Editors' Suggestion

Constrained Dynamics and Directed Percolation

Aydin Deger,[1,*] Achilleas Lazarides,[1,†] and Sthitadhi Roy[2,‡]
[1]Interdisciplinary Centre for Mathematical Modelling and Department of Mathematical Sciences, Loughborough University, Loughborough, Leicestershire LE11 3TU, United Kingdom
[2]International Centre for Theoretical Sciences, Tata Institute of Fundamental Research, Bengaluru 560089, India

(Received 28 June 2022; accepted 7 October 2022; published 31 October 2022)

In a recent work [A. Deger *et al.*, Phys. Rev. Lett. **129**, 160601 (2022).] we have shown that kinetic constraints can completely arrest many-body chaos in the dynamics of a classical, deterministic, translationally invariant spin system with the strength of the constraint driving a dynamical phase transition. Using extensive numerical simulations and scaling analyses we demonstrate here that this constraint-induced phase transition lies in the directed percolation universality class in both one and two spatial dimensions.

DOI: 10.1103/PhysRevLett.129.190601

Kinetic constraints have long emerged as a prominent avenue towards impeding ergodicity [1–7], complementary to and fundamentally different from that of quenched randomness and originally introduced in the context of the dynamics of glass forming systems at low temperatures. The underlying mechanism of dynamical slowing down is the decrease of the effective connectivity in configuration space by forbidding processes based on local constraints [8]. This is to be contrasted to the mechanisms of glassy relaxation in disordered systems, involving topographic features of the potential energy landscape. In many-body quantum systems it has been established in the last few years that constraints can lead to slow relaxation and also stabilize a many-body localized phase, not only at low temperatures but also at infinite temperatures [8–11].

This motivates the following question: what is the fate of *classical* many-body chaos in the presence of constrained dynamics? A signature of chaos is that infinitesimally small perturbations grow exponentially, dramatically changing the state at later times. In extended many-body systems and for spatially localized perturbations, this effect spreads ballistically in space, eventually resulting in global changes in the state [12]. We have recently showed that constraints in the dynamics can, if strong enough, fully arrest this spreading, confining the effect of a perturbation to a small region and thus suppressing chaos [13]. The nature of the transition between these chaotic and frozen phases, in particular its universality class, has however remained an open question.

Via numerical simulations and scaling analyses, we provide a sharp answer to this question: the phase transition belongs to the directed percolation (DP) universality class [14–17]. This constitutes the central result of this work, and is remarkable because the models we consider are translation invariant, with deterministic dynamical rules, quite differently from conventional percolation models. To the best of our knowledge, the appearance of DP universality in such a clean, deterministic setting has not been reported before.

The DP problem is an anisotropic variant of the standard percolation problem such that the percolating cluster can grow only in a given direction [18]. Equivalently, it can be understood as a cluster growth process on a graph with bonds directed along the given direction. This naturally endows the problem with a dynamical interpretation with time as one of the directions of the graph along which the bonds are directed. In order to map our problem onto a DP problem, we note that at late times, in the nonchaotic phase the spins freeze while in the chaotic phase they remain dynamically active. Based on this observation, we map the dynamically active and frozen spins to occupied and empty sites on the space-time lattice for the DP problem.

We consider constrained dynamics of a periodically driven classical Heisenberg spin system in spatial dimensions $d = 1$ and $d = 2$. Driving ensures that this model has no conserved quantities, including total energy. The Hamiltonian within a period T is given by

$$H(t) = \begin{cases} \sum_{\langle i,j\rangle} S_i^z S_j^z + h\sum_i S_i^z; & t \in [0, T/2) \\ g\sum_i S_i^x; & t \in [T/2, T) \end{cases}, \quad (1)$$

where $\langle i, j\rangle$ denotes a pair of nearest-neighbor sites. The equations for the stroboscopic dynamics of the spins in the presence of kinetic constraints are then

$$\vec{S}_i(t+T) = \mathsf{R}_\mathsf{x}[\gamma_{x,i}(t)] \cdot \mathsf{R}_\mathsf{z}[\gamma_{z,i}(t)] \cdot \vec{S}_i(t), \quad (2)$$

where $\mathsf{R}_{\mathsf{x(z)}}[\gamma_{x(z),i}]$ denotes rotation matrices about the $x(z)$ axis by an angle $\gamma_{x(z),i}$. These angles are given by

$$\gamma_{z,i}(t) = \Theta_i(t)\left[\sum_{j\in\langle i,j\rangle} S_j^z(t) + h\right]T/2,$$

$$\gamma_{x,i}(t) = \Theta_i(t) g T/2, \tag{3}$$

where $\Theta_i(t)$ encodes the kinetic constraint via a Heaviside step function

$$\Theta_i(t) = \Theta[\cos\theta_c - \min_{j|j\in\langle i,j\rangle} S_j^z(t)]. \tag{4}$$

This constraint means that the spin at site i rotates under the dynamics only if at least one of its neighboring spins lies outside the spherical sector subtended by a polar angle θ_c. We will call such a spin *active*. As a result, a spin is *frozen* and does not evolve dynamically if all its neighbors lie inside the spherical sector. The constraint (4) is inspired by the Fredrickson-Andersen model of constrained Ising spin glasses [1,2]. The physics is that dynamics is locally forbidden in a region if it is surrounded by immobile high-density regions, modeled by up-spins and allowed if there are some mobile low-density regions, modeled by down-spins, in the neighborhood. We generalize this via Eq. (4) to the case of Heisenberg spins. The angle θ_c therefore parametrizes the strength of the constraint and, as we will show, tunes the system across a dynamical phase transition at $\theta_{c,\mathrm{crit}}$ between an *active* phase at $\theta_c < \theta_{c,\mathrm{crit}}$ and a *frozen* phase at $\theta_c > \theta_{c,\mathrm{crit}}$. To distinguish between these and to map the problem to DP, which is usually discussed in terms of binary variables, we define an indicator function $\sigma_i(t)$, which we call the *activity.* It takes a value 1 if the spin at site i is active at time t and 0 otherwise; in other words, $\sigma_i(t) = \Theta_i(t)$. In terms of this, we define the density of active sites at time t as

$$\rho(t) = N^{-1}\sum_i \sigma_i(t), \tag{5}$$

where N is the total number of spins.

Numerically simulating the dynamics starting from an all-active initial state, we find that in the active phase, $\theta_c < \theta_{c,\mathrm{crit}}$, there is always a finite density of active sites at arbitrarily long times, as illustrated in Fig. 1(left). This implies that the long-time state is evolving dynamically and fluctuating. On the other hand, in the frozen phase, $\theta_c > \theta_{c,\mathrm{crit}}$, the system goes into a state at late times wherein the σ_i stops fluctuating with t. In other words, the state described in terms of σ_i gets *absorbed* into a frozen one. The constraint-induced dynamical phase transition is therefore an *absorbing phase transition*. We find that, in the frozen phase, but near the critical point, the typical absorbing state is one where $\sigma_i = 0$ for all sites [Fig. 1 (right)]. Once the system reaches such a state, since $\Theta_i = 0$ for all i in Eqs. (2) and (3), the dynamics is completely frozen.

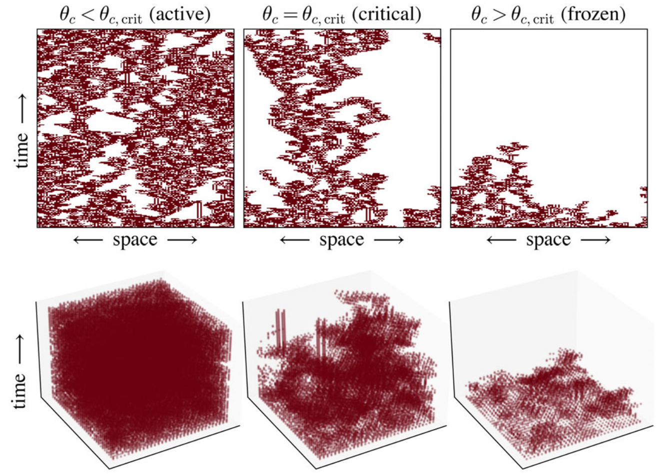

FIG. 1. Instances of active clusters for constrained spin dynamics in the active phase (left), at criticality (middle), and in the frozen phase (right) in the space-time graph in $1+1$D (top) and $2+1$D (bottom). In all cases we start from an all-sites-active configuration. In the active phase, there is always a finite density of active sites at arbitrarily long times whereas in the frozen phase the system goes into an absorbing state where all spins are frozen.

We also find other absorbing states where there exist temporally persistent, spatially finite, and dynamically stable configurations surrounded by inactive spins—reminiscent of breathers. These evolve regularly without spreading and persist forever. However, due to the extreme diluteness of such active sites, they are statistically irrelevant for the scaling behavior near the critical point [19]. We can therefore posit that $\rho(t)$ in the limit of $t\to\infty$ is a valid order parameter, with the active and frozen phases characterized by $\rho_\infty \equiv \rho(t\to\infty) \to$ finite and vanishing values respectively.

Finally, for the dynamics of $\sigma_i(t)$ to be a bonafide DP problem, we need to argue that the active sites necessarily form a contiguous cluster in the space-time graph connecting all active sites at time t to the initially active sites at $t = 0$. In other words, that the dynamics cannot spawn active clusters in a background of frozen sites. This is straightforwardly argued for based on the locality of the constraints: a spin can change state over a period if and only if at least one of its neighbors is active. Therefore if a spin is inactive, $\sigma_i(t) = 0$ it may only become active $\sigma_i(t+T) = 1$ if either of $\sigma_{i\pm1}(t) = 1$. This implies that any active spin $\sigma_i(t) = 1$ has one of its parents active, $\sigma_{i\pm1}(t-T) = 1$, and so on up to the initial time $t = 0$. This implies that the active sites form a contiguous cluster in the space-time graph. Therefore, the active phase of the constrained dynamics corresponds to the percolating phase as the cluster of active sites percolates all the way to infinite time whereas the frozen phase corresponds to the non-percolating phase as the cluster of active sites dies out.

From the above discussion, we conclude that the constraint-induced dynamical phase transition can be described as a continuous, absorbing phase transition with a one-component order parameter, short-ranged dynamical rules and no symmetries except translation invariance. It therefore satisfies three out of the four conjectured requirements by Janssen and Grassberger [20,21] for the transition to be in the DP universality class. The one requirement that our model does not satisfy is the presence of a unique absorbing state (any configuration of the spins inside the cones is an absorbing state). Nevertheless the DP universality is known to be extremely robust against such violations of the aforementioned requirements [22–27]. In the following, using extensive numerical simulations and scaling analyses we firmly establish that our constraint-induced transition does indeed lie in the DP universality class.

Before delving into the results, let us briefly recapitulate the scaling forms and the critical exponents for the DP universality class. Since the DP transition is a continuous phase transition, the order parameter goes to zero continuously with an exponent β from the active side as

$$\rho_\infty \sim \Delta^\beta \qquad \Delta \equiv \theta_{c,\mathrm{crit}} - \theta_c. \tag{6}$$

In addition to a correlation length, ξ_x, diverging as $\xi_x \sim |\Delta|^{-\nu_x}$, we also have a correlation time, ξ_t, which diverges with a different exponent $\xi_t \sim |\Delta|^{-\nu_t}$. This also defines the dynamical exponent $z = \nu_t/\nu_x$ which relates the rescaling of space and time under rescaling of the parameter Δ that tunes the phase transition. The DP transition is thus described in terms of the three independent critical exponents (β, ν_x, ν_t), which are strictly defined in the steady state. Since the true steady state is not accessible in numerical calculations, we obtain these exponents from dynamical scaling as follows. With an initial condition where all sites are active, scale invariance at the critical point suggests $\rho(t, \theta_{c,\mathrm{crit}})$ decaying as a power law for an infinite system, $\rho(t, \theta_{c,\mathrm{crit}}) \sim t^{-\alpha}$. Usual considerations of critical scaling imply that corrections away from this limit

TABLE I. Summary of the DP universality class critical exponents for $d = 1$ and $d = 2$ taken from Ref. [16].

d/exponents	β	ν_x	ν_t	α	z
$d = 1$	0.276	1.097	1.734	0.159	1.581
$d = 2$	0.584	0.734	1.295	0.451	1.76

are captured via universal scaling functions of $t/\xi_t \sim t|\Delta|^{\nu_t}$ and of t/L^z,

$$\rho(t) \sim t^{-\alpha} f(t|\Delta|^{\nu_t}, tL^{-z}). \tag{7}$$

where L is the linear size of the system and $N = L^d$ with d with the spatial dimension.

For very large systems such that $tL^{-z} \ll 1$, asymptotically in the active phase such that $t \gg \xi_t$, $\rho(t)$ saturates to a constant and hence we expect the scaling function f in Eq. (7) to be such that the time dependence in $\rho(t)$ drops out in this limit. This implies $f(y_1, y_2 \ll 1) \sim y_1^\alpha$ for $y_1 \gg 1$ and hence $\rho(t \to \infty) \sim \Delta^{\alpha\nu_t}$ in the active phase. Comparing this to Eq. (6), we find the relation between the exponents $\alpha = \beta/\nu_t$. Therefore, by performing a scaling analysis on the data for $\rho(t)$, one can extract the exponents α, ν_t, and z and hence the three fundamental exponents β, ν_x, and ν_t. Table I summarizes the known values of these exponents.

Let us now turn to our results for the constrained spin dynamics described via Eq. (1) through Eq. (4) for both $d = 1$ and $d = 2$. We evolve our system using the full dynamics for $\{\vec{S}_i(t)\}$, then calculate $\{\sigma_i(t)\}$ and then $\rho(t)$. For the former, we consider a chain and for the latter, a square lattice. The results for the $1 + 1$D case are shown in Fig. 2 whereas those for the $2 + 1$D case in Fig. 3. Our initial conditions are chosen randomly except for ensuring that all spins are active at $t = 0$. This is done by initializing randomly the polar and azimuthal angles of the spins, $\arccos(S_i^z)$ and $\arctan(S_i^x/S_i^y)$, from uniform distributions

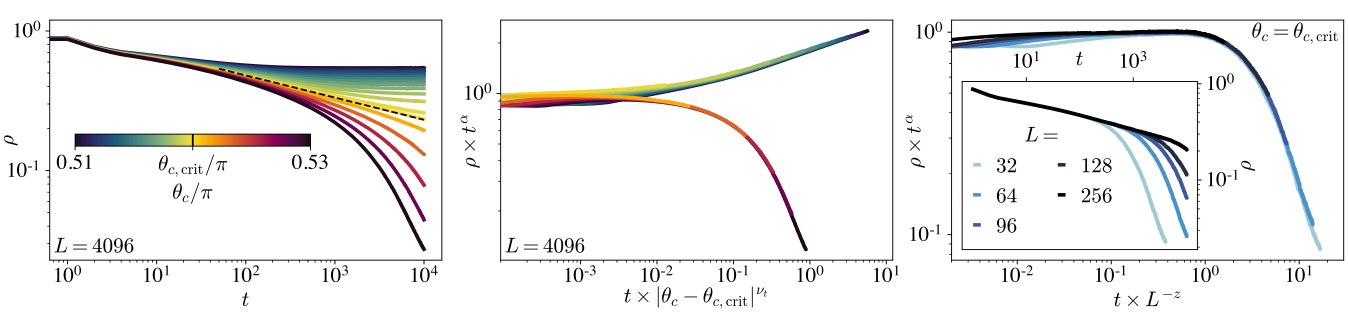

FIG. 2. Critical scaling in the $1 + 1$D case. Left: The density of active sites ρ as a function of t for different values of θ_c. At the critical point, ρ decays with t following a power law with an exponent $\alpha = 0.159$ consistent with DP in $1 + 1$D (black dashed line). Middle: Scaling ρ in the left panel with t^α and plotting it against $t|\theta_c - \theta_{c,\mathrm{crit}}|^{\nu_t}$ with $\theta_{c,\mathrm{crit}} \approx 0.5234\pi$ shows perfect scaling collapse for the DP universality exponents of $\alpha = 0.159$ and $\nu_t = 1.734$. Right: Finite-size scaling at the critical point by plotting ρt^α against tL^{-z} with the DP universality exponent, $z = 1.581$, again shows excellent collapse. The inset shows the unscaled data. Results for $g = 0.4$, $h = 0.1$, and $T = 2\pi$.

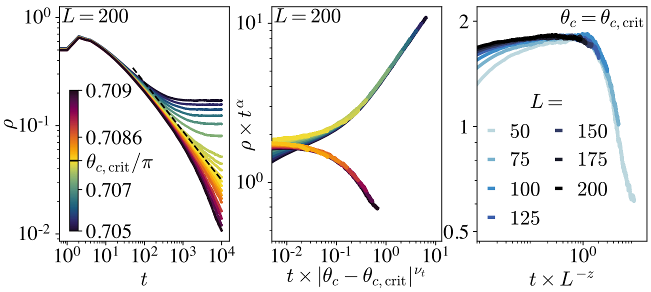

FIG. 3. Critical scaling in the 2 + 1D case. Analogous figure to Fig. 2 but for $d = 2$. In the left panel, the dashed black line corresponds to $t^{-\alpha}$ with $\alpha = 0.451$. The middle panel shows the collapse of ρt^{α} onto a universal function of $t|\theta_c - \theta_{c,\mathrm{crit}}|^{\nu_t}$ with $\theta_{c,\mathrm{crit}} \approx 0.7084\pi$ and $\nu_t = 1.295$, consistent with DP universality. Right panel shows finite-size scaling collapse at $\theta_{c,\mathrm{crit}}$ of ρt^{α} as function of tL^{-z} with $z = 1.76$, the DP universality value. Results for $g = 0.4$, $h = 0.1$, and $T = 2\pi$.

$\in (\theta_c, \pi]$ and $\in [0, 2\pi)$, respectively. In what follows, we set $T = 2\pi$, $h = 0.1$, and $g = 0.4$ without loss of generality.

In the left panels we show $\rho(t)$, defined in Eq. (5), as a function of t for different values of θ_c straddling $\theta_{c,\mathrm{crit}}$ for the largest sizes in our simulations; $L = 4096$ for $d = 1$ and $L = 200$ for $d = 2$. For $\theta_c < \theta_{c,\mathrm{crit}}$, the data on logarithmic axes curves upwards from a power law indicating its tendency to saturate to a finite value at $t \to \infty$, signifying the active or percolating phase. On the other hand, for $\theta_c > \theta_{c,\mathrm{crit}}$, the deviation of $\rho(t)$ from the power-law curves downwards indicating a rapid decay of $\rho(t \to \infty) \to 0$, a signature of the frozen or nonpercolating phase. We therefore estimate the critical point from the $\rho(t)$ data as the θ_c value which has the minimal curvature. At the so-estimated, $\theta_c = \theta_{c,\mathrm{crit}}$, the data show a perfect power law indicated by the straight line with almost no curvature on logarithmic axes. The black dashed lines show the expected power laws from the DP universality exponents (see Table I). They are in excellent agreement with the data, which in turn confirm that the α exponent is the same as that of DP universality.

The sizes we considered ensure $tL^{-z} \ll 1$ and hence the L dependence in the scaling function (7) can be ignored. We confirm this by ensuring the data in Figs. 2 (left) and 3 (left) are converged with L. In this limit $\rho(t)t^{\alpha}$ is a universal function of $t|\theta_c - \theta_{c,\mathrm{crit}}|^{\nu_t}$. Upon rescaling $\rho(t)$ with t^{α} and plotting it against $t|\theta_c - \theta_{c,\mathrm{crit}}|^{\nu_t}$ with α and ν_t from Table I and $\theta_{c,\mathrm{crit}}$ extracted as above, we find that the data for all θ_c collapse onto two universal curves, one for each phase. This is shown in the middle panels in Figs. 2 and 3 for $d = 1$ and $d = 2$, respectively, which confirms that the ν_t exponent is also the same as that of DP universality.

Finally, we consider the finite-size scaling at the critical point to extract the dynamical exponent z. At criticality, $\Delta = 0$, and hence the scaling function (7) implies that $\rho(t)t^{\alpha}$ is a universal function of tL^{-z}. In the right panels of Figs. 2 and 3, we plot $\rho(t)t^{\alpha}$ as a function of tL^{-z} at $\theta_c = \theta_{c,\mathrm{crit}}$ with z from Table I and find that the curves for several L collapse on top each other. This confirms that the dynamical exponent z is also the same as that of DP universality.

The analyses presented in the three panels together in Fig. 2 for $d = 1$ and Fig. 3 for $d = 2$ thus show that the three exponents α, ν_t, and z, and hence by extension the three independent exponents β, ν_x, and ν_t for our constraint-induced dynamical phase transition are the same as those for the DP universality class. We therefore conclude that the transition lies in the same universality class—this constitutes the central result of this work.

Note that the dynamics in our model are completely translation invariant and deterministic. In the past absorbing transitions in deterministic systems were found to have nonuniversal model-dependent critical exponents [28], which makes the appearance of DP universality in our case remarkable. To intuitively understand this result, we introduce a stochastic cellular automaton (CA). This is to be viewed as a coarse-grained version of the spin system, having broadly the same features and constraints in its dynamics. This CA involves a Boolean variable per site, τ_i, representing whether the spin at i is inside (0) or outside (1) the cone. The dynamical rule is then that $\tau_j(t+1) = \tau_j(t)$ if both parents $\tau_{j\pm1}(t) = 0$ or $\tau_j(t+1)$ is 1 or 0 with probabilities $1 - p$ and p if either or both $\tau_{j\pm1}(t) = 1$. Numerically, we find this CA to display a transition in the DP class [19]. Note that this occurs even though our simplified CA model replaces the continuous spin degree of freedom with a two-state variable and the deterministic by stochastic dynamics, and in so doing completely misses spatiotemporal correlations in (or, history dependence of) the dynamics.

Our work should also be connected to and contrasted with transitions belonging to the DP universality found in certain cellular automatons in a different setting [29,30]. Apart from the fundamental difference from our work that the dynamics in these models was stochastic, the mapping to DP was based on two copies of the system and the active or inactive sites were defined based on whether the Ising variables were different or the same in the two copies. In this sense it was a damage spreading process [31–33], while in our work the mapping requires only one copy of the system and the growing DP cluster corresponds to a growing cluster of dynamically active.

To summarize, we have established that the constraint-induced dynamical phase transition in clean, deterministic systems from a chaotic phase to one where the dynamics is arrested [13] can be mapped onto a DP problem and the transition lies in the conventional DP universality class. A question of interest for future work is under what conditions such transitions may fall out of this universality class. A natural setting to look for such cases would be systems with long-ranged power-law interactions and constraints.

In particular, one may write a model where the constraints, such as those in Eq. (4) are long-ranged—then an active site at i can activate another at $i + r$ with a probability with a strength which falls off as a power law with r. Whether such a model has a transition or not, and if it does, does it share features with the DP transition in long-ranged spreading processes [34–36] is a question of interest.

A. D. and A. L. acknowledge support from EPSRC Grant No. EP/V012177/1 and S. R. acknowledges support from an ICTS-Simons Early Career Faculty Fellowship by a grant from the Simons Foundation (677895, R. G.). The numerical simulations were performed on the Contra cluster at ICTS-TIFR.

[*]a.deger@lboro.ac.uk
[†]a.lazarides@lboro.ac.uk
[‡]sthitadhi.roy@icts.res.in

[1] G. H. Fredrickson and H. C. Andersen, Kinetic Ising Model of the Glass Transition, Phys. Rev. Lett. **53**, 1244 (1984).
[2] G. H. Fredrickson and H. C. Andersen, Facilitated kinetic Ising models and the glass transition, J. Chem. Phys. **83**, 5822 (1985).
[3] J. Jäckle and S. Eisinger, A hierarchically constrained kinetic Ising model, Z. Phys. B Condens. Matter **84**, 115 (1991).
[4] P. Sollich and M. R. Evans, Glassy Time-Scale Divergence and Anomalous Coarsening in a Kinetically Constrained Spin Chain, Phys. Rev. Lett. **83**, 3238 (1999).
[5] P. Sollich and M. R. Evans, Glassy dynamics in the asymmetrically constrained kinetic Ising chain, Phys. Rev. E **68**, 031504 (2003).
[6] F. Ritort and P. Sollich, Glassy dynamics of kinetically constrained models, Adv. Phys. **52**, 219 (2003).
[7] J. P. Garrahan, P. Sollich, and C. Toninelli, Kinetically constrained models, in *Dynamical Heterogeneities in Glasses, Colloids, and Granular Media*, edited by L. Berthier, G. Biroli, J.-P. Bouchaud, L. Cipelletti, and W. van Saarloos(Oxford University Press, Oxford, 2011).
[8] S. Roy and A. Lazarides, Strong ergodicity breaking due to local constraints in a quantum system, Phys. Rev. Res. **2**, 023159 (2020).
[9] M. van Horssen, E. Levi, and J. P. Garrahan, Dynamics of many-body localization in a translation-invariant quantum glass model, Phys. Rev. B **92**, 100305(R) (2015).
[10] Z. Lan, M. van Horssen, S. Powell, and J. P. Garrahan, Quantum Slow Relaxation and Metastability Due to Dynamical Constraints, Phys. Rev. Lett. **121**, 040603 (2018).
[11] N. Pancotti, G. Giudice, J. I. Cirac, J. P. Garrahan, and M. C. Bañuls, Quantum East Model: Localization, Nonthermal Eigenstates, and Slow Dynamics, Phys. Rev. X **10**, 021051 (2020).
[12] A. Das, S. Chakrabarty, A. Dhar, A. Kundu, D. A. Huse, R. Moessner, S. S. Ray, and S. Bhattacharjee, Light-Cone Spreading of Perturbations and the Butterfly Effect in a Classical Spin Chain, Phys. Rev. Lett. **121**, 024101 (2018).
[13] A. Deger, S. Roy, and A. Lazarides, Arresting Classical Many-Body Chaos by Kinetic Constraints, Phys. Rev. Lett. **129**, 160601 (2022).
[14] W. Kinzel, Percolation structures and processes, in *Annals of the Israel Physical Society*, edited by G. Deutscher, R. Zallen, and J. Adler (Adam Hilger, Bristol, 1983), Vol. 5.
[15] W. Kinzel, Phase transitions of cellular automata, Z. Phys. B **58**, 229 (1985).
[16] H. Hinrichsen, Non-equilibrium critical phenomena and phase transitions into absorbing states, Adv. Phys. **49**, 815 (2000).
[17] M. Henkel, H. Hinrichsen, and S. Lübeck, *Non-Equilibrium Phase Transitions: Volume 1: Absorbing Phase Transitions*, Theoretical and Mathematical Physics (Springer, Netherlands, 2008).
[18] S. R. Broadbent and J. M. Hammersley, Percolation processes, Math. Proc. Cambridge Philos. Soc. **53**, 629 (1957).
[19] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.129.190601 for discussions pertaining to (i) statistical irrelevance of rare active sites, and (ii) a stochastic automaton model as a coarse-grained model of the constrained spin dynamics.
[20] H. K. Janssen, On the nonequilibrium phase transition in reaction-diffusion systems with an absorbing stationary state, Z. Phys. B **42**, 151 (1981).
[21] P. Grassberger, On phase transitions in Schlögl's second model, Z. Phys. B **47**, 365 (1982).
[22] I. Jensen, Critical Behavior of the Pair Contact Process, Phys. Rev. Lett. **70**, 1465 (1993).
[23] I. Jensen, Critical behaviour of a surface reaction model with infinitely many absorbing states, J. Phys. A **27**, L61 (1994).
[24] E. V. Albano, Irreversible phase transitions into non-unique absorbing states in a multicomponent reaction system, Physica (Amsterdam) **214A**, 426 (1995).
[25] M. A. Muñoz, G. Grinstein, R. Dickman, and R. Livi, Critical Behavior of Systems with Many Absorbing States, Phys. Rev. Lett. **76**, 451 (1996).
[26] M. A. Muñoz, G. Grinstein, R. Dickman, and R. Livi, Infinite numbers of absorbing states: Critical behavior, Physica (Amsterdam) **103D**, 485 (1997).
[27] M. A. Muñoz, G. Grinstein, and R. Dickman, Phase structure of systems with infinite numbers of absorbing states, J. Stat. Phys. **91**, 541 (1998).
[28] J. M. Houlrik, I. Webman, and M. H. Jensen, Mean-field theory and critical behavior of coupled map lattices, Phys. Rev. A **41**, 4210 (1990).
[29] S.-W. Liu, J. Willsher, T. Bilitewski, J.-J. Li, A. Smith, K. Christensen, R. Moessner, and J. Knolle, Butterfly effect and spatial structure of information spreading in a chaotic cellular automaton, Phys. Rev. B **103**, 094109 (2021).
[30] J. Willsher, S.-W. Liu, R. Moessner, and J. Knolle, Measurement-induced phase transition in a classical, chaotic many-body system, Phys. Rev. B **106**, 024305 (2022).
[31] S. A. Kauffman, Emergent properties in random complex automata, Physica (Amsterdam) **10D**, 145 (1984).
[32] O. Martin, Lyapunov exponents of stochastic dynamical systems, J. Stat. Phys. **41**, 249 (1985).

[33] B. Derrida and D. Stauffer, Phase transitions in two-dimensional Kauffman cellular automata, Europhys. Lett. **2**, 739 (1986).

[34] P. Grassberger, Spreading of epidemic processes leading to fractal structures, in *Fractals in Physics* (Elsevier, Amsterdam, 1986), pp. 273–278.

[35] H. K. Janssen, K. Oerding, F. Van Wijland, and H. J. Hilhorst, Lévy-flight spreading of epidemic processes leading to percolating clusters, Eur. Phys. J. B **7**, 137 (1999).

[36] H. Hinrichsen and M. Howard, A model for anomalous directed percolation, Eur. Phys. J. B **7**, 635 (1999).