Broken Global Symmetries and Defect Conformal Manifolds

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Just as exactly marginal operators allow one to deform a conformal field theory along the space of theories known as the conformal manifold, appropriate operators on conformal defects allow for deformations of the defects. When a defect breaks a global symmetry, there is a contact term in the conservation equation with an exactly marginal defect operator. The resulting defect conformal manifold is the symmetry breaking coset, and its Zamolodchikov metric is expressed as the two-point function of the exactly marginal operator. As the Riemann tensor on the conformal manifold can be expressed as an integrated four-point function of the marginal operators, we find an exact relation to the curvature of the coset space. We confirm this relation against previously obtained four-point functions for insertions into the $1/2$ BPS Wilson loop in $\mathcal{N}=4$ SYM and 3D $\mathcal{N}=6$ theory and the $1/2$ BPS surface operator of the 6D $\mathcal{N}=(2,0)$ theory.

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Introduction and summary.—Amongst all operators of a conformal field theory (CFT), exactly marginal operators hold a special place, allowing for continuous deformations of the theory, forming a space of CFTs known as the conformal manifold. Those are common in supersymmetric theories, but otherwise not. In this Letter we point out that in the presence of conformal defects, one can define a similar notion of *defect conformal manifold*, and it naturally arises whenever a global symmetry is broken by the defect with supersymmetry or without.

Theories with conformal boundaries or defects are ubiquitous and play an important role both in condensed matter physics and in string theory. They form a defect CFT (DCFT) involving operators on and off the defect. A relatively unexplored topic (notable exceptions are [1–5]) are marginal deformations of DCFTs by defect operators.

For a defect of dimension d, exactly marginal defect operators $\mathbb{O}_i$ have scaling dimension d, and the correlation function of defect operators ϕ in the deformed theory can be expressed as

$$\langle\langle \phi\phi' \dots \rangle\rangle_{\zeta^i} = \langle\langle e^{-\int \zeta^i \mathbb{O}_i d^d x} \phi\phi' \dots \rangle\rangle_0 \tag{1}$$

where ζ^i are local coordinates on the defect conformal manifold and the double bracket notation represents the correlation function in the DCFT normalized by the expectation value of the defect without insertions.

If the theory has a global symmetry G with current $J^{\mu a}$, broken by the defect to G', its conservation equation is modified to

$$\partial_\mu J^{\mu a} = \mathbb{O}_i(x_\parallel)\delta^{ia}\delta^{D-d}(x_\perp), \tag{2}$$

where i is an index for the broken generators, $x_\parallel$ the directions along the defect, and $x_\perp$ the transverse ones.

In a theory in D dimensions, $J^{\mu a}$ has dimension $D-1$. Therefore $\mathbb{O}_i$ has dimension d, so in the undeformed theory

$$\langle\langle \mathbb{O}_i(x_\parallel)\mathbb{O}_j(0)\rangle\rangle = \frac{C_{\mathbb{O}}\delta_{ij}}{x_\parallel^{2D}}. \tag{3}$$

$C_{\mathbb{O}}$ is fixed by the normalization of $J^{\mu a}$ and determines the Zamolodchikov metric locally as $g_{ij} = C_{\mathbb{O}}\delta_{ij}$ [6].

For $\phi = \phi' = \mathbb{O}_i$, Eq. (1) extends the Zamolodchikov metric beyond the flat space approximation. Differentiating Eq. (1) with respect to ζ^i gives the Riemann tensor [7]

$$\begin{aligned} R_{ijkl} = \int d^d x_1 d^d x_2 [&\langle\langle \mathbb{O}_j(x_1)\mathbb{O}_k(x_2)\mathbb{O}_i(0)\mathbb{O}_l(\infty)\rangle\rangle_c \\ &- \langle\langle \mathbb{O}_j(0)\mathbb{O}_k(x_2)\mathbb{O}_i(x_1)\mathbb{O}_l(\infty)\rangle\rangle_c], \end{aligned} \tag{4}$$

where $\langle\langle \dots \rangle\rangle_c$ indicates the connected correlator. This integral can be reduced to an integral over cross ratios [8]. See Eqs. (15), (30), and (33) below.

When $\mathbb{O}_i$ arise from symmetry breaking, the defect conformal manifold is the coset G/G'. It should be stressed that this statement is not evident in the correlation function calculation explained above, which is local on the conformal manifold. Rather it is a nontrivial statement that

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allows us to predict the form of the curvature (and higher derivatives of the metric).

Furthermore, the size of the coset is set by $C_{\mathbb{O}}$ which appears in the metric and curvature tensor, making Eq. (4) a nontrivial identity for integrated correlators. In the remainder of this Letter we apply this idea in three examples: 1D DCFT of $1/2$ Bogomol'yni-Prasad-Sommerfield (BPS) Wilson loops in $\mathcal{N}=4$ supersymmetry Yang-Mills (SYM) in 4D and in the $\mathcal{N}=6$ theory in 3D and the 2D DCFT of surface operators in the 6D $\mathcal{N}=(2,0)$ theory. We derive explicit expressions for the Riemann tensor and verify it with known results for the four-point functions.

Maldacena-Wilson loops.—The $1/2$ BPS Wilson loop along the Euclidean time direction in $\mathcal{N}=4$ SYM is

$$W=\mathrm{Tr}\mathcal{P}e^{\int(iA_0+\Phi_6)dt}. \tag{5}$$

The case of the $1/2$ BPS circular loop has some subtle differences [9,10], but here they are immaterial.

The defect CFT point of view on this observable was developed in Refs. [11–19]. The lowest dimension insertions are the six scalar fields Φ_I. Of them, Φ_6 is marginally irrelevant, "going up" the renormalization group flow to the UV non-BPS Wilson loop with no scalar coupling [10,18,20–24].

The remaining five scalars are to leading order $\mathbb{O}_i$ of Eq. (2), the finite deformations being broken SO(6) rotations

$$\Phi_6\to\cos\theta\Phi_6+\sin\theta\Phi_i\zeta^i/|\zeta|. \tag{6}$$

It is natural to identify $|\zeta|=2\tan(\theta/2)$, extending the local metric in Eq. (3) to the conformally flat metric on S^5

$$g_{ij}=\frac{C_{\mathbb{O}}\delta_{ij}}{(1+|\zeta|^2/4)^2}. \tag{7}$$

The two-point function of Φ_i is indeed as in Eq. (3) with C_Φ twice the bremsstrahlung function related to the expectation value of the circular Wilson loop [12,15,25,26]

$$C_\Phi=\frac{1}{\pi^2}\lambda\partial_\lambda\log\langle W_\circ\rangle=\frac{\sqrt{\lambda}}{2\pi^2}\frac{I_2(\sqrt{\lambda})}{I_1(\sqrt{\lambda})}+o(1/N^2), \tag{8}$$

where λ is the 't Hooft coupling and I_n are modified Bessel functions. At weak and strong coupling, this is

$$C_\Phi=\begin{cases}\frac{\lambda}{8\pi^2}-\frac{\lambda^2}{192\pi^2}+\frac{\lambda^3}{3072\pi^2}-\frac{\lambda^4}{46080\pi^2}+O(\lambda^5),\\ \frac{\sqrt{\lambda}}{2\pi^2}-\frac{3}{4\pi^2}+\frac{3}{16\pi^2\sqrt{\lambda}}+\frac{3}{16\pi^2\lambda}+O(\lambda^{-3/2}).\end{cases} \tag{9}$$

To express the four-point function of Φ_i, we define $\Phi^{(n)}=t_n^i\Phi_i(x_n)$ where t_n^i are constant five vectors. Then [19,27]

$$\langle\!\langle\Phi^{(1)}\Phi^{(2)}\Phi^{(3)}\Phi^{(4)}\rangle\!\rangle=C_\Phi^2\frac{t_{12}t_{34}}{x_{12}^2x_{34}^2}\mathcal{G}(\chi;\sigma,\tau),$$
$$\mathcal{G}(\chi;\sigma,\tau)=\sigma h_2(\chi)+\tau h_1(\chi)+h_0(\chi), \tag{10}$$

with $t_{nm}\equiv t_n\cdot t_m$, $x_{nm}=x_n-x_m$ and the cross ratios

$$\chi=\frac{x_{12}x_{34}}{x_{13}x_{24}}, \tag{11}$$

$$\sigma=\frac{t_{13}t_{24}}{t_{12}t_{34}}=\alpha\bar{\alpha},\qquad\tau=\frac{t_{14}t_{23}}{t_{12}t_{34}}=(1-\alpha)(1-\bar{\alpha}). \tag{12}$$

The functions in Eq. (10) are fixed by superconformal symmetry to take the form

$$h_0=\chi^2(f/\chi-f/\chi^2)',\qquad h_1=-\chi^2(f/\chi)',$$
$$h_2=\chi^2\mathbb{F}-\chi^2(f-f/\chi)', \tag{13}$$

where f is a function of χ, prime is the derivative with respect to χ, and $\mathbb{F}$ does not depend on χ and is determined from the topological sector of the correlators which occurs for the choice $\alpha=\bar{\alpha}=1/\chi$ [19,28,29].

Under crossing symmetry, $h_{0,1,2}$ transform as

$$\chi^2h_2(1-\chi)=(1-\chi)^2h_2(\chi),$$
$$\chi^2h_1(1-\chi)=(1-\chi)^2h_0(\chi),$$
$$\chi^2h_0(1-\chi)=(1-\chi)^2h_1(\chi). \tag{14}$$

As the four-point function depends only on the cross ratio χ, one can perform one of the two integrals in the curvature [Eq. (4)] explicitly and reduce the formula to [8]

$$R_{ijkl}=-\mathrm{RV}\int_{-\infty}^{+\infty}d\eta\log|\eta|[\langle\!\langle\Phi_i(1)\Phi_j(\eta)\Phi_k(\infty)\Phi_l(0)\rangle\!\rangle_c$$
$$+\langle\!\langle\Phi_i(0)\Phi_j(1-\eta)\Phi_k(\infty)\Phi_l(1)\rangle\!\rangle_c]. \tag{15}$$

RV denotes a particular prescription for regularizing and subtracting the divergences—a hard-sphere (point-splitting) cutoff followed by minimal subtraction [8].

We can further reduce the integral to $\eta\in(0,1)$, but need to account for the subtlety that in 1D the order of the insertions is meaningful, so

$$\langle\!\langle\Phi_i(1)\Phi_j(\eta)\Phi_k(\infty)\Phi_l(0)\rangle\!\rangle_c$$
$$=\begin{cases}\langle\!\langle\Phi_j(\eta)\Phi_l(0)\Phi_i(1)\Phi_k(\infty)\rangle\!\rangle, & \eta<0,\\ \langle\!\langle\Phi_l(0)\Phi_j(\eta)\Phi_i(1)\Phi_k(\infty)\rangle\!\rangle, & 0<\eta<1,\\ \langle\!\langle\Phi_l(0)\Phi_i(1)\Phi_j(\eta)\Phi_k(\infty)\rangle\!\rangle, & \eta>1.\end{cases} \tag{16}$$

To illustrate the calculation, we consider the contribution to Eq. (15) from the region $\eta\in(0,1)$. Using Eq. (10) and replacing η with the cross ratio χ [Eq. (11)] we find

$$-\int_0^1 d\chi \left(\frac{\log\chi}{\chi^2} [g_{li}g_{jk}h_2(\chi) + g_{lk}g_{ij}h_1(\chi) + g_{lj}g_{ik}h_0(\chi)] + \frac{\log(1-\chi)}{\chi^2} [g_{li}g_{jk}h_2(\chi) + g_{lj}g_{ik}h_1(\chi) + g_{lk}g_{ij}h_0(\chi)] \right). \quad (17)$$

Here we see all three tensor structures of bilinears of the metric, but after combining all three regions, the result must have the same tensor structure as a Riemann tensor. Finally using the crossing relations [Eq. (14)] we find

$$R_{ijkl} = 2(g_{ik}g_{jl} - g_{il}g_{jk}) \int_0^1 \frac{d\chi}{\chi^2} \log\chi (h_2 + h_1 - 2h_0). \quad (18)$$

Comparing with Eq. (10), the integrand can be written as

$$-\frac{2\log\chi}{\chi^2} \mathcal{G}(\chi; \sigma^*, \tau^*), \qquad \sigma^* = \tau^* = -1/2. \quad (19)$$

Equation (18) has the structure of the curvature of the maximally symmetric space $S^5 = \mathrm{SO}(6)/\mathrm{SO}(5)$ [Eq. (7)]. The integral in Eq. (18) is then related to the radius of the sphere. In Appendix A we simplify the integral to

$$\int_0^1 d\chi \left[\left(1 - \frac{2}{\chi^3}\right) f - \left(1 + \frac{1}{\chi}\right) \mathbb{F} \right]. \quad (20)$$

f and $\mathbb{F}$ were calculated at strong coupling by explicit world-sheet Witten diagrams [18] and extended up to fourth order in Ref. [27] based on the formalism in Refs. [16,19]. This results in

$$\mathbb{F} = -\frac{3}{\sqrt{\lambda}} + \frac{45}{8}\frac{1}{\lambda^{3/2}} + \frac{45}{4}\frac{1}{\lambda^2} + O(\lambda^{-5/2}). \quad (21)$$

Writing f in a power series

$$f(\chi, \lambda) = \sum_{n=1}^{\infty} \lambda^{-\frac{n}{2}} f^{(n)}(\chi), \quad (22)$$

the first one is [18]

$$f^{(1)} = -(1-\chi^2)\log(1-\chi) + \frac{\chi^3(2-\chi)}{(1-\chi)^2}\log(\chi) - \frac{\chi(1-2\chi)}{1-\chi}. \quad (23)$$

The integral in Eq. (20) can be computed for $f^{(1)}$ as well as for $f^{(2)}$, $f^{(3)}$, and $f^{(4)}$ found in Ref. [27], by integration by parts. We find for the Ricci scalar R of Eq. (18):

$$\frac{R}{20} = \frac{2\pi^2}{\sqrt{\lambda}} + \frac{3\pi^2}{\lambda} + \frac{15\pi^2}{4\lambda^{3/2}} + \frac{15\pi^2}{4\lambda^2} + O(\lambda^{-5/2}). \quad (24)$$

This exactly agrees with the large λ expansion of $1/C_\Phi$, whose inverse is in Eq. (9), as expected for a sphere of radius $\sqrt{C_\phi}$.

The relation between the integrated four-point function and C_Φ can also be deduced from the integral identities guessed in Ref. [30], as shown in Appendix B. Checks against weak coupling expressions [30–32] were also performed there.

1/2 BPS loop in 3D $\mathcal{N} = 6$ theory.—Another line defect with known four-point function is the 1/2 BPS Wilson loop of the $\mathcal{N} = 6$ theory in 3D [33,34]. The SU(4) R symmetry is broken by the defect to SU(3), so the defect conformal manifold is $\mathbb{CP}^3$. Now the marginal operators are chiral and have the structure of a supermatrix. The Zamolodchikov metric takes the form

$$g_{i\bar{j}} = \langle\langle \mathbb{O}_i(0)\bar{\mathbb{O}}_{\bar{j}}(1) \rangle\rangle = 4B_{1/2}\delta_{ij}, \quad (25)$$

where $B_{1/2} = \sqrt{2\lambda}/4\pi + \cdots$ is the bremsstrahlung function for these operators [35–37].

For the four-point function we need to distinguish two orderings [34]:

$$\langle\langle \mathbb{O}_i(x_1)\bar{\mathbb{O}}_{\bar{j}}(x_2)\mathbb{O}_k(x_3)\bar{\mathbb{O}}_{\bar{l}}(x_4) \rangle\rangle = \frac{g_{i\bar{j}}g_{k\bar{l}}K_1 - g_{i\bar{l}}g_{k\bar{j}}K_2}{x_{12}^2 x_{34}^2},$$
$$\langle\langle \mathbb{O}_i(x_1)\bar{\mathbb{O}}_{\bar{j}}(x_2)\bar{\mathbb{O}}_{\bar{k}}(x_3)\mathbb{O}_l(x_4) \rangle\rangle = \frac{g_{i\bar{j}}g_{l\bar{k}}H_1 - g_{i\bar{k}}g_{l\bar{j}}H_2}{x_{12}^2 x_{34}^2}. \quad (26)$$

Here $x_1 < x_2 < x_3 < x_4$ and K_i, H_i depend on the cross ratio χ [Eq. (11)]. Other orderings can be determined by conformal invariance.

The curvature now splits according to chirality:

$$R_{ij\bar{k}\bar{l}} = (g_{i\bar{l}}g_{j\bar{k}} - g_{i\bar{k}}g_{j\bar{l}})\mathcal{R}_1,$$
$$R_{i\bar{j}k\bar{l}} = (g_{i\bar{l}}g_{k\bar{j}} + g_{i\bar{j}}g_{k\bar{l}})\mathcal{R}_2. \quad (27)$$

Plugging the expressions from Eq. (26) into Eq. (15) and accounting for the ordering in Eq. (16), we find

$$\mathcal{R}_1 = \int_0^1 \frac{d\chi}{\chi^2} \left[\log\frac{\chi}{1-\chi}(K_1 + K_2) + 2\log\chi(H_1 + H_2) \right],$$
$$\mathcal{R}_2 = \int_0^1 \frac{d\chi}{\chi^2} [\log(1-\chi)(2H_1 - 2H_2 - K_1) + \log\chi(2H_2 + K_2)]. \quad (28)$$

The functions H_i and K_i are expressed in terms of functions $h(\chi)$ defined for $\chi \in (0, 1)$ and $f(z)$ with $z = \chi/(\chi - 1) < 0$ as

$$H_1 = \chi^2[\chi(h/\chi)']', \qquad H_2 = \chi^2(\chi h')',$$
$$K_1 = z^2[z(f/z)']', \qquad K_2 = z^2(zf')'. \quad (29)$$

In Appendix C we show, based on crossing symmetry and assumptions on the behavior of the functions in the limits $\chi \to 0,\, 1$, that $\mathcal{R}_1 = 0$. Likewise, using Eq. (C6), we find after repeated integration by parts

$$\mathcal{R}_2 = -2\int_0^1 \frac{h(\chi)}{\chi(1-\chi)}d\chi + 2\int_{-\infty}^0 \frac{f(z)}{(z-1)^2}dz. \qquad (30)$$

In Ref. [34] the functions h and f were evaluated at first order at strong coupling from the analytic bootstrap

$$h^{(1)} \propto -\frac{(1-\chi)^3}{\chi}\log(1-\chi) + \chi(3-\chi)\log\chi + \chi - 1,$$
$$f^{(1)} \propto -\frac{(1-z)^3}{z}\log(1-z) + z(3-z)\log|z| + z - 1, \qquad (31)$$

with the same proportionality constant $1/(2\pi\sqrt{2\lambda})$, determined by explicit Witten diagram calculations. Evaluating the integral [Eq. (30)], we find $\mathcal{R}_2 = \pi/\sqrt{2\lambda}$, so the Ricci scalar agrees to leading order with $12/2B_{1/2}$, as expected for $\mathbb{CP}^3$. This calculation also serves as an independent derivation of the proportionality constant without relying on the AdS/CFT correspondence and could be used to determine further unknowns in higher loop calculations.

Surface operators in 6D.—The 6D $\mathcal{N} = (2,0)$ theory has 1/2 BPS surface operators [38] with the geometry of the plane or the sphere. In the absence of a Lagrangian description, we cannot write an expression like Eq. (5), yet many properties of the surface operators are known. In particular, they carry a representation of the A_{N-1} algebra of the theory [39–41], and we focus on the fundamental representation, described by an M2-brane in $\mathrm{AdS}_7 \times S^4$ [42].

The defect CFT approach to surface operators was developed in Ref. [43]. In this case the scalar $\mathbb{O}_i$ (2) is associated to breaking of SO(5) R symmetry and is of dimension 2. As shown in Ref. [43], the normalization constant $C_{\mathbb{O}}$ in Eq. (3) is now related to the anomaly coefficients c and a_2 [44–51] by

$$C_{\mathbb{O}} = \frac{c}{\pi^2} = -\frac{a_2}{\pi^2} = \frac{1}{\pi^2}\left(N - \frac{1}{2} - \frac{1}{2N}\right). \qquad (32)$$

The curvature tensor [Eq. (4)] is now written in terms of the complex cross ratio [8]

$$R_{ijkl} = -2\pi \mathrm{RV}\int d^2\eta \log|\eta| \langle\!\langle \mathbb{O}_i(1)\mathbb{O}_j(\eta)\mathbb{O}_k(\infty)\mathbb{O}_l(0)\rangle\!\rangle_c. \qquad (33)$$

We further simplify the integral by mapping $|\eta| > 1$ to $|\eta| < 1$ by a conformal transformation, giving

$$R_{ijkl} = -2\pi\int_{|\eta|<1} d^2\eta \log|\eta| [\langle\!\langle \mathbb{O}_l(0)\mathbb{O}_j(\eta)\mathbb{O}_i(1)\mathbb{O}_k(\infty)\rangle\!\rangle_c - \langle\!\langle \mathbb{O}_l(0)\mathbb{O}_i(\eta)\mathbb{O}_j(1)\mathbb{O}_k(\infty)\rangle\!\rangle_c]. \qquad (34)$$

In this expression, η is equal to the cross ratio χ as defined in Eq. (36).

The analog of Eq. (10), Eq. (26) is

$$\langle\!\langle \mathbb{O}^{(1)}\mathbb{O}^{(2)}\mathbb{O}^{(3)}\mathbb{O}^{(4)}\rangle\!\rangle = C_{\mathbb{O}}^2 \frac{t_{12}t_{34}}{\vec{x}_{12}^4\vec{x}_{34}^4}\mathcal{G}(\chi,\bar{\chi};\alpha,\bar{\alpha}),$$
$$\mathcal{G} = \sigma h_2(\chi,\bar{\chi}) + \tau h_1(\chi,\bar{\chi}) + h_0(\chi,\bar{\chi}) + \mathcal{G}_2, \qquad (35)$$

where the cross ratios are as in Eq. (12) and

$$U = \frac{\vec{x}_{12}^2\vec{x}_{34}^2}{\vec{x}_{13}^2\vec{x}_{24}^2} = \chi\bar{\chi}, \qquad V = \frac{\vec{x}_{14}^2\vec{x}_{23}^2}{\vec{x}_{13}^2\vec{x}_{24}^2} = (1-\chi)(1-\bar{\chi}). \qquad (36)$$

The crossing equations for h_i are as in Eq. (14) but with $|\chi|^2$ and $|1-\chi|^2$.

$\mathcal{G}_2$ in Eq. (35) is parity odd, and using the symmetry of the integration domain in Eq. (34), it does not contribute to the curvature tensor. This is easy to verify for the expression in Eq. (38) by changing the integration variables to U, V. The same should hold to all orders.

The curvature tensor is then

$$R_{ijkl} = -2\pi(g_{ik}g_{jl} - g_{il}g_{jk})\int_{|\chi|<1} d^2\chi \frac{\log|\chi|}{|\chi|^2}(h_0 - h_2). \qquad (37)$$

The integrand is in fact the parity even part of $\mathcal{G}\log|\chi|/|\chi|^2$ with $\sigma^* = -1$ and $\tau^* = 0$.

The four-point function was calculated to first order at large N from the M2-brane with the geometry of AdS_3 in AdS_7 [52] resulting in

$$h_0 = \frac{6}{N}U^2[(V - U + 1)\bar{D}_{3333} + U\bar{D}_{3322} - \bar{D}_{2222}],$$
$$h_1 = \frac{6}{N}U^2[(U - V + 1)\bar{D}_{3333} + \bar{D}_{3223} - \bar{D}_{2222}].$$
$$h_2 = \frac{6}{N}U^2[(U + V - 1)\bar{D}_{3333} + \bar{D}_{3232} - \bar{D}_{2222}],$$
$$\mathcal{G}_2 = -\frac{9}{2N}U^2(\chi - \bar{\chi})(\alpha - \bar{\alpha})\bar{D}_{3333}. \qquad (38)$$

The $\bar{D}$ functions are given in Ref. [53]. The expressions here are 16 times smaller than in Ref. [52] because of a difference in the normalization of $\mathbb{O}_i$ compared with the S^4 coordinates y_i in Ref. [52].

By numerical integration, we confirm that

$$R_{ijkl} = \frac{N}{\pi^2}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}), \qquad (39)$$

as expected for $S^4 = \mathrm{SO}(5)/\mathrm{SO}(4)$ of radius squared $C_{\mathbb{O}} \sim N/\pi^2$ in the large N limit (32).

Discussion.—The main result of this Letter is that much like exactly marginal bulk operators, exactly marginal defect operators lead to defect conformal manifolds with Zamolodchikov metrics and with Riemann curvature given by an integrated four-point function [Eq. (4)]. In analogy to Goldstone's theorem [54,55], such marginal defect operators are guaranteed to exist when the defect breaks a global symmetry. Thus unlike bulk marginal operators, exactly marginal defect operators are ubiquitous.

We checked Eq. (4) against known four-point functions in three different examples and found a match with the curvature of the metric as in Eq. (7). While these examples are in supersymmetric theories, symmetry breaking defects exist in many CFTs. For example, for the critical O(N) model [56] defects were studied in Refs. [57–61], and our analysis applies there and possibly has experimental signatures.

Analogous constraints can be found for higher point functions (see, e.g., Ref. [32]). The fully integrated correlators are again derivatives of the Zamolodchikov metric, and therefore fixed by the metric of the manifold.

This integral constraint can be incorporated into bootstrap algorithms. This was implemented in the numerical analysis in Ref. [30] leading to far improved accuracy. Likewise, it can be implemented in analytic studies, replacing the need for Witten diagram calculations in the 3D $\mathcal{N} = 6$ example [34] and extending it to higher orders.

The same analysis can be applied to Wilson loops and surface operators in higher dimensional representations, where a lot of the required calculations have already been done [45–50,62–69].

Richer defect conformal manifolds can arise from less symmetric symmetry breaking than the examples discussed here. Such defects will have a variety of marginal operators with different two-point functions, and one could find integral constraints for different components of the Riemann tensor.

Defect conformal manifolds do not require broken symmetries. One natural setting is in 3D theories, where line operators are known to have multiple marginal couplings [70–75]. It is also natural to look at systems with both defect and bulk marginal operators to construct richer structures. Some work in that direction is in Ref. [5].

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Appendix A. Simplifying the integral of $f(\chi)$.—RTo simplify the integral in Eq. (18), we plug in the expressions [Eq. (13)] to find

$$\int_0^1 d\chi\, \log\chi \left(\chi\mathbb{F} - f - \frac{2f}{\chi} + \frac{2f}{\chi^2}\right)'. \tag{A1}$$

Integrating by parts gives the boundary term

$$\log\chi \left(\chi\mathbb{F} - f - \frac{2f}{\chi} + \frac{2f}{\chi^2}\right)\Big|_0^1. \tag{A2}$$

Noticing the boundary behavior [27],

$$f(\chi) \sim \begin{cases} -\mathbb{F}\chi^2/2, & \chi \to 0, \\ \mathbb{F}/2, & \chi \to 1, \end{cases} \tag{A3}$$

the only nonvanishing term is a divergence $\mathbb{F}\log 0$ which we express as an integral and combine with the result of integration by parts:

$$-\int_0^1 d\chi \left(\frac{\mathbb{F}}{\chi} + \mathbb{F} - \frac{f}{\chi} - \frac{2f}{\chi^2} + \frac{2f}{\chi^3}\right). \tag{A4}$$

The crossing relation $\chi^2 f(1-\chi) = -(1-\chi)^2 f(\chi)$ leads to the integral identities

$$\int_0^1 d\chi \frac{f}{\chi^2} = 0, \qquad \int_0^1 d\chi \frac{f}{\chi} = \int_0^1 d\chi f. \tag{A5}$$

This finally allows us to further simplify Eq. (A4) to Eq. (20).

Appendix B: Relation to integral identities of Ref. [30].—A linear combination of the two integral constraints noticed in Ref. [30] is

$$\int_0^1 d\chi \left(3\frac{f}{\chi} - 2\frac{\delta G}{\chi^2}(1 + \log\chi)\right) = \frac{1}{2C_\Phi} + 3\mathbb{F}, \tag{B1}$$

where δG, in the notations of Sec. (Maldacena-Wilson loops), is

$$\frac{\delta G}{\chi^2} = \mathbb{F} - \partial_\chi \left[\left(1 - \frac{1}{\chi} + \frac{1}{\chi^2}\right) f\right]. \tag{B2}$$

Using Eq. (A3), the left-hand side of Eq. (B1) is

$$2\mathbb{F} + \mathbb{F}\log\epsilon + \int_0^1 d\chi \left[3\frac{f}{\chi} - 2\mathbb{F}(1 + \log\chi) - 2\left(\frac{1}{\chi} - \frac{1}{\chi^2} + \frac{1}{\chi^3}\right) f\right]. \tag{B3}$$

Then using Eq. (A5), this is

$$\int_0^1 d\chi \left[\left(2-\frac{1}{\chi}\right)\mathbb{F}+\left(1-\frac{2}{\chi^3}\right)f\right], \tag{B4}$$

and finally using our result for the integral [Eq. (20)], this is the right-hand side of Eq. (B1), proving it.

Appendix C: Integral identities for H_i, K_i.—To simplify the integrals in Eq. (28) we note that the crossing relation $(1-\chi)^2K_1(\chi)=-\chi^2K_2(1-\chi)$ implies

$$\int_0^1 \frac{d\chi}{\chi^2}\log\chi K_i = -\int_0^1 \frac{d\chi}{\chi^2}\log(1-\chi)K_{3-i}, \tag{C1}$$

which yields

$$\mathcal{R}_1 = 2\int_0^1 \frac{d\chi}{\chi^2}\left[\log\frac{\chi}{1-\chi}K_1+\log\chi(H_1+H_2)\right]. \tag{C2}$$

Using Eq. (29), this is a total derivative. Furthermore, assuming the asymptotics

$$h(\chi)\sim\begin{cases} a_0\chi+a_1\chi\log\chi, & \chi\to 0,\\ a_2(\chi-1), & \chi\to 1,\end{cases}$$
$$f(z)\sim\begin{cases} b_0z+b_1z\log|z|, & z\to 0,\\ b_2+b_3\log|z|, & z\to-\infty,\end{cases} \tag{C3}$$

we find

$$\int_0^1 \frac{d\chi}{\chi^2}\log\chi H_2 = 0 \tag{C4}$$

and

$$\mathcal{R}_1 = 2a_0-2b_0. \tag{C5}$$

This indeed vanishes in the perturbative analytic bootstrap, where $a_0=b_0=-b_2$ as a consequence of the crossing of h and a braiding relation to f [34].

By using Eqs. (C1), (C3), and (C4) we can also simplify the second line of Eq. (28) to

$$\mathcal{R}_2 = 2\int_0^1 \frac{d\chi}{\chi^2}\log(1-\chi)(H_1-H_2-K_1). \tag{C6}$$

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