Complex Contact Interaction for Systems with Short-Range Two-Body Losses

Ce Wang,[1,2] Chang Liu,[1] and Zhe-Yu Shi[3,*]

[1] *Institute for Advanced Study, Tsinghua University, Beijing 100084, China*

[2] *School of Physics Science and Engineering, Tongji University, Shanghai 200092, China*

[3] *State Key Laboratory of Precision Spectroscopy, East China Normal University, Shanghai 200062, China*

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The concept of contact interaction is fundamental in various areas of physics. It simplifies physical models by replacing the detailed short-range interaction with a zero-range contact potential that reproduces the same low-energy scattering parameter, i.e., the s-wave scattering length. In this Letter, we generalize this concept to open quantum systems with short-range two-body losses. We show that the short-range two-body losses can be effectively described by a *complex* scattering length. However, in contrast to closed systems, the dynamics of an open quantum system is governed by the Lindblad master equation the includes a non-Hermitian Hamiltonian as well as an extra recycling term. We thus develop proper methods to regularize both terms in the master equation in the contact (zero-range) limit. We then apply our regularized complex contact interaction to study the dynamic problem of a weakly interacting and dissipating Bose-Einstein condensate. It is found that the physics is greatly enriched because the scattering length is continued from the real axis to the complex plane. For example, we show that a strong dissipation may prevent an attractive Bose-Einstein condensate from collapsing. We further calculate the particle decay in this system to the order of $(\text{density})^{3/2}$ which resembles the celebrated Lee-Huang-Yang correction to the ground state energy of interacting Bose gases [Lee and Yang, Phys. Rev. **105**, 1119 (1957); Lee, Huang, and Yang, Phys. Rev. **106**, 1135 (1957)]. Possible methods for tuning the complex scattering length in cold atomic gas experiments are also discussed.

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Separation of scales appears in many physical systems. It allows us to construct models that are simple enough yet able to capture the fundamental pictures of the physics effectively. For example, separation of length scales happens in systems such as ultracold atomic gases and nuclear systems where the ranges of the interparticle interactions are much smaller than other length scales (e.g., the interparticle distance and the thermal de Broglie wavelength). The complicated short-range interactions can then be replaced by a zero-range contact potential, once the latter can reproduce the same physical behavior for a low-energy collision process. Such contact potential has been considered as the fundamental model in nuclear and cold atom physics since the pioneering works of Bethe, Peierls [1], and Fermi [2].

In scattering theory, the low-energy scattering data are described by the (real) s-wave scattering length a [3]. Given the input s-wave scattering length, there are three approaches that can describe the zero-range contact interaction in the literature, which include the Bethe-Peierls model, pseudopotential, and renormalized delta potential.

Behte-Peierls model.—In the study of deuteron scattering theories, Bethe and Peierls suggest that the effect of a short-range potential $V(r)$ may be replaced by a boundary condition at $r=0$ [1]. It is shown that the zero-energy solution for the two-body relative wave function is $\varphi(\mathbf{r}_{\mathrm{rel}})=1/r_{\mathrm{rel}}-1/a$ outside the interaction range r_0. Thus if we are only interested in the low-energy physics in such systems, the interaction can be replaced by a boundary condition on the many-body wave function [4],

$$\psi(\underline{\mathbf{r}}_N)\simeq\left(\frac{1}{r_{ij}}-\frac{1}{a}\right)A(\underline{\mathbf{r}}_N^{(ij)},\mathbf{R}_{ij}),\qquad r_{ij}\to 0,\tag{1}$$

where A could be an arbitrary function, $\mathbf{R}_{ij}=((\mathbf{r}_i+\mathbf{r}_j)/2)$ and $\mathbf{r}_{ij}=\mathbf{r}_i-\mathbf{r}_j$ are the center of mass and relative coordinates of particle i and j, $\underline{\mathbf{r}}_N$ represents all the coordinates in $\{\mathbf{r}_1,\ldots,\mathbf{r}_N\}$, and $\underline{\mathbf{r}}_N^{(ij)}$ represents all the coordinates except $\mathbf{r}_i$ and $\mathbf{r}_j$.

Pseudopotential.—Introduced by Fermi, the pseudopotential models the short-range interaction through a delta potential and an extra operator which regularizes the wave function near the origin [2,5,6],

$$U(\mathbf{r})=\frac{4\pi\hbar^2 a}{m}\delta(\mathbf{r})\partial_r r,\tag{2}$$

with m the particle mass. It can be shown that this pseudopotential is equivalent to posing the boundary condition [Eq. (1)] at the origin [6,7].

TABLE I. Three approaches regularizing the complex contact interaction. We denote coordinates $\{\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N\}$ by $\underline{\mathbf{r}}_N$. $\underline{\mathbf{r}}_N^{(\alpha\beta)}$ stands for all the coordinates in $\underline{\mathbf{r}}_N$ except the two with indices α, β.

	Contact interaction	Complex contact interaction	Recycling term
Bethe-Peierls model	$\psi(\underline{\mathbf{r}}_N) \simeq ((1/r_{\alpha\beta}) - (1/a))\, A(\underline{\mathbf{r}}_N^{(\alpha\beta)}, \mathbf{R}_{\alpha\beta})$	$\rho_{jl}(\underline{\mathbf{r}}_j, \underline{\mathbf{r}}'_l) \simeq ((1/r_{\alpha\beta}) - (1/a_c))\, ((1/r'_{\mu\nu}) - (1/a_c^*)) \times B_{jl}(\underline{\mathbf{r}}_j^{(\alpha\beta)}, \mathbf{R}_{\alpha\beta}; \underline{\mathbf{r}}_l'^{(\mu\nu)}, \mathbf{R}'_{\mu\nu})$	$\mathrm{Im}(4\pi\hbar^2/ma_c)\sqrt{(j+2)(j+1)(l+2)(l+1)} \times \int_{\mathbf{R}} B_{j+2,l+2}(\underline{\mathbf{r}}_j, \mathbf{R}; \underline{\mathbf{r}}'_l, \mathbf{R})$
Pseudopotential	$U(\mathbf{r}) = (4\pi\hbar^2 a/m)\delta(\mathbf{r})\partial_r r$	$U_c(\mathbf{r}) = (4\pi\hbar^2 a_c/m)\delta(\mathbf{r})\partial_r r$	$(4\pi\hbar^2 \lvert a_i\rvert/m)\sqrt{(j+2)(j+1)(l+2)(l+1)} \times \int_{\mathbf{R},\mathbf{r},\mathbf{r}'} \delta(\mathbf{r})\delta(\mathbf{r}')\partial_r r \partial_{r'} r' \rho_{j+2,l+2}$
Renormalization relation	$(m/4\pi\hbar^2 a) = (1/g) + (1/\Omega)\sum_{\mathbf{k}}(1/2\epsilon_{\mathbf{k}})$	$(m/4\pi\hbar^2 a_c) = (1/(g - i\gamma)) + (1/\Omega)\sum_{\mathbf{k}}(1/2\epsilon_{\mathbf{k}})$	$\gamma \int_{\mathbf{r}} \hat{\psi}_{\mathbf{r}}^2 \hat{\rho} \hat{\psi}_{\mathbf{r}}^{\dagger 2}$

Renormalized delta potential.—Another way to regularize the delta potential is to use the renormalization method developed in quantum field theory. Given $V(\mathbf{r}) = g\delta(\mathbf{r})$, one can calculate the on shell two-body T matrix $t(E)$ and compare it with the low-energy scattering amplitude $f(E)$ via $t(E) = -(4\pi\hbar^2/m)f(E)$. This relates the coupling constant g to the s-wave scattering length a through the renormalization relation [8],

$$\frac{1}{g} = \frac{m}{4\pi\hbar^2 a} - \frac{1}{\Omega}\sum_{\mathbf{k}} \frac{1}{2\epsilon_{\mathbf{k}}}. \tag{3}$$

Here $\epsilon_{\mathbf{k}} = (\hbar^2 k^2/2m)$ is the single-particle dispersion, and Ω is the system volume. It is worth noting that the momentum summation in the rhs leads to ultraviolet divergence that needs to be properly canceled in any practical calculation.

The three descriptions of the contact potential are equivalent, as they are able to reproduce the same two-body scattering data. These simplified models are more suitable for many-body calculations and thus set the foundation of many successful theories in cold atom physics, from the ground state energy correction of weakly interacting Bose-Einstein condensates (BECs) [9,10] to the BEC-BCS crossover in two-component Fermi gases [8,11,12]. Inspired by this idea, we generalize the concept of contact potential to open many-body systems with short-range two-body losses that can be described by Lindblad master equations.

We first discuss the structure of a general master equation which governs the system dynamics. By taking the limit of interaction and loss range $r_0 \to 0$, we show that the only important low-energy parameter that remains is a complex scattering length a_c. We further develop the methods to regularize or renormalize the interactions and two-body losses in the Lindblad master equation, which are listed in Table I. We then apply our model to a bosonic system with weak interaction and loss. The experimental method of tuning the complex scattering length a_c is also discussed [13].

The effect of few-particle losses in cold atomic gases has attracted great attention in recent years [14–22]. However, the necessity of regularization of the Lindblad master equation is usually overlooked. It is not until recently that Bouchoule *et al.* has raised the question of regularization of one-particle losses in three-dimensional systems [23].

The Lindblad master equation.—Consider an open system of interacting bosons subject to (finite-range) two-body losses; the evolution of the density matrix $\hat{\rho}$ is governed by the Lindblad master equation $\partial_t\hat{\rho} = \mathcal{L}\hat{\rho}$ with the Lindbladian ($\hbar = 1$) [24]

$$\mathcal{L}\hat{\rho} = \frac{1}{i}[\hat{H}, \hat{\rho}] - \frac{1}{2}\int_{\mathbf{r}_1,\mathbf{r}_2} V_i(r_{12})\{\hat{\psi}_{\mathbf{r}_1}^\dagger \hat{\psi}_{\mathbf{r}_2}^\dagger \hat{\psi}_{\mathbf{r}_2} \hat{\psi}_{\mathbf{r}_1}, \hat{\rho}\} + \mathcal{J}\hat{\rho},$$

where $\hat{\psi}_{\mathbf{r}}$ is the annihilation operator at position $\mathbf{r}$, $\hat{H}$ is the usual Hermitian Hamiltonian of interacting bosons, and

$$\hat{H} = -\int_{\mathbf{r}} \hat{\psi}_{\mathbf{r}}^\dagger \frac{\nabla^2}{2m}\hat{\psi}_{\mathbf{r}} + \frac{1}{2}\int_{\mathbf{r}_1,\mathbf{r}_2} V_r(r_{12})\hat{\psi}_{\mathbf{r}_1}^\dagger \hat{\psi}_{\mathbf{r}_2}^\dagger \hat{\psi}_{\mathbf{r}_2} \hat{\psi}_{\mathbf{r}_1}. \tag{4}$$

We assume the two-body loss rate V_i is a function that depends on the interparticle distance. The recycling term $\mathcal{J}\hat{\rho}$ is

$$\mathcal{J}\hat{\rho} = \int_{\mathbf{r}_1,\mathbf{r}_2} V_i(r_{12})\hat{\psi}_{\mathbf{r}_1}\hat{\psi}_{\mathbf{r}_2}\hat{\rho}\hat{\psi}_{\mathbf{r}_2}^\dagger \hat{\psi}_{\mathbf{r}_1}^\dagger. \tag{5}$$

The interaction V_r and the two-body loss rate V_i are assumed to be finite ranged and vanish at $r > r_0$. It is also required that $V_i \geq 0$ inside r_0, which is necessary to guarantee the positive definiteness of the density matrix.

The master equation may be regarded as the evolution under a non-Hermitian Hamiltonian $\hat{H}_{\mathrm{eff}}$ together with the recycling term, i.e., $\partial_t\hat{\rho} = (1/i)(\hat{H}_{\mathrm{eff}}\hat{\rho} - \hat{\rho}\hat{H}_{\mathrm{eff}}^\dagger) + \mathcal{J}\hat{\rho}$, where the $\hat{H}_{\mathrm{eff}}$ is equivalent to the Hermitian Hamiltonian $\hat{H}$, but with the real potential V_r replaced by a complex one $V_c = V_r - iV_i$.

The complex scattering length.—The special form of the jump operator $\hat{\psi}_{\mathbf{r}_1}\hat{\psi}_{\mathbf{r}_2}$ leads to a hierarchical structure of the Lindbladian $\mathcal{L}$. To see this, note that the bosonic Fock space naturally defines orthogonal projections $\hat{P}_l, l = 0, 1, 2, \ldots$ which project any state to the l-boson subspace $\mathcal{H}_l$. For any linear operator $\hat{O}$, we thus have decomposition $\hat{O} = \sum_{j,l} \hat{O}_{jl}$ with $\hat{O}_{jl} \equiv \hat{P}_j \hat{O} \hat{P}_l$ an operator that maps a state in $\mathcal{H}_l$ to $\mathcal{H}_j$. Because the jump operator $\hat{\psi}_{\mathbf{r}_1}\hat{\psi}_{\mathbf{r}_2}$ always annihilates two particles, one can show that the master equation may be decomposed to a series of hierarchy equations for $\hat{\rho}_{jl}$,

$$\partial_t \hat{\rho}_{jl} = \frac{1}{i}(\hat{H}_{\text{eff}}\hat{\rho}_{jl} - \hat{\rho}_{jl}\hat{H}^{\dagger}_{\text{eff}}) + \mathcal{J}\hat{\rho}_{j+2,l+2}. \tag{6}$$

The hierarchical structure allows us to consider a "two-body" problem in the presence of two-body loss. If we start with an initial density matrix $\hat{\rho}(0)$ that contains two bosons, i.e., $\hat{\rho}(0) = \hat{\rho}_{22}(0)$, it is clear from Eq. (6) that the only nonvanishing blocks of $\hat{\rho}(t)$ will be $\hat{\rho}_{22}$ and $\hat{\rho}_0$, which satisfy

$$\partial_t \hat{\rho}_{22} = \frac{1}{i}(\hat{H}_{\text{eff}}\hat{\rho}_{22} - \hat{\rho}_{22}\hat{H}^{\dagger}_{\text{eff}}), \tag{7}$$

$$\partial_t \hat{\rho}_{00} = \mathcal{J}\hat{\rho}_{22} = -\partial_t \text{tr}\hat{\rho}_{22}. \tag{8}$$

We see that the evolution of the two-particle density matrix $\hat{\rho}_{22}$ is fully described by the non-Hermitian Hamiltonian $\hat{H}_{\text{eff}}$. This means the "two-body" problem may be solved in the same manner as the usual two-body problem except that the potential $V_c(r)$ is complex. Consider the s-wave zero-energy wave function in relative coordinates $\varphi(r)$. It is then clear that

$$\varphi(r) = \frac{1}{r} - \frac{1}{a_c}, \quad \text{for } r \geq r_0, \tag{9}$$

because the system is noninteracting in this region.

Equation (9) gives the definition of the complex scattering length a_c. Furthermore, it can be shown that $\text{Im}(a_c^{-1}) = m\int_0^{r_0} r^2 dr V_i(r)|\varphi(r)|^2$ [25]. Together with the constrain $V_i \geq 0$, we conclude that $\text{Im}(a_c)$ is always negative in the presence of two-body loss. We thus write a_c as $a_c = a_r + ia_i$ with $a_i < 0$. Similar results may also be obtained by considering the open channel wave function of a multichannel Hermitian two-body model [26–31]. In that case, the closed channels of the model play the role of a dissipative reservoir (see the Supplemental Material SM for more details).

Complex Bethe-Peierls model.—To generalize the Bethe-Peierls boundary condition, we first write the Lindblad equation in the first quantization formalism. Acting $\langle \underline{\mathbf{r}}_j | \cdot | \underline{\mathbf{r}}'_l \rangle$ on both sides of Eq. (6) ($|\underline{\mathbf{r}}_l\rangle \equiv (1/\sqrt{l!})\hat{\psi}^{\dagger}_{\mathbf{r}_1} \ldots \hat{\psi}^{\dagger}_{\mathbf{r}_l}|0\rangle$), we obtain

$$\partial_t \rho_{jl} = (1/i)(H_{\text{eff}}(\underline{\mathbf{r}}_j) - H^{\dagger}_{\text{eff}}(\underline{\mathbf{r}}'_l))\rho_{jl} + \mathcal{J}\rho_{j+2,l+2}, \tag{10}$$

where $\rho_{jl}(\underline{\mathbf{r}}_j, \underline{\mathbf{r}}'_l) \equiv \langle \underline{\mathbf{r}}_j | \hat{\rho}_{jl} | \underline{\mathbf{r}}'_l \rangle$ is the first quantized density matrix and $H_{\text{eff}}(\underline{\mathbf{r}}_j) = \sum_{\alpha=1}^{j} -(\nabla_\alpha^2/2m) + \sum_{1\leq\alpha<\beta\leq j} V_c(r_{\alpha\beta})$ is the first quantized Hamiltonian. The recycling term is given by

$$\mathcal{J}\rho_{j+2,l+2} = \sqrt{(j+2)(j+1)(l+2)(l+1)} \times \int_{\mathbf{x},\mathbf{y}} V_i(|\mathbf{x}-\mathbf{y}|)\rho_{j+2,l+2}(\underline{\mathbf{r}}_j, \mathbf{x}, \mathbf{y}; \underline{\mathbf{r}}'_l, \mathbf{x}, \mathbf{y}). \tag{11}$$

From Eq. (10), we notice that in the region where all the particles are apart from each other such that $r_{\alpha\beta}, r'_{\alpha\beta} > r_0$ for all possible distinct pairs α, β, the evolution of ρ_{jl} is governed by a noninteracting H_{eff} plus the recycling term $\mathcal{J}\rho_{j+2,l+2}$. In the zero-range limit $r_0 \to 0$, this region fills the whole domain of ρ_{jl}; one thus expects that the effect of the complex interaction V_c can be replaced by a boundary condition at $r_{\alpha\beta} \to 0$.

To be more concrete, we consider a system with mean interparticle distance d and energy per particle $(k^2/2m)$, and focus on the density matrix with a pair of particles (α and β) close to each other such that $r_{\alpha\beta} \ll d, k^{-1}$. In this region, the two-body scattering process dominates, and every other term in Eq. (10) besides the two-body relative kinetic energy and interaction $V_c(r_{\alpha\beta})$ can be ignored [32]. Then the Lindblad equation reduces to

$$0 \simeq -\frac{\nabla^2_{\mathbf{r}_{\alpha\beta}}}{m}\rho_{jl} + V_c(r_{\alpha\beta})\rho_{jl}, \tag{12}$$

which is nothing but the zero-energy two-body Schrödinger equation in the relative coordinate $\mathbf{r}_{\alpha\beta}$.

Because of the centrifugal barrier of higher partial waves, ρ_{jl} is dominated by the s-wave two-body wave function $\varphi(r)$. We thus have $\rho_{jl} \propto \varphi(r_{\alpha\beta})$ when $r_{\alpha\beta} \to 0$. The same proof may also be applied to the region $r'_{\mu\nu} \ll d, k^{-1}$, which leads to following asymptotic form of $\rho_{jl}(\underline{\mathbf{r}}_j, \underline{\mathbf{r}}_l)$ when $r_{\alpha\beta}, r'_{\mu\nu} \to 0$,

$$\rho_{jl} \simeq \varphi(r_{\alpha\beta})\varphi(r'_{\mu\nu})B_{jl}(\underline{\mathbf{r}}_j^{(\alpha\beta)}, \mathbf{R}_{\alpha\beta}; \underline{\mathbf{r}}_l'^{(\mu\nu)}, \mathbf{R}'_{\mu\nu}) \tag{13}$$

with B_{jl} an arbitrary function.

Taking the limit of $r_0 \to 0$, we obtain the boundary condition

$$\rho_{jl} \simeq \left(\frac{1}{r_{\alpha\beta}} - \frac{1}{a_c}\right)\left(\frac{1}{r'_{\mu\nu}} - \frac{1}{a_c^*}\right) \times B_{jl}(\underline{\mathbf{r}}_j^{(\alpha\beta)}, \mathbf{R}_{\alpha\beta}; \underline{\mathbf{r}}_l'^{(\mu\nu)}, \mathbf{R}'_{\mu\nu}), \qquad r_{\alpha\beta}, r'_{\mu\nu} \to 0. \tag{14}$$

The recycling term can be calculated by substituting Eq. (13) into Eq. (11), which leads to

$$\mathcal{J}\rho_{j+2,l+2} = \mathrm{Im}\left(\frac{4\pi\hbar^2}{ma_c}\right)\sqrt{(j+2)(j+1)(l+2)(l+1)} \times \int_{\mathbf{R}} B_{j+2,l+2}(\underline{\mathbf{r}}_j, \mathbf{R}; \underline{\mathbf{r}}'_l, \mathbf{R}) \tag{15}$$

where we restored $\hbar$.

The boundary condition [Eq. (14)] together with the recycling term [Eq. (15)] determine the evolution of density matrix ρ_{jl} in the zero-range limit. They thus can be viewed as the complex analog of the Bethe-Peierls boundary condition [Eq. (1)].

Complex pseudopotential.—Given the boundary condition [Eq. (14)], it is straightforward to apply the standard regularization method [6,7] and show that the short-range complex interaction V_c (V_c^*) in H_{eff} ($H_{\mathrm{eff}}^\dagger$) can also be replaced by a complex pseudopotential U_c (U_c^*) with

$$U_c(\mathbf{r}) = \frac{4\pi\hbar^2 a_c}{m}\delta(\mathbf{r})\partial_r r. \tag{16}$$

Similarly, the recycling term [Eq. (15)] can be written in terms of the regularized operators

$$\mathcal{J}\rho_{j+2,l+2} = \frac{4\pi\hbar^2|a_i|}{m}\sqrt{(j+2)(j+1)(l+2)(l+1)} \times \int_{\mathbf{R},\mathbf{r},\mathbf{r}'} \delta(\mathbf{r})\delta(\mathbf{r}')\partial_r r\partial_{r'} r'\rho_{j+2,l+2}, \tag{17}$$

where $\hat{\rho}_{j+2,l+2}$ stands for $\rho_{j+2,l+2}(\underline{\mathbf{r}}_j, \mathbf{R}+(\mathbf{r}/2), \mathbf{R}-(\mathbf{r}/2); \underline{\mathbf{r}}'_l, \mathbf{R}+(\mathbf{r}'/2), \mathbf{R}-(\mathbf{r}'/2))$.

Renormalized contact potential.—Following the conventional renormalization approach, we first write the short-range complex potential V_c as a delta potential,

$$V_c = (g - i\gamma)\delta(\mathbf{r}), \tag{18}$$

with g (γ) being the real (imaginary) coupling constant.

It is then straightforward to calculate the two-body scattering amplitude [33],

$$f(k) = -\frac{m}{4\pi}\frac{1}{(g-i\gamma)^{-1} + \frac{1}{\Omega}\sum_{\mathbf{k}}(2\epsilon_{\mathbf{k}})^{-1} + \frac{ikm}{4\pi}}. \tag{19}$$

Comparing this formula with the standard low-energy expansion of the scattering amplitude $f(k) = -1/(a_c^{-1} + ik)$, we find the renormalization relation,

$$\frac{1}{g-i\gamma} = \frac{1}{g_0 - i\gamma_0} - \frac{1}{\Omega}\sum_{\mathbf{k}}\frac{1}{2\epsilon_{\mathbf{k}}}, \tag{20}$$

where we have defined $g_0 - i\gamma_0 \equiv (4\pi\hbar^2 a_c/m)$ as being the renormalized complex coupling constant. And the second quantized recycling term is simply

$$\mathcal{J}\hat{\rho} = \gamma\int_{\mathbf{r}} \hat{\psi}_{\mathbf{r}}^2\hat{\rho}\hat{\psi}_{\mathbf{r}}^{\dagger 2}. \tag{21}$$

We list the results for the three regularization approaches in Table I. It is worth noting that the renormalization relation [Eq. (20)] has been derived recently using effective field theory [34]. In two recent works, it has also been applied for the calculation of non-Hermitian Hamiltonians [35,36].

Application to Bose gases.—To demonstrate the validity of our regularized model, we study the quench dynamics of BECs subjected to weak interaction and loss, i.e., $n|a_c|^3 \ll 1$ where n is the boson density.

We shall use the renormalized delta potential approach for this many-body problem. Writing the original Lindbladian in momentum space, we obtain

$$H_{\mathrm{eff}} = \sum_{\mathbf{k}}\epsilon_{\mathbf{k}}\hat{a}_{\mathbf{k}}^\dagger\hat{a}_{\mathbf{k}} + \frac{g-i\gamma}{2\Omega}\sum_{\mathbf{k},\mathbf{k}',\mathbf{p}}\hat{a}_{\mathbf{k}+\mathbf{p}}^\dagger\hat{a}_{\mathbf{k}'-\mathbf{p}}^\dagger\hat{a}_{\mathbf{k}'}\hat{a}_{\mathbf{k}}, \tag{22}$$

and the recycling term

$$\mathcal{J}\hat{\rho} = \frac{\gamma}{\Omega}\sum_{\mathbf{k},\mathbf{k}',\mathbf{p}}\hat{a}_{\mathbf{k}'}\hat{a}_{\mathbf{k}}\hat{\rho}\hat{a}_{\mathbf{k}'-\mathbf{p}}^\dagger\hat{a}_{\mathbf{k}+\mathbf{p}}^\dagger, \tag{23}$$

where $a_{\mathbf{k}}^\dagger \equiv (1/\sqrt{\Omega})\int_{\mathbf{r}} e^{i\mathbf{k}\cdot\mathbf{r}}\psi_{\mathbf{r}}^\dagger$.

We consider a system of N bosons initially condense in the zero momentum state, such that a large fraction of bosons still remains in the condensate when t is small, i.e., the depletion $(N-N_0)/N \ll 1$ where $N_0 \equiv \langle\hat{a}_0^\dagger\hat{a}_0\rangle$ is the number of particles in the condensate. Then we may apply the Bogoliubov approximation [37] and substitute $\hat{a}_0, \hat{a}_0^\dagger$ in the Lindbladian by $\sqrt{N - \sum_{\mathbf{k}\neq 0}\hat{a}_{\mathbf{k}}^\dagger\hat{a}_{\mathbf{k}}}$. This leads to a quadratic Bogoliubov Lindbladian that describes the dynamics of the noncondensed bosons,

$$\mathcal{L}_B\hat{\rho}' = \frac{1}{i}\left[\hat{H}_B, \hat{\rho}'\right] - 2\gamma_0 n\sum_{\mathbf{k}\neq 0}\left\{\hat{a}_{\mathbf{k}}^\dagger\hat{a}_{\mathbf{k}}, \hat{\rho}'\right\} + 4\gamma_0 n\sum_{\mathbf{k}\neq 0}\hat{a}_{\mathbf{k}}\hat{\rho}'\hat{a}_{\mathbf{k}}^\dagger. \tag{24}$$

Here $\hat{\rho}'$ is the reduced density matrix for the noncondensed bosons. We see that the Lindbladian $\mathcal{L}_B$ describes an open system governed by $\hat{H}_B$ and single-particle loss with loss rate $4\gamma_0 n$. Here H_B is a Hermitian Hamiltonian,

$$\hat{H}_B = \sum_{\mathbf{k}\neq 0}\left((\epsilon + g_0 n)\hat{a}_{\mathbf{k}}^\dagger\hat{a}_{\mathbf{k}} + \frac{g_0 n - i\gamma_0 n}{2}\hat{a}_{\mathbf{k}}^\dagger\hat{a}_{-\mathbf{k}}^\dagger + \mathrm{H.c.}\right). \tag{25}$$

Similar to the conventional Bogoliubov approximation approach [38], we replaced all the bare coupling constants g, γ by renormalized values g_0, γ_0.

Remarks on $\mathcal{L}_B$.—We emphasize that the recycling term $\mathcal{J}\hat{\rho}$ is essential for deriving the correct many-body Lindbladian $\mathcal{L}_B$, as part of the recycling term such that $(\gamma/\Omega)\hat{a}_0\hat{a}_0\hat{\rho}\hat{a}^\dagger_{\mathbf{k}}\hat{a}^\dagger_{-\mathbf{k}}$ becomes $\gamma n\hat{\rho}\hat{a}^\dagger_{\mathbf{k}}\hat{a}^\dagger_{-\mathbf{k}}$ and constitutes the Hermitian Hamiltonian H_B after the approximation. This demonstrates that the recycling term $\mathcal{J}\hat{\rho}$ indeed plays an important role in the many-body dynamics, and it is crucial to regularize it accordingly. Moreover, we note that the total density n is time dependent due to the breakdown of particle number conservation. However, for systems with $n|a_c|^3 \ll 1$, one can simply substitute it by the mean-field value $n(t) = n(0)/[1 + 2\gamma_0 n(0)t]$, which gives the correct results to the order we desire (see the derivation below).

To solve the quadratic Lindbladian $\mathcal{L}_B$, we may consider the dynamics of the SU(1,1) generators for the conventional Bogoliubov Hamiltonian, $A_0^{\mathbf{k}} = \frac{1}{2}(N_{\mathbf{k}} + N_{-\mathbf{k}} + 1)$, $A_1^{\mathbf{k}} = \frac{1}{2}(\hat{a}^\dagger_{\mathbf{k}}\hat{a}^\dagger_{-\mathbf{k}} + \text{H.c.})$, and $A_2^{\mathbf{k}} = (1/2i)(\hat{a}^\dagger_{\mathbf{k}}\hat{a}^\dagger_{-\mathbf{k}} - \text{H.c.})$ [39].

The dynamics of $A_i^{\mathbf{k}}$ may be calculated by $(d/dt)\langle A_i^{\mathbf{k}}\rangle = \text{tr}(\partial_t\hat{\rho}' A_i^{\mathbf{k}}) = \text{tr}(\mathcal{L}_B\hat{\rho}' A_i^{\mathbf{k}})$, which leads to a closed matrix equation

$$\dot{\mathbf{A}}^{\mathbf{k}} = -2\begin{pmatrix} 2\gamma_0 n & \gamma_0 n & -g_0 n \\ \gamma_0 n & 2\gamma_0 n & \epsilon_{\mathbf{k}} + g_0 n \\ -g_0 n & -\epsilon_{\mathbf{k}} - g_0 n & 2\gamma_0 n \end{pmatrix}\mathbf{A}^{\mathbf{k}} + \begin{pmatrix} 2\gamma_0 n \\ 0 \\ 0 \end{pmatrix} \tag{26}$$

with $\mathbf{A}^{\mathbf{k}} \equiv (\langle A_0^{\mathbf{k}}\rangle, \langle A_1^{\mathbf{k}}\rangle, \langle A_2^{\mathbf{k}}\rangle)^T$. We note that Eq. (26) reduces to the conventional equation of motion for the SU(1,1) generators in the $a_i \to 0$ limit [40,41].

The matrix in Eq. (26) needs to be solved numerically for the density n is time dependent, while a lot of information can be extracted by considering the short-time dynamics near an arbitrary time t_0 where we may approximate the density by a constant $n(t) \simeq n(t_0) + O(t - t_0)$. In this case, the solution to the matrix equation can be written as

$$\mathbf{A}^{\mathbf{k}}(t) \simeq \mathbf{A}_s^{\mathbf{k}} + \sum_{j=0}^{2} \mathbf{C}_j e^{2i(t-t_0)\xi_{j,\mathbf{k}}}. \tag{27}$$

Here $\mathbf{A}_s^{\mathbf{k}}$ is the quasisteady value for the SU(1,1) generators whose elements are listed in the Supplemental Material SM, $\mathbf{C}_j$ are constant vectors that depend on the initial value of $\mathbf{A}^{\mathbf{k}}$ at $t = t_0$, and $2i\xi_{j,\mathbf{k}}$ represent the three eigenvalues of the 3-by-3 matrix in Eq. (26). The eigenvalues can be calculated explicitly:

$$\xi_{0,\mathbf{k}} = 2i\gamma_0 n, \qquad \xi_{(1,2),\mathbf{k}} = 2i\gamma_0 n \pm \sqrt{\epsilon_{\mathbf{k}}^2 + 2g_0 n\epsilon_{\mathbf{k}} - \gamma_0^2 n^2}.$$

Clearly, $\xi_{(1,2),\mathbf{k}}$ reduce to the excitation energies of Bogoliubov modes in the $\gamma_0 \to 0$ limit, and the imaginary $\xi_{0,\mathbf{k}}$ indicates that $\mathcal{L}_B$ only has one steady state, i.e., the vacuum [42].

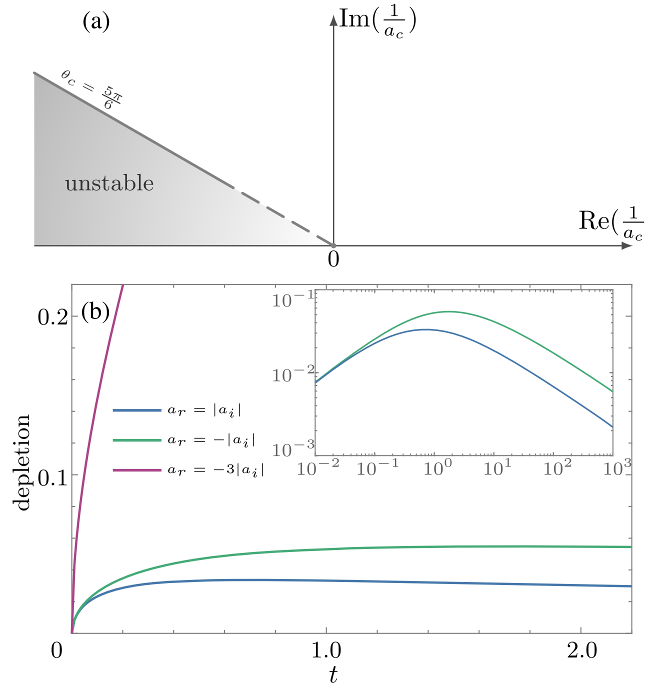

FIG. 1. (a) The phase diagram on the complex a_c^{-1} plane for BECs subjected to weak interaction and two-body loss. The system is unstable for $\theta \equiv \arg(a_c^{-1}) > (5\pi/6)$. (b) The depletion $(1/N)\sum_{\mathbf{k}\neq 0} N_{\mathbf{k}}$ as a function of time (in unit of $(1/\gamma_0 n(0))$). The initial condition is $\mathbf{A}_0^{\mathbf{k}} = (\frac{1}{2}, 0, 0)^T$; the parameters are $g_0 = \gamma_0$ (blue, stable), $g_0 = -\gamma_0$ (green, stable), and $g_0 = -3\gamma_0$ (purple, unstable). Inset: the long-time behavior of depletion.

Even though the system eventually evolves to the vacuum, it is still possible to discuss the stability of the Bose gas in time period $t \lesssim [\gamma_0 n(0)]^{-1}$ where many bosons still remain in the system. The excitation energies $\xi_{(1,2),\mathbf{k}}$ provide this stability information.

Note that a negative imaginary part in $\xi_{j,\mathbf{k}}$ represents an exponentially growth of that mode. In the conventional analysis on BECs with no loss ($\gamma_0 = 0$), the atomic cloud is unstable whenever the argument under the square root is negative for some $\mathbf{k}$, i.e., when $g_0 < 0$. However, in the presence of losses ($\gamma_0 > 0$), there is a competition between the leading $2i\gamma_0 n$ term and the imaginary part from the square roots in $\xi_{(1,2),\mathbf{k}}$. This depicts the competition between the inelastic process which tends to suppress the particle number with finite momentum $\mathbf{k}$ and the elastic scattering process which tends to keep exciting finite momentum particles from the condensate. For $g_0 > -\sqrt{3}\gamma_0$, $\text{Im}(\xi_{(1,2),\mathbf{k}}) > 0$ for all momenta, the number of excitations always decays, and the system keeps evolving toward the quasisteady state $\mathbf{A}_s^{\mathbf{k}}$, while for $g_0 < -\sqrt{3}\gamma_0$, $\text{Im}(\xi_{(1,2),\mathbf{k}}) < 0$ for small momenta. The system is unstable against the strong attraction in this region, and the Bogoliubov modes as well as the depletion $(1/N)\sum_{\mathbf{k}} N_{\mathbf{k}} \equiv \sum_{\mathbf{k}}(\langle A_0^{\mathbf{k}}\rangle - \frac{1}{2})$ keep growing. Experimentally, these growing Bogoliubov modes would cause a burst of jets of finite momentum atoms. Such "Bosenova" phenomena have already been observed in BECs with pure attraction ($g_0 < 0$, $\gamma_0 = 0$) [43,44].

The different behaviors define a critical angle $\theta_c = (5\pi/6)$ for $\arg(a_c^{-1})$ which separates the complex a_c^{-1} plane into two regions. To demonstrate the difference of dynamics in these regions, we numerically solve the matrix Eq. (26) for different g_0/γ_0 and plot the depletion as a function of time in Fig. 1(b). One can see that for $g_0/\gamma_0 < -\sqrt{3}$, the depletion quickly grows and reaches O(1) where the Bogoliubov approximation becomes invalid, in contrast to the cases with $g_0/\gamma_0 > -\sqrt{3}$ where the depletion remains small. In cold atom systems, we estimate that the time unit $[\gamma_0 n(0)]^{-1} \gtrsim 10$ ms near an optical Feshbach resonance [45,46]. We thus expect that this dynamical behavior may be observed in experiments.

Another interesting physical quantity that worth noting is the particle decay rate $\dot{N}$. In the Appendix, we use the Bogoliubov approximation to calculate the decay rate $\dot{N}$ for the quasisteady state. Using the renormalization relation given in Eq. (20), it is found that

$$\langle \dot{N} \rangle_s = -\frac{8\pi\hbar^2 |a_i| nN}{m}[1 + c_\theta (n|a_c|^3)^{1/2}] \qquad (28)$$

with $c_\theta = 2\sqrt{2\pi}(\cos(2\theta)/\sqrt{\sin(\theta_c - \theta)} + 2\cos\theta\sqrt{\sin(\theta_c - \theta)})$ and $\theta = \arg(a_c^{-1}) \in (0, \theta_c)$.

We note that the leading term in $\dot{N}$ may be viewed as the mean-field effect due to the two-body loss, which gives particle decay on the mean-field level $n(t) \simeq n(0)/[1 + 2\gamma_0 n(0)t]$ [47], while the next term in the order of $(n|a_c|^3)^{1/2}$ is an analog to the celebrated Lee-Huang-Yang correction for weakly interacting Bose gas [9,10].

Finally, we comment on the experiment control of a_c. Complex scattering lengths have been observed in cold atom experiments through optical Feshbach resonance [31,46]. The optical Feshbach resonance couples the open scattering channel to a closed channel molecule with a finite lifetime, which results in a complex scattering length that can be tuned via controlling the detuning and the intensity of the optical fields. Indeed, we develop a resonant two-channel model with a finite lifetime closed channel dimer and show that the complex scattering length a_c can be experimentally tuned across the entire lower half of the complex plane [13].

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*zyshi@lps.ecnu.edu.cn

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