Incompressible Polar Active Fluids with Quenched Random Field Disorder in Dimensions $d > 2$

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We present a hydrodynamic theory of incompressible polar active fluids with quenched random field disorder. This theory shows that such fluids can overcome the disruption caused by the quenched disorder and move coherently, in the sense of having a nonzero mean velocity in the hydrodynamic limit. However, the scaling behavior of this class of active systems cannot be described by linearized hydrodynamics in spatial dimensions between 2 and 5. Nonetheless, we obtain the exact dimension-dependent scaling exponents in these dimensions.

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One of the most important themes of condensed matter physics is the competition between order and disorder. One of the most powerful results on this topic is the Mermin-Wagner-Hohenberg theorem [1,2], which states that *equilibrium* systems *cannot* spontaneously break a continuous symmetry in spatial dimensions $d \le 2$ at nonzero temperature. Much of the current interest in "active matter" is stimulated by the discovery [3–6] that nonequilibrium "movers" *can* spontaneously break a continuous symmetry (rotation invariance) in the presence of noise even in $d = 2$, by "flocking"—that is, moving coherently with a nonzero spatially averaged velocity $\langle \boldsymbol{v}(\boldsymbol{r}, t)\rangle \neq \mathbf{0}$.

In equilibrium systems, even *arbitrarily weak* quenched (i.e., static) random fields destroy long-ranged ferromagnetic order in all spatial dimensions $d \le 4$ [7–10]. This raises the questions: what is the effect of disorder on *active* materials [11–20] and, more precisely, can an *ordered* polar active fluid form when quenched random field disorder is present?

The answers to these questions are crucial for understanding how coherent motion is possible in any realistic biophysical situation. Consider, for example, a large cluster of cells moving through an extracellular polymerized matrix. That matrix will inevitably contain local random spatial heterogeneities that are fixed on the experimentally relevant timescale (i.e., quenched) [21]. Is there a maximal cluster size, or can arbitrarily large clusters move coherently in this disordered matrix?

In this Letter, we investigate these questions for *incompressible* polar active fluids. Models assuming incompressibility have been extensively and successfully used to describe cell layers [22] and bacterial fluids [23–26]. While these systems are generally spatiotemporally chaotic [27–32], which is accounted for in the models by the introduction of a negative viscosity, the same model with a positive viscosity should account for coherent motion as observed in cellular clusters, for instance. Because of either steric interaction in the high packing limit [33] or cell-cell avoidance by long-distance sensing through fast-diffusing signaling molecules, incompressibility is natural in cellular materials. Further, an even wider class of living materials ranging from intracellular gels [34–37] to cells [38–41] to cell layers and aggregates [39,42] have been modeled as two-component incompressible active fluids [43–46]. If the birth and death of active components are taken into account in these materials (as they should be in most of these systems at long enough timescales), all of them are again described by the model we consider.

We show in this Letter that incompressible active fluids can move coherently even through disordered matrices in all spatial dimensions $d > 2$, i.e., a polar phase survives in the presence of a finite amount of quenched random field disorder. Furthermore, we find that for $2 < d < 5$, there is a *breakdown of linearized hydrodynamics*, just as there is in simple thermal fluids [47] for $d \le 2$, and flocks without quenched disorder for $d \le 4$ [4,5]. That is, the spatiotemporal scaling of fluctuations in these systems is *not* correctly given by a linear theory, due to strong nonlinear coupling between large fluctuations. Nonetheless, there *is* universal scaling of correlations in this range of spatial dimensions, and we have been able to determine its scaling exponents *exactly*.

In previous papers [48], we have shown that incompressible polar active fluids retain long-range order, even in

$d=2$, in the presence of quenched random field disorder. Since the effect of fluctuations is expected to decrease with increasing dimensionality, this would seem to directly imply long-range order for all $d>2$ as well. However, the incompressible flock in $d=2$ is qualitatively distinct from that in higher dimensions [49,50] since it lacks a true "soft" or hydrodynamic mode for most directions of wave vector because incompressibility constrains the dynamics to a *much* greater degree in $d=2$. As a result, the findings in Ref. [48] do not automatically imply long-range order in $d>2$. Our conclusion here that there *is* long-range order in all $d>2$ is therefore nontrivial and new.

In the following, we will first present a hydrodynamic theory of incompressible polar active fluids with *both* annealed disorder (which represents endogenous fluctuations due to, e.g., errors made by a motile agent while attempting to follow its neighbors [3]) and quenched random field disorder. We then apply a dynamic renormalization group (DRG) analysis to obtain the exponents that fully characterize the scaling behavior of the system in the moving phase. Specifically, choosing our coordinates so that the x axis is along the mean velocity $\langle \mathbf{v}\rangle$ of the flock (i.e., $\langle \mathbf{v}\rangle = v_0\hat{\mathbf{x}}$), and defining the fluctuation $\mathbf{u}(\mathbf{r},t)$ of the velocity at the point $\mathbf{r}$ at time t away from this mean velocity via $\mathbf{u}(\mathbf{r},t)=\mathbf{v}(\mathbf{r},t)-v_0\hat{\mathbf{x}}$, we find that the two point correlations $\langle \mathbf{u}(\mathbf{r},t)\cdot\mathbf{u}(\mathbf{0},0)\rangle$ of these fluctuations is of the form

$$\langle \mathbf{u}(\mathbf{r},t)\cdot\mathbf{u}(\mathbf{0},0)\rangle = r_\perp^{2\chi} G_Q\left(\frac{|x|}{r_\perp^{\zeta}}\right) + r_\perp^{2\chi'} G_A\left(\frac{|x-\gamma t|}{r_\perp^{\zeta'}}, \frac{|t|}{r_\perp^{z'}}\right), \tag{1}$$

where G_Q and G_A are universal scaling functions, "$\perp$" denotes directions perpendicular to $\hat{\mathbf{x}}$, γ is a model-dependent nonuniversal speed, and the universal scaling exponents are given by

$$\zeta = \frac{d+1}{3} = \frac{4}{3}, \qquad \chi = \frac{2-d}{3} = -\frac{1}{3}, \tag{2a}$$

$$\zeta' = \frac{2(d+1)}{d+7} = \frac{4}{5}, \qquad z' = \frac{4(d+1)}{d+7} = \frac{8}{5}, \tag{2b}$$

$$\chi' = -\left[\frac{d^2+4d-9}{2(d+7)}\right] = -\frac{3}{5} \tag{2c}$$

for spatial dimensions between 2 and 5, where the final equalities hold in the physically relevant case $d=3$.

Hydrodynamic description.—We start with the hydrodynamic equation of motion (EOM) of a generic incompressible polar active fluid with both quenched and annealed fluctuations constructed using symmetry arguments [4,5]. The *only* hydrodynamic variable we need to account for is the velocity field $\mathbf{v}$. However, in contrast to the Navier-Stokes equations for passive incompressible fluids, $\mathbf{v}$ is hydrodynamic not because it is conserved—it is not, since momentum is not conserved—but because it is a broken symmetry variable (more precisely, certain components of it are). Our EOM also contains terms that violate momentum conservation and Galilean invariance because the motile agents move through a frictional (and disordered) medium. Furthermore, because the system is nonequilibrium, many terms forbidden in equilibrium are allowed here [51]. These considerations imply the following EOM [4,5]:

$$\partial_t \mathbf{v} + \lambda_1(\mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla P - (\mathbf{v}\cdot\nabla P_1)\mathbf{v} + U(|\mathbf{v}|)\mathbf{v} + \mu_1\nabla^2\mathbf{v} + \mu_2(\mathbf{v}\cdot\nabla)^2\mathbf{v} + \mathbf{f}_Q + \mathbf{f}_A, \tag{3}$$

where the "pressure" P acts as a Lagrange multiplier to enforce the incompressibility constraint: $\nabla\cdot\mathbf{v}=0$, the "anisotropic" pressure is an arbitrary function of the speed $|\mathbf{v}|$, and $U(|\mathbf{v}|)<0$ for $|\mathbf{v}|>v_0$ and $U(|\mathbf{v}|)>0$ for $|\mathbf{v}|<v_0$; these last two inequalities ensure that the system has a nonzero preferred speed v_0, which allows it to be in the ordered phase. Furthermore, $\mathbf{f}_Q$ and $\mathbf{f}_A$ are respectively the quenched and annealed noises, which have zero means and correlations of the form

$$\langle f_Q^i(\mathbf{r},t) f_Q^j(\mathbf{r}',t')\rangle = 2D_Q\delta_{ij}\delta^d(\mathbf{r}-\mathbf{r}'), \tag{4a}$$

$$\langle f_A^i(\mathbf{r},t) f_A^j(\mathbf{r}',t')\rangle = 2D_A\delta_{ij}\delta^d(\mathbf{r}-\mathbf{r}')\delta(t-t'), \tag{4b}$$

where the indices i, j enumerate the spatial coordinates. In the EOM, Eq. (3), we have only included terms that are relevant to the universal scaling behavior, based on the DRG analysis below.

We focus on the broken-symmetry moving phase, and consider the local velocity deviation $\mathbf{u}(\mathbf{r},t)$, from the mean flow $v_0\hat{\mathbf{x}}$: $\mathbf{u}=\mathbf{v}-v_0\hat{\mathbf{x}}$, whose EOM is obtained from Eq. (3) by keeping only relevant terms (some of which, however, are nonlinear):

$$\partial_t u_x = -\partial_x P - (\gamma+b)\partial_x u_x - \alpha\left(u_x + \frac{u^2}{2v_0}\right) + f_Q^x + f_A^x, \tag{5a}$$

$$\partial_t \mathbf{u}_\perp = -\nabla_\perp P - \gamma\partial_x\mathbf{u}_\perp - \lambda_1(\mathbf{u}_\perp\cdot\nabla_\perp)\mathbf{u}_\perp + \mathbf{f}_Q^\perp + \mathbf{f}_A^\perp - \frac{\alpha}{v_0}\left(u_x + \frac{u^2}{2v_0}\right)\mathbf{u}_\perp + \mu_\perp\nabla_\perp^2\mathbf{u}_\perp + \mu_x\partial_x^2\mathbf{u}_\perp, \tag{5b}$$

where $\gamma \equiv \lambda_1 v_0$, $\alpha \equiv -v_0(dU/d|\mathbf{v}|)_{|\mathbf{v}|=v_0}$, $b \equiv v_0^2(dP_1/d|\mathbf{v}|)_{|\mathbf{v}|=v_0}$, $\mu_\perp=\mu_1$, and $\mu_x=\mu_1+\mu_2 v_0^2$.

Linear theory.—First we examine the linearized version of Eqs. (5a) and (5b). In terms of the spatiotemporally Fourier-transformed field $\mathbf{u}(\mathbf{q},\omega)=(2\pi)^{-(d+1)/2}\int dt d^d r\, e^{-i(\mathbf{q}\cdot\mathbf{r}-\omega t)}\mathbf{u}(\mathbf{r},t)$, the linearized EOMs read

$$[-i(\omega-(\gamma+b)q_x)+\alpha]u_x = -iq_x P + f_Q^x + f_A^x, \tag{6a}$$

$$[-\mathrm{i}(\omega-\gamma q_x)+\Gamma(\boldsymbol{q})]u_L=-\mathrm{i}q_\perp P+f_Q^L+f_A^L, \quad (6b)$$

$$[-\mathrm{i}(\omega-\gamma q_x)+\Gamma(\boldsymbol{q})]\boldsymbol{u}_T=\boldsymbol{f}_Q^T+\boldsymbol{f}_A^T, \quad (6c)$$

where we have decomposed $\boldsymbol{u}_\perp$ into a single "longitudinal" component $u_L(\boldsymbol{q},\omega)\hat{\boldsymbol{q}}_\perp$ along $\hat{\boldsymbol{q}}_\perp$ and $(d-2)$ "transverse" components $\boldsymbol{u}_T(\boldsymbol{q},\omega)$ normal to $\hat{\boldsymbol{q}}_\perp$, i.e., $\boldsymbol{u}_\perp(\boldsymbol{q},\omega)=u_L(\boldsymbol{q},\omega)\hat{\boldsymbol{q}}_\perp+\boldsymbol{u}_T(\boldsymbol{q},\omega)$, and made the same decomposition for $\boldsymbol{f}_{A/Q}$. We have also introduced the $\boldsymbol{q}$-dependent damping coefficient:

$$\Gamma(\boldsymbol{q})\equiv\mu_\perp q_\perp^2+\mu_x q_x^2. \quad (7)$$

We now calculate the autocorrelation functions in this linear theory. Since the EOM of $\boldsymbol{u}_T$ is completely decoupled from the other two modes, its autocorrelation function can be obtained immediately:

$$\langle\boldsymbol{u}_T(\boldsymbol{q},\omega)\cdot\boldsymbol{u}_T(\boldsymbol{q}',\omega')\rangle=C_A^T(\boldsymbol{q},\omega)\delta(\omega+\omega')\delta(\boldsymbol{q}+\boldsymbol{q}')+C_Q^T(\boldsymbol{q})\delta(\omega)\delta(\omega')\delta(\boldsymbol{q}+\boldsymbol{q}'), \quad (8)$$

where

$$C_A^T(\boldsymbol{q},\omega)=\frac{2D_A(d-2)}{(\omega-\gamma q_x)^2+[\Gamma(\boldsymbol{q})]^2}, \quad (9a)$$

$$C_Q^T(\boldsymbol{q})=\frac{4\pi(d-2)D_Q}{\gamma^2q_x^2+[\Gamma(\boldsymbol{q})]^2}, \quad (9b)$$

and the subscripts A and Q denote the annealed and quenched parts, respectively. The correlation of $\boldsymbol{u}_T$ constitutes the most divergent part of the velocity correlator, since u_x is the "massive" mode, and u_L is "almost massive" because u_x is enslaved to it by the incompressibility condition $q_xu_x+q_\perp u_L=0$. This renders it impossible, for most directions of $\boldsymbol{q}$, to create a nonzero u_L without also creating a massive u_x field along with it. We explicitly calculate the autocorrelations of u_x and u_L in the Supplemental Material (SM) [52].

Using Eqs. (8) and (9), the fluctuations of $\boldsymbol{u}$ in real space and time can be obtained by integrating over all wave vectors $\boldsymbol{q}$ and frequencies ω. Performing the frequency integral gives

$$\langle|\boldsymbol{u}(\boldsymbol{r},t)|^2\rangle=\frac{(d-2)}{(2\pi)^d}\int d^d\boldsymbol{q}\left[\frac{D_A}{\Gamma(\boldsymbol{q})}+\frac{2D_Q}{\gamma^2q_x^2+[\Gamma(\boldsymbol{q})]^2}\right]. \quad (10)$$

In the infrared limit ($\boldsymbol{q}\to\boldsymbol{0}$), the second term in the integrand (due to the quenched disorder D_Q) is more divergent and thus dominates the fluctuations in the system. The integral of this term is logarithmically divergent in $d=3$, which implies quasi-long-range orientational order at this lower critical dimension. Further, the scaling of Eqs. (9) and (10) yields the scaling exponents for the quenched and annealed fluctuations in this linear theory:

$$\zeta_{\rm lin}=2, \qquad \chi_{\rm lin}=\frac{3-d}{2}, \quad (11a)$$

$$\zeta'_{\rm lin}=1, \qquad \chi'_{\rm lin}=\frac{2-d}{2}, \qquad z'_{\rm lin}=2. \quad (11b)$$

However, all of the above conclusions are modified by the nonlinearity in the EOM when $d<5$. In particular, the flock moves coherently, i.e., has long-range order, for all $d>2$. That is, the nonlinearity changes the lower critical dimension $d_{\rm LC}$ of this system from the linear theory's prediction $d_{\rm LC}=3$ to $d_{\rm LC}=2$.

Nonlinear theory.—As indicated by the linear theory, fluctuations in $\boldsymbol{u}$ are dominated by those of $\boldsymbol{u}_\perp$ (more precisely the transverse components of $\boldsymbol{u}_\perp$, i.e., $\boldsymbol{u}_T$). The full EOM of $\boldsymbol{u}_\perp$, Eq. (5b), after eliminating all irrelevant terms [52], becomes

$$\partial_t\boldsymbol{u}_\perp=-\nabla_\perp P-\gamma\partial_x\boldsymbol{u}_\perp-\lambda_1(\boldsymbol{u}_\perp\cdot\nabla_\perp)\boldsymbol{u}_\perp+\mu_\perp\nabla_\perp^2\boldsymbol{u}_\perp+\mu_x\partial_x^2\boldsymbol{u}_\perp+\boldsymbol{f}_Q^\perp+\boldsymbol{f}_A^\perp. \quad (12)$$

We will now obtain exact scaling exponents from Eq. (12) using a DRG argument [47]. In this DRG analysis, we first decompose the field $\boldsymbol{u}_\perp$ into the rapidly varying and slowly varying parts, which are supported in the small- and large-momentum space, respectively. We then average the EOM over the rapidly varying fields to get an effective EOM for the slowly varying fields. In this process the various coefficients in the EOM get renormalized and this renormalization can be represented by Feynman diagrams. We will therefore refer to all corrections that arise due to this part of the DRG process as "graphical corrections." Next we rescale the time, lengths, and the field as follows:

$$t\to te^{z\ell}, \quad x\to xe^{\zeta\ell}, \quad r_\perp\to r_\perp e^{\ell}, \quad \boldsymbol{u}_\perp\to\boldsymbol{u}_\perp e^{\chi\ell}, \quad (13)$$

to restore the supporting momentum space (i.e., the Brillouin zone) back to its original size. This procedure is repeated infinitely, leading to the following recursion relations for the various coefficients:

$$\frac{d\mu_\perp}{d\ell}=(z-2+\eta_\perp)\mu_\perp, \quad (14a)$$

$$\frac{d\mu_x}{d\ell}=(z-2\zeta)\mu_x, \quad (14b)$$

$$\frac{d\gamma}{d\ell}=(z-\zeta)\gamma, \quad (14c)$$

$$\frac{d\lambda_1}{d\ell}=(z+\chi-1)\lambda_1, \quad (14d)$$

$$\frac{dD_Q}{d\ell} = [2z - 2\chi - \zeta - (d-1)]D_Q, \tag{14e}$$

$$\frac{dD_A}{d\ell} = [z - 2\chi - \zeta - (d-1)]D_A, \tag{14f}$$

where $\eta_\perp$ represents the graphical correction to $\mu_\perp$—the only graphical correction to the DRG flow equations above. We explain why there are no other graphical corrections in the SM [52].

This quantity $\eta_\perp$ is a function of *all* of the parameters $\mu_\perp$, μ_x, γ, λ_1, and D_Q. We only know how to calculate its dependence on those parameters in perturbation theory, which can be organized by Feynman graphs, as described in [47].

However, because only $\mu_\perp$ gets any graphical corrections, we can actually determine the value $\eta_\perp(\mu_\perp, \mu_x, \gamma, \lambda_1, D_Q)$ *must* take on at the fixed point of the DRG *without* calculating this functional dependence at all! We will explain how this is done below.

Having established the form of the DRG recursion relations, Eq. (14)—that is, the fact that *only* $\mu_\perp$ gets any graphical corrections [those denoted by $\eta_\perp$ in Eq. (14a)]—we will now show that the quenched random field disorder is *always* relevant at the "annealed" fixed point that controls the ordered phase in the absence of quenched disorder, even when graphical corrections are taken into account, and determine the universal scaling exponents, Eq. (2), in the presence of quenched random field disorder *exactly*.

Note that the *form* of the recursion relations is exactly the same in the absence of quenched disorder as in its presence; that is, the recursion relations, Eq. (14), continue to hold, albeit with different values for $\eta_\perp$ depending on whether quenched disorder is present or not. This is because the arguments presented in the SM [52] for the quenched problem apply equally well to the annealed problem. (The argument for the nonrenormalization of λ_1 is different in the annealed case [49], but the result stands.) Therefore, the same conclusion holds: only $\mu_\perp$ gets graphically corrected. The only differences that the absence of quenched disorder makes are (1) the graphical correction $\eta_\perp$ will now be generated entirely by the *annealed* noise rather than the quenched noise, and (2) the values of the exponents z, ζ, and χ will change to the values found in the study of the annealed problem [49]. Those values were determined in [49] by choosing z, ζ, and χ to fix μ_x, $\mu_\perp$, and D_A, since those parameters control the dominant fluctuations in the absence of quenched disorder. To see that only these parameters matter in the annealed problem, one need simply inspect the annealed contribution (i.e., the D_A term) in Eq. (10).

Making this choice, and noting that the DRG eigenvalues of D_A and D_Q [i.e., the terms in square brackets in Eqs. (14e) and (14f)] differ by precisely z, it follows that, since we are choosing z, ζ, and χ to make the DRG eigenvalue of D_A vanish, that the eigenvalue for D_Q is given by z. Since z for incompressible flocks *without* quenched disorder is always positive [$z = [2(d+1)/5]$ for $d \le 4$ and $z = 2$ for $d > 4$ [49]], it follows that the quenched noise is always strongly relevant, i.e., it will change the long-distance and time scaling of fluctuations.

We can calculate the new scaling that ensues in the presence of *quenched* noise by much the same reasoning that we just outlined for the annealed problem. The only change is that it is now $\mu_\perp$, γ, and D_Q that we must keep constant at this fixed point, since they control the dominant (i.e., quenched) fluctuations in Eq. (10). The coefficient of the relevant nonlinear term λ_1 must also be fixed at this stable fixed point. This implies that the right-hand sides of the recursion relations Eqs. (14a), (14c), (14e), and (14d) for $\gamma, \mu_\perp, D_Q$, and λ_1 must vanish. This requirement leads to four linear equations for the three exponents z, χ, and ζ, and the graphical correction $\eta_\perp$:

$$z - 2 + \eta_\perp = 0, \qquad z - \zeta = 0, \tag{15a}$$

$$2z - 2\chi - \zeta - (d-1) = 0, \qquad z + \chi - 1 = 0. \tag{15b}$$

Solving these equations we find

$$z = \zeta = \frac{d+1}{3}, \qquad \chi = \frac{2-d}{3}, \qquad \eta_\perp = \frac{5-d}{3}. \tag{16}$$

We see that ζ and χ differ from those obtained from the linear theory, Eq. (11a), and only become equal to those linear values at the upper critical dimension $d = 5$. Furthermore, $\chi < 0$, which implies long-range order, for all $d > 2$. At exactly two dimensions, our present analysis no longer holds since the only "soft" dimension is coupled directly to the "hard" dimension (i.e., along the direction of collection motion) through the incompressibility condition, and a completely different formulation of the problem is required, as described in [48]. We note that $d = 2$ is also a singular limit of incompressible flocks *without* quenched disorder [49,50] (see Fig. 1 of the SM [52]).

The alert reader might be puzzled that we were able to obtain this result without actually calculating the functional dependence of the graphical correction $\eta_\perp$ to $\mu_\perp$ on the parameters γ, $\mu_\perp$, D_Q, and λ_1. We elaborate on what makes this possible in the SM [52]; for now, we simply note that similar arguments are used in every problem for which exact exponents can be obtained, and they invariably do not require the actual calculation of the graphical corrections to any parameters. Indeed, such a calculation can never give *exact* results, since all graphical calculations are inherently perturbative in nature [53]. Examples of such problems include the Navier-Stokes equation forced by a momentum nonconserving noise [47], the one-dimensional Kardar-Parisi-Zhang equation [54], and incompressible flocks in $d > 2$ *without* quenched disorder [49].

Scaling behavior.—Using the exponents, Eq. (16), we now derive the $\boldsymbol{u}$-$\boldsymbol{u}$ correlation function, Eq. (1), and the exponents, Eq. (2), and discuss the scaling behavior of the correlation function in different limits. The dominant part of the $\boldsymbol{u}$-$\boldsymbol{u}$ correlation function in Fourier space is displayed in Eq. (8), with μ_x, γ, and $D_{A,Q}$ given by their "bare" values, since there are no graphical corrections to them, $\mu_\perp$ is now a $\boldsymbol{q}$-dependent quantity:

$$\mu_\perp(\boldsymbol{q}) = \mu_{\perp 0}\left(\frac{q_\perp}{\Lambda}\right)^{-\eta_\perp} f_{\mu_\perp}\left[\frac{q_x/\Lambda'}{(q_\perp/\Lambda)^\zeta}\right], \tag{17}$$

where $f_{\mu_\perp}$ is a scaling function with the limiting behaviors

$$f_{\mu_\perp}(s) \propto \begin{cases} \text{constant}, & s \ll 1, \\ s^{-\eta_\perp/\zeta}, & s \gg 1. \end{cases} \tag{18}$$

Here, Λ is the nonuniversal ultraviolet cutoff, and $\Lambda' = (\mu_{\perp 0}/\gamma)\Lambda^2$. The subscript "0" in $\mu_\perp$ denotes the bare value.

Fourier-transforming $\langle \boldsymbol{u}_T(\boldsymbol{q},\omega)\cdot \boldsymbol{u}_T(\boldsymbol{q}',\omega')\rangle$, we obtain

$$\langle \boldsymbol{u}(\boldsymbol{r},t)\cdot \boldsymbol{u}(\boldsymbol{0},0)\rangle = C_A(\boldsymbol{r},t) + C_Q(\boldsymbol{r}), \tag{19}$$

where

$$C_A(\boldsymbol{r},t) = \int \frac{d\omega d^d q}{(2\pi)^{d+1}} e^{i(\boldsymbol{q}\cdot\boldsymbol{r}-\omega t)} \times \left\{\frac{2(d-2)D_A}{(\omega-\gamma q_x)^2 + [\mu_x q_x^2 + \mu_\perp(\boldsymbol{q}) q_\perp^2]^2}\right\}, \tag{20a}$$

$$C_Q(\boldsymbol{r}) = \int \frac{d^d q}{(2\pi)^d}\left\{\frac{2(d-2)D_Q}{\gamma^2 q_x^2 + [\mu_\perp(\boldsymbol{q}) q_\perp^2]^2}\right\} e^{i\boldsymbol{q}\cdot\boldsymbol{r}} \tag{20b}$$

are the correlations coming from the annealed and quenched noises, respectively.

For $C_Q(\boldsymbol{r},t)$, by changing the variables of integration to: $\boldsymbol{k}_\perp \equiv \boldsymbol{q}_\perp(r_\perp\Lambda)$ and $k_x \equiv q_x(r_\perp\Lambda)^\zeta$, Eq. (20b) can be written as

$$C_Q(\boldsymbol{r}) = r_\perp^{2\chi} G_Q\left(\frac{|x|}{r_\perp^\zeta}\right), \tag{21}$$

where G_Q is a scaling function given in [52].

For $C_A(\boldsymbol{r},t)$, the annealed part of the correlation function, the dominant contribution to the integral in Eq. (20a) comes from the region in which the two terms inside the square brackets in the denominator become comparable:

$$\mu_{x0} q_x^2 \sim \mu_\perp(\boldsymbol{q}) q_\perp^2. \tag{22}$$

Since $\mu_\perp(\boldsymbol{q})$ diverges at small $\boldsymbol{q}$ [see Eq. (17)], Eq. (22) implies $q_x \gg q_\perp$ and hence $q_x \gg q_\perp^\zeta$ since $\zeta > 1$ for $d > 2$ [see Eq. (16)]. Using this in Eq. (17) we get

$$\mu_\perp(\boldsymbol{q}) = \mu_{\perp 0}\left(\frac{q_x}{\Lambda'}\right)^{-\frac{\eta_\perp}{\zeta}}. \tag{23}$$

Inserting Eq. (23) into Eq. (20a), introducing $\omega' = \omega - \gamma q_x$, and further changing variables of integration $\boldsymbol{k}_\perp \equiv \boldsymbol{q}_\perp r_\perp$, $k_x \equiv q_x(r_\perp\Lambda)^{\zeta'}$, $\Omega \equiv \omega'(r_\perp\Lambda)^{z'}$, we obtain

$$C_A(\boldsymbol{r},t) = r_\perp^{2\chi'} G_A\left(\frac{|x-\gamma_0 t|}{r_\perp^{\zeta'}}, \frac{|t|}{r_\perp^{z'}}\right), \tag{24}$$

where ζ', z', χ' are given in Eq. (2b), and G_A is a scaling function given in [52].

Inserting Eqs. (21) and (24) into Eq. (19) gives Eq. (1). We now delineate its scaling behavior in distinct regimes. Since $\chi > \chi'$ and $(\chi/\zeta) > (\chi'/\zeta')$, the equal-time correlation is dominated by the contribution from the quenched fluctuations. Specifically,

$$\langle \boldsymbol{u}(\boldsymbol{r},0)\cdot \boldsymbol{u}(\boldsymbol{0},0)\rangle = r_\perp^{2\chi} G_Q\left(\frac{x}{r_\perp^\zeta}\right) \propto \begin{cases} r_\perp^{2\chi}, & |x| \ll r_\perp^\zeta, \\ |x|^{\frac{2\chi}{\zeta}}, & |x| \gg r_\perp^\zeta. \end{cases} \tag{25}$$

On the other hand, the time dependence of the correlation is solely determined by the annealed fluctuations, since the quenched fluctuations are constant in time. However, the quenched fluctuations do affect the equal-position correlation indirectly by renormalizing the diffusion coefficient $\mu_\perp$, which is one of the controlling parameters of the annealed fluctuations [see Eq. (20a)]. As a result, the difference between the equal-position correlation function at time t and its value at $t=0$ is given by

$$\langle \boldsymbol{u}(\boldsymbol{0},t)\cdot \boldsymbol{u}(\boldsymbol{0},0)\rangle - \langle \boldsymbol{u}(\boldsymbol{0},0)\cdot \boldsymbol{u}(\boldsymbol{0},0)\rangle = C_A(\boldsymbol{0},t) = A|t|^\theta, \tag{26}$$

where A is a nonuniversal constant and

$$\theta = \frac{2\chi'}{\zeta'} = -\left[\frac{d^2+4d-9}{2(d+1)}\right] = -\frac{3}{2}, \tag{27}$$

with the last equality holding in the physical case $d = 3$. We give the detailed argument for this expression for θ in the SM [52]. In Fig. 1 of the SM [52], we show how some of the scaling exponents vary with spatial dimension, and how they compare with those in the purely annealed case [49,50].

Summary and outlook.—We have considered the effects of quenched random field disorder in incompressible polar active fluids in the flocking phase, and shown that the quenched disorder makes the scaling behavior of the system very different from that predicted by linearized hydrodynamics, and from that of an incompressible polar active fluid with only annealed disorder. Crucially, we demonstrate that flocks are not inevitably destroyed by random-field disorder. While this Letter focuses on a one-component active fluid in the incompressible limit, an interesting future direction would be to consider the hydrodynamic behavior of active suspensions, which are two-component (swimmers and solvent) systems that are only incompressible as a whole.

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