a large number N of independent identically distributed random variables $\hat{\sigma}_i$ with zero mean and finite variance, their sum $\hat{S} = \sum_i \hat{\sigma}_i$ has fluctuations of order $\sqrt{N}$, and the asymptotic probability distribution function (PDF) of $\hat{S}/\sqrt{N}$ is a Gaussian law with finite variance. Most importantly, this result is independent of the probability law of the $\hat{\sigma}$'s, and the normal distribution plays the role of an attractor for the addition of an increasing number of random variables. The Gaussian distribution is therefore said to be stable and this is the most basic manifestation of what physicists call universality. The CLT has been generalized to the case where either the mean or the variance of the σ_i-law diverges: In this case, once it has been properly normalized, $\hat{S}$ is distributed according to one of the celebrated Lévy-stable laws [3–5] that generalize the Gaussian law of the CLT.

The CLT can also be generalized to situations where the $\hat{\sigma}_i$ are correlated [2,6]. If the correlation matrix G_{ij} decays sufficiently fast with a given "distance" r_{ij} between $\hat{\sigma}_i$ and $\hat{\sigma}_j$, such that $\sum_i G_{ij}$ is finite in the limit $N \to \infty$, the correlations are said to be weak. Then, the system behaves as if it were made of uncorrelated finite size clusters of $\hat{\sigma}_i$ and $\hat{S}$ still has fluctuations of order $\sqrt{N}$. The CLT applies again, and the distribution of $\hat{S}/\sqrt{N}$ is still Gaussian.

On the other hand, when $\sum_i G_{ij}$ diverges as $N \to \infty$, the fluctuations of $\hat{S}/\sqrt{N}$ also diverge and the $\hat{\sigma}_i$ are said to be strongly correlated. Such situations are encountered in many different contexts, from critical systems to out-of-equilibrium dynamics such as disease propagation, surface growth, or turbulence. Our understanding of stable laws is much scarcer in this case. Nevertheless, it is reasonable to assume that properly normalized, $\hat{S}/f(N)$ should here again follow a stable law. Assuming universality, these laws, which are neither Gaussian nor Lévy, should depend only on a small number of parameters, such as the dimension of the system and its symmetries. These stable laws for strongly correlated variables have been observed experimentally or estimated numerically with relative ease [7–20]. On the theoretical side, a few exact results have been obtained in some specific models [21–29]. In generic models, the connections between CLT, stable laws, and the fixed points of the renormalization group (RG) have been identified [30–32] since the early days of the Kadanoff-Wilson version of the RG [33]. However, it appears that these connections have remained at the conceptual level and have not been transformed into a set of techniques for calculating PDFs applicable to strongly correlated systems, but in isolated cases with *ad hoc* methods [34–40]. Furthermore, the connection between RG and CLTs raises two paradoxes: (1) the PDF—being an observable—is RG-scheme independent, whereas fixed points are not; (2) as discussed below, there is a family of critical rate functions, indexed by a real number ζ, but only one RG fixed point. We show here that the functional RG (FRG) in its modern version [41,42] is the right framework to solve these paradoxes and compute quantitatively the PDF of strongly correlated random variables.

Let us briefly review the concepts fleshed out above in the context of the Ising model in the vicinity of its second order phase transition, on which we will focus from now on. The Hamiltonian of the ferromagnetic Ising model is

$H = -J\sum_{\langle ij\rangle}\hat{\sigma}_i\hat{\sigma}_j$ with $J > 0$, $\hat{\sigma}_i = \pm 1$, and $\langle ij\rangle$ label nearest neighbor sites on a hypercubic d-dimensional lattice of linear size L with periodic boundary conditions. A second order phase transition occurs in the Ising model at some finite temperature T_c in $d > 1$ (we focus on the non-mean-field case $d < 4$). At fixed temperature $T \gtrsim T_c$ and for $r_{ij} \gg a$, where a is the lattice spacing, the correlation function of the spins behaves as $G_{ij} \sim r_{ij}^{-d+2-\eta}\exp(-r_{ij}/\xi_\infty)$, where $\eta \geq 0$ is the anomalous dimension of the spin field, and ξ_∞ is the correlation length of the infinite system (at zero magnetic field), which diverges at the transition as $|t|^{-\nu}$, $t = T - T_c$. The condition of weak correlations is thus equivalent to the finiteness of ξ_∞.

We are interested below in the PDF of the normalized total spin defined as $\hat{s} = L^{-d}\sum_i\hat{\sigma}_i$, the average of which is the magnetization $m = \langle\hat{s}\rangle$. The fluctuations of $\hat{s}$ are measured by $\langle\hat{s}^2\rangle$: $\langle\hat{s}^2\rangle = L^{-d}\chi$, defining the magnetic susceptibility χ. For fixed $T \gtrsim T_c$, $\chi \sim \xi_\infty^{2-\eta}$ independent of L for $L \gg \xi_\infty$. This implies that the fluctuations of $\hat{S}/\sqrt{N}$ are of order one: The system is weakly correlated. A precise calculation of the PDF is obtained from a saddle point approximation that becomes asymptotically exact when $L \to \infty$. As expected, it shows that the CLT holds and that the PDF becomes indeed Gaussian in this limit: $P(\hat{s} = s) \propto \exp[-(L^d/2\chi)s^2]$ for $T > T_c$ and $L \to \infty$ (at fixed $sL^{d/2}$)] [43].

The argument above collapses at T_c and fixed L, where $\xi_\infty \gg L$, because χ scales with L as $\chi \sim \int^L d^d r\, r^{-d+2-\eta} \sim L^{2-\eta}$, which diverges when $L \to \infty$. This implies that $\langle\hat{s}^2\rangle \sim L^{-d+2-\eta}$ and that the fluctuations of $\hat{S}/\sqrt{N}$ diverge as $L^{(2-\eta)/2} = N^{(2-\eta)/2d}$. The spins are strongly correlated and the standard CLT no longer holds: The saddle point approximation fails and P has no reason to be a Gaussian anymore. However, the scaling of the fluctuations of $\hat{s}$ suggest that P is a universal function of the scaling variable $\tilde{s} = sL^{(d-2+\eta)/2}$ [44].

It is rarely stressed that there is not only one PDF at criticality, but an infinity corresponding to the inequivalent ways to take the limit $L \to \infty$ and $T \to T_c^+$, i.e., $\xi_\infty \to \infty$, see Fig. 1 [45]. Indeed, choose any sequence $T_L > T_c$ converging to T_c, such that $\zeta = L/\xi_\infty(T_L)$ is constant. Then, for instance, if $\xi_\infty(T_L) \ll L$ and from the discussion above, $\langle\hat{s}^2\rangle \sim L^{-d}\chi \sim L^{-d+2-\eta}\zeta^{\eta-2}$. Once again, and even though $\xi_\infty(T_L)$ is finite at any L, the spins become more and more strongly correlated as $T_L \to T_c$. Therefore, the PDF must be nontrivial for all values of ζ even in the limit $\zeta \gg 1$ [i.e., $\xi_\infty(T_L) \ll L$]. In this limit, we expect to recover some Gaussian-like features for typical values of s because the system looks for all $T_L > T_c$ as a collection of uncorrelated small blocks of spins of sizes $\xi_\infty(T_L)$. However, some non-Gaussianity should remain in the tails of the PDF reminiscent of the strong correlations present at criticality where χ is diverging.

Assuming scaling, the PDF must depend on ξ_∞ and L only through the ratio ζ, which we parametrize as

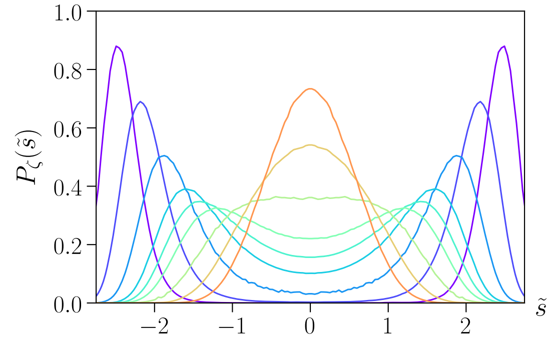

FIG. 1. Different critical PDFs of the 3D Ising model as functions of $\tilde{s} = L^{(d-2+\eta)/2}s$, obtained from Monte Carlo simulations with periodic boundary conditions with $L = 128$ for various $\zeta = \mathrm{sgn}(T - T_c)L/\xi_\infty(|T - T_c|)$, with ζ between -4 and 4 by step of one (from bottom curve to top curve at the center).

$$P_\zeta(\hat{s} = s) \approx e^{-L^d I(s,\xi_\infty,L)} \approx e^{-I_\zeta(\tilde{s})}. \tag{1}$$

This relation defines the rate function $I(s, \xi_\infty, L)$, as it is known in large deviation theory (also known as the "constraint effective potential" in quantum field theory [46–48]), as well as its scaling function $I_\zeta(\tilde{s})$. Notice that we could define as well a family of universal critical PDFs when coming from the low-temperature phase, $T \to T_c^-$. To tackle both cases at once, we define $\zeta = \mathrm{sgn}(t)L/\xi_\infty(|t|)$. We show in Fig. 1 some of these PDFs obtained numerically (see below) in $d = 3$ with periodic boundary conditions.

These probabilistic arguments do not allow for computing $I_\zeta(\tilde{s})$. In the following, we show that the FRG yields a general formalism for such calculations. Being interested in universal PDFs, we replace the lattice Ising model by a $\mathbb{Z}_2$-invariant field theory for which $\hat{s} = L^{-d}\int_{\mathbf{x}}\hat{\phi}(\mathbf{x})$ and thus,

$$P(\hat{s} = s) = \mathcal{N}\int \mathcal{D}\hat{\phi}\,\delta(s - \hat{s})\exp(-\mathcal{H}[\hat{\phi}]), \tag{2}$$

with $\mathcal{N}$ a normalization factor. Noting that the delta function can be replaced by a infinitely peaked Gaussian, $\delta(z) \propto \exp(-M^2z^2/2)$ with $M^2 \to \infty$, the PDF can be interpreted as the partition function $\mathcal{Z}_M$ of a system with Hamiltonian $\mathcal{H}_M[\hat{\phi}] = \mathcal{H}[\hat{\phi}] + M^2/2(\int_{\mathbf{x}}[\hat{\phi}(\mathbf{x}) - s])^2$, that is, $P(s) \propto \lim_{M\to\infty}\mathcal{Z}_M$. Note that $M = 0$ corresponds to the standard partition function $\mathcal{Z}$ of the model (at finite size L).

For a critical theory, the computation of $\mathcal{Z}_M$ is plagued with the singularities induced by the long-distance (small-wavelength) fluctuations. The modern version of the FRG, tailored to deal with this difficulty [49–51], consists of freezing out these modes in the partition function while leaving unchanged the others and by gradually decreasing to zero the scale k that separates the low and high wave number modes: This generates the RG flow of partition functions or, equivalently, of Hamiltonians.

A one-parameter family of models with partition functions $\mathcal{Z}_{M,k}[h]$ is thus built by changing the original Hamiltonian $\mathcal{H}_M$ into $\mathcal{H}_M + \Delta\mathcal{H}_k - h.\hat{\phi}$, where h is a magnetic field (or source) and the dot in $h.\hat{\phi}$ implies an integral over space or momentum: $\mathcal{Z}_{M,k}[h] = \int \mathcal{D}\hat{\phi}\, \exp(-\mathcal{H}_M - \Delta\mathcal{H}_k + h.\hat{\phi})$. Here, $\Delta\mathcal{H}_k$ is the term designed to effectively freeze the low wave number fluctuations $\hat{\phi}(|\mathbf{q}| < k)$ while leaving unchanged the high wave number modes $\hat{\phi}(|\mathbf{q}| > k)$. It is chosen to be quadratic: $\Delta\mathcal{H}_k = 1/2\hat{\phi}.R_k.\hat{\phi}$ with $R_k(\mathbf{x},\mathbf{y})$ such that (i) when $k \sim a^{-1}$, $R_{k\sim a^{-1}}(|\mathbf{q}|)$ is very large for all $|\mathbf{q}|$, which implies that all fluctuations are frozen, and (ii) $R_{k=0}(|\mathbf{q}|) \equiv 0$ so that all fluctuations are integrated over and $\mathcal{Z}_{M,k=0}[h] = \mathcal{Z}_M[h]$. Varying the scale k between a^{-1} and zero induces the RG flow of $\mathcal{Z}_{M,k}[h]$, in which fluctuations of wave numbers $|\mathbf{q}| > k$ are progressively integrated over.

Actual calculations of $\mathcal{Z}_{M,k}[h]$ require one to perform approximations that are known to be controlled only when working with the (slightly modified) Legendre transform of $\log \mathcal{Z}_{M,k}[h]$ with respect to h, $\Gamma_{M,k}[\phi]$ [52–54], defined as

$$\Gamma_{M,k}[\phi] = -\ln \mathcal{Z}_{M,k}[h] + h.\phi - \frac{1}{2}\phi.R_k.\phi - \frac{M^2}{2}\{\phi - s\}^2, \tag{3}$$

where $\{\phi - s\}^2 \equiv (\int_{\mathbf{x}}[\phi(\mathbf{x}) - s])^2$ and $\phi(\mathbf{x}) = \langle\hat{\phi}(\mathbf{x})\rangle = \delta\mathcal{Z}_{M,k}/\delta h(\mathbf{x})$. It can also be defined as (see Supplemental Material [55])

$$e^{-\Gamma_{M,k}[\phi]} = \int \mathcal{D}\hat{\phi}\, e^{-\mathcal{H}[\hat{\phi}] - \frac{1}{2}(\hat{\phi}-\phi).R_{M,k}.(\hat{\phi}-\phi) + \frac{\delta\Gamma_{M,k}}{\delta\phi}.(\hat{\phi}-\phi)}, \tag{4}$$

where $R_{M,k}(\mathbf{x},\mathbf{y}) = R_k(\mathbf{x},\mathbf{y}) + M^2$, or in momentum space $R_{M,k}(\mathbf{q}) = R_k(\mathbf{q}) + M^2\delta_{\mathbf{q},0}$, with $\mathbf{q} = (2\pi/L)\mathbf{n}$ and $\mathbf{n} \in \mathbb{Z}^d$. Equation (4) has the advantage of explicitly showing that $\Gamma_{M,k}$ does not depend on s. Up to the replacement of R_k by $R_{M,k}$, $\Gamma_{M,k}$ is formally identical to the usual scale-dependent effective action Γ_k introduced in the FRG [42], and, indeed, $\Gamma_k[\phi] = \Gamma_{M=0,k}[\phi]$. The exact RG equation satisfied by $\Gamma_{M,k}[\phi]$ is the usual Wetterich equation in the presence of the regulator $R_{M,k}$,

$$\partial_k\Gamma_{M,k}[\phi] = \frac{1}{2}\int_{\mathbf{x},\mathbf{y}} \partial_k R_{M,k}(\mathbf{x},\mathbf{y})(\Gamma^{(2)}_{M,k} + R_{M,k})^{-1}(\mathbf{x},\mathbf{y}), \tag{5}$$

where $\Gamma^{(2)}_{M,k} = \Gamma^{(2)}_{M,k}[\mathbf{x},\mathbf{y};\phi] = \delta^2\Gamma_{M,k}/\delta\phi(\mathbf{x})\delta\phi(\mathbf{y})$.

Defining $\check{\Gamma}_k[\phi] = \lim_{M\to\infty}\Gamma_{M,k}[\phi]$, the additional k-independent term $M^2\delta_{\mathbf{q},0}$ completely freezes the zero-momentum mode $\int_{\mathbf{x}}\hat{\phi}(\mathbf{x})$ in $\check{\Gamma}_k[\phi]$, and we show in the Supplemental Material [55] that, when evaluated in constant field $\phi(\mathbf{x}) = s$, $L^{-d}\check{\Gamma}_k[s] = I_k(s)$ is a scale-dependent rate function such that $P(s) \propto \lim_{k\to 0}\exp[-L^d I_k(s)]$. [In contrast, when evaluated in a constant field $\phi(\mathbf{x}) = m$, the effective action $\Gamma_k[\phi = m]$ is $L^d U_k(m)$, where $U_k(m)$ is the k-dependent effective potential that becomes the true effective potential at $k = 0$. In particular, $\Gamma[\phi] = \Gamma_{k=0}[\phi]$ being the Legendre transform of $\ln\mathcal{Z}[h]$, the effective potential $U(m) = U_{k=0}(m)$ is a convex function of m [42]. Note that both Γ and I are RG-scheme independent by construction.] Our aim in the following is to compute $\check{\Gamma}_k[\phi = s]$ and to evaluate it at $k = 0$. For this purpose, we now study the flow of $\check{\Gamma}_k$ comparing it with the better known flow of Γ_k.

For $\zeta \ll 1$ and $a^{-1} \gg k \gg 1/L \gg 1/\xi_\infty$, the regulator R_k effectively freezes the zero-momentum mode in Γ_k, which makes its flow identical to that of $\check{\Gamma}_k$, up to corrections of order $(kL)^{-d}$. In this range of k, the system is self-similar because both a and L play no role in the flows. It follows that both U_k and I_k obey a scaling form $I_k(\phi) \simeq U_k(\phi) = k^d\tilde{U}^*(\phi k^{-(d-2+\eta)/2})$, where $\tilde{U}^*$ is k independent; that is, it is the dimensionless fixed point potential of the RG flow of Γ_k [42]. It is a nonconvex function that has the typical double well form, see below.

When k becomes of order $2\pi/L$, the flows of U_k and I_k start to differ significantly. On the one hand, the flow of U_k becomes essentially that of a zero-dimensional system (corresponding to the fluctuations of the zero-momentum mode only), and $\lim_{k\to 0} U_k(m)$ becomes convex with a curvature at small m given by $\chi^{-1} \propto L^{-2+\eta}$. On the other hand, the flow of $I_k(s)$ stops typically for $k \lesssim 2\pi/L$ because in this quantity the zero-momentum mode is frozen by the $M \to \infty$ term. In particular, this allows for a nonconvex shape of $I(s) = I_{k=0}(s)$, and $L^d I(s)$ is found to naturally be a function of $\tilde{s} = sL^{(d-2+\eta)/2}$.

The above picture is modified when $\zeta \gg 1$ ($T > T_c$, $L \gg \xi_\infty$), because the system size can no longer play any significant role when $\xi_\infty \ll L$. In particular, the flows of U_k and I_k rapidly stop for $k \lesssim 1/\xi_\infty$ and it makes no difference whether the zero mode is completely frozen or not. Approaching criticality from the disordered phase, we therefore find that $I_{k=0}(s) \simeq U_{k=0}(m = s)$. These functions are convex with positive curvature $\chi^{-1} \propto \xi_\infty^{-2+\eta}$ at $s = 0$. Working at fixed ζ, we thus have $I_{k=0}(s) \propto \zeta^{2-\eta}L^{d-2+\eta}s^2$ at small s. The PDF is therefore Gaussian at small s as in the CLT, which is expected because the system looks like a collection of uncorrelated clusters of spins of extension ξ_∞. However, since the susceptibility diverges at $T = T_c$ as $\xi_\infty^{-2+\eta}$, the fluctuations are anomalously large compared with the usual CLT because they are of order $L^{-[(d-2+\eta)/2]}$ instead of the standard $L^{-d/2}$. Varying ζ then generates a smooth family of rate functions, the shapes of which depend on the competition between L and ξ_∞ in the flow. Furthermore, $I_\zeta(\tilde{s})$ behaves as $\tilde{s}^{2d/(d-2+\eta)}$ at large $\tilde{s}$, a behavior inherited from $\tilde{U}^*$ [55].

To compute in practice the rate function and find its specific shape depending on ζ, it is necessary to solve the

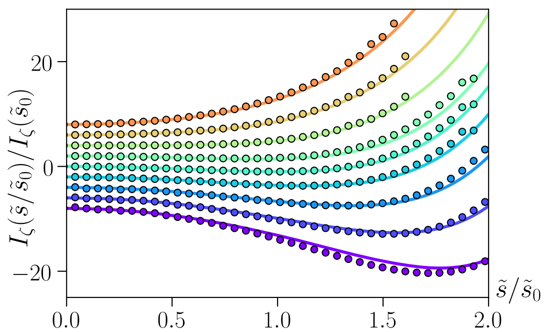

FIG. 2. Normalized rate functions $I_\zeta(\tilde{s})$ of the 3D Ising model obtained from the FRG (full line) and MC simulations (symbols) performed on the cubic lattice with periodic boundary conditions for ζ between −4 and 4 (from bottom to top, same color code as in Fig. 1). The normalization point $\tilde{s}_0$ is the position of the minimum of $I_{\zeta=0}$. The rate functions have been shifted for better visibility.

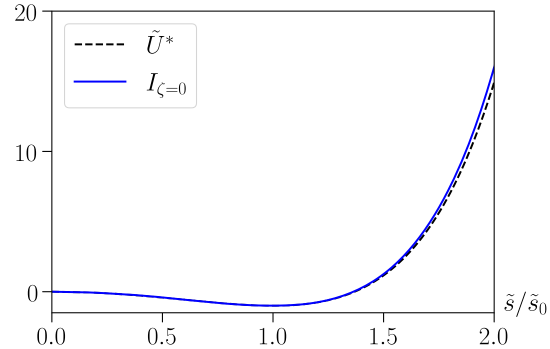

FIG. 3. Scaling rate function at $\zeta = 0$ obtained from the FRG (blue) as a function of $\tilde{s}$, and the fixed point potential obtained with the same regulator as function of $\tilde{\phi} = \tilde{s}$. Both have been normalized such that their minimum is −1 at 1. The difference is only visible in the tail.

flow equation (5). This cannot be done exactly, and it is necessary to perform approximations. Here, we focus on the simplest of such approximations, which nevertheless allows for a functional calculation of the rate function, the so-called local potential approximation (LPA). It amounts to using the ansatz $\check{\Gamma}_k[\phi] = \int_{\mathbf{x}}\{\frac{1}{2}(\partial\phi)^2 + I_k[\phi(\mathbf{x})]\}$ and projecting the flow equation onto this ansatz. The corresponding LPA flow equation is then closed for the scale-dependent rate function and reads

$$\partial_k I_k(s) = \frac{1}{2L^d}\sum_{\mathbf{q}\neq 0}\frac{\partial_k R_k(\mathbf{q})}{\mathbf{q}^2 + R_k(\mathbf{q}) + I_k''(s)}, \qquad (6)$$

and we use the "exponential regulator" $R_k(\mathbf{q}) = \alpha k^2 e^{-\mathbf{q}^2/k^2}$ with $\alpha = 4.65$ [55]. Note that at the LPA, the anomalous dimension vanishes, $\eta = 0$, but since $\eta \ll 1$ for the three-dimensional Ising model, we expect the approximation to correctly capture the shape of the rate function. The scaling functions $I_\zeta(\tilde{s}) = I_{\zeta,k=0}(\tilde{s})$ obtained from integrating the LPA flow, Eq. (6), are shown as solid lines in Fig. 2 for various ζ, see the Supplemental Material [55]. We have verified that the resulting rate functions obey the expected scaling, are functions of $\tilde{s}$ and ζ only, and only very weakly depend on the regulator function R_k [55].

In the same figure, we compare our FRG results to the rate functions obtained from Monte Carlo (MC) simulations on the cubic lattice with periodic boundary conditions, using a Wolff algorithm [60] with histogram reweighting [61], also used to generate Fig. 1, see Ref. [55]. Since lattice and field theory calculations use different units, it requires rescalings of the x axis (magnitude of the total spin s) and y axis (associated with the microscopic length scales, since $I(s)$ is a density) in the plot of $I(s)$, Fig. 2. Importantly, these model-dependent lengths are independent of ζ and should be determined from only one value of ζ (we use $\zeta = 0$). We find that to compare the rate function obtained from MC simulation for a given ζ_{MC} to that obtained from LPA at ζ_{LPA} necessitates a rescaling of ζ_{LPA}, with $\zeta_{\mathrm{MC}} \simeq 0.9\zeta_{\mathrm{LPA}}$ [55]. We attribute this to errors in the computation of ξ_∞ induced by LPA. With this small caveat, we find a very good agreement between simulations and FRG on the whole range of $\zeta \in [-4, 4]$. Note that the rate functions become strictly convex for $\zeta \gtrsim 2.2$.

It is interesting to note that $I_{\zeta=0}$ is very similar to the fixed point potential, when properly normalized, see Fig. 3. This could explain why the fixed point of the RG has long been thought to describe the critical PDFs [34,38,40]. However, this cannot be true exactly because the dependence of the fixed point effective potential $\tilde{U}^*$ on the choice of regulator R_k cannot be normalized out. This can be shown explicitly in the large N limit of the critical $O(N)$ model [62]. Reciprocally, our Letter confirms that the RG is deeply related to probability theory, since computing a fixed point is actually almost synonymous to computing the $I_{\zeta=0}$ but for the zero mode, which is excluded in the rate function. This elucidates the long-standing paradoxes arising from the confusion between the fixed point potential $\tilde{U}^*$ and $I_{\zeta=0}$, which, although very closely related, are conceptually different.

Our Letter raises many questions and paves the way to many applications that we want to briefly review below. For instance, the method can be generalized to all pure statistical systems at thermal equilibrium, with probably very good results, at least when the LPA is accurate, that is, when η is small. The generalization to disordered and/or out-of-equilibrium systems, where very little is known about the computation of critical PDFs, certainly requires one to adapt the formalism. This should be feasible since the FRG already yields fairly accurate results for such problems like the random field $O(N)$ models [63–65], reaction-diffusion models [66,67], and the Kardar-Parisi-Zhang equation [68–71] to mention just a few [42]. Also, the coexistence region in the low-temperature phase is highly nontrivial, scaling as a surface term, and necessitates one to go beyond the LPA, which does not capture domain walls. This could explain why our results do not agree as well with MC simulations for large and negative ζ. However, the LPA can be systematically improved via a derivative expansion or the Blaizot–Mendez-Galain–Wschebor approximation scheme [72,73]. The study of the convergence along the lines of [53,54] for the rate function is left for future work.

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