Operational Interpretation of Quantum Fisher Information in Quantum Thermodynamics

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In the framework of quantum thermodynamics preparing a quantum system in a general state requires the consumption of two distinct resources, namely, work and energetic coherence. It has been shown that the work cost of preparing a quantum state is determined by its free energy. Considering a similar setting, here we determine the coherence cost of preparing a general state when there are no restrictions on work consumption. More precisely, the coherence cost is defined as the minimum rate of consumption of systems in a pure coherent state, that is needed to prepare copies of the desired system. We show that the coherence cost of any system is determined by its quantum Fisher information about the time parameter, hence introducing a new operational interpretation of this central quantity of quantum metrology. Our resource-theoretic approach also reveals a previously unnoticed connection between two fundamental properties of quantum Fisher information.

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The information-theoretic approach to quantum thermodynamics and, more specifically, the resource-theoretic approach [1] has proven to be extremely fruitful. This, for instance, has lead to the discovery of new aspects of quantum coherence in thermodynamics (see, e.g., Refs. [2–9]). In this approach one studies the interconvertibility of systems under a limited set of operations, which presumably can be implemented with negligible thermodynamic costs. A popular choice is the set of thermal operations, i.e., those that can be implemented by coupling the system to a thermal bath via energy-conserving unitaries [10,11].

From a thermodynamics point of view, preparing a general quantum state requires consumption of both work and energetic coherence, i.e., coherence between states with different energies, which can also be understood as asymmetry with respect to time translations [3,12–14]. In the resource-theoretic framework of quantum thermodynamics, it has been shown that the work cost of preparing many independent and identically distributed (IID) copies of any quantum system is determined by its free energy [11]. On the other hand, characterizing the coherence cost of preparing quantum systems has remained an open question [9,15].

In this Letter, we settle this question and show that the coherence cost of preparing a quantum system in a general state is determined by the quantum Fisher information (QFI) [16–19] of the system about the time parameter (see Theorem 2). More precisely, to prepare copies of the desired system in the IID regime, the minimum rate of consumption of systems in a fixed pure coherent state is determined by the ratio of QFI's of the desired system to the input pure system (see Fig. 1). Interestingly, a similar result does not hold for the reverse process, called coherence distillation: for generic mixed input states the rate of conversion to pure coherent states is zero [6].

Hence, our result reveals a novel operational interpretation of QFI, which is the central quantity of quantum metrology [20,21]. Remarkably, our resource-theoretic approach also clarifies a close connection between two different fundamental properties of QFI, namely QFI as a convex roof of variance and QFI as the variance of purification of state. While QFI has been extensively studied in quantum metrology, to our knowledge this connection has not been appreciated before.

To focus on coherence as a resource independent of work, one can supplement thermal operations with a battery or work reservoir that can provide an unlimited amount of work (in other words, one can make work a free resource). It has been shown (see, e.g., [6,22,23]) that in this way one can implement all and only time-translationally invariant (TI) operations [23–26], i.e., completely positive trace-preserving maps satisfying the covariance condition,

$$e^{-iH_{\text{out}}t}\mathcal{E}_{\text{TI}}(\sigma)e^{iH_{\text{out}}t} = \mathcal{E}_{\text{TI}}(e^{-iH_{\text{in}}t}\sigma e^{iH_{\text{in}}t}), \tag{1}$$

for all density operators σ and all times t. Here, H_{in} and H_{out} are, respectively, the input and output Hamiltonians. TI operations cannot generate (energetic) coherence: to prepare systems containing coherence via TI operations, one needs an input that contains coherence. On the other hand, preparing incoherent states, i.e., those that commute with the system Hamiltonian, does not require consuming coherence. In summary, to understand coherence as a resource independent of work, we study state conversions

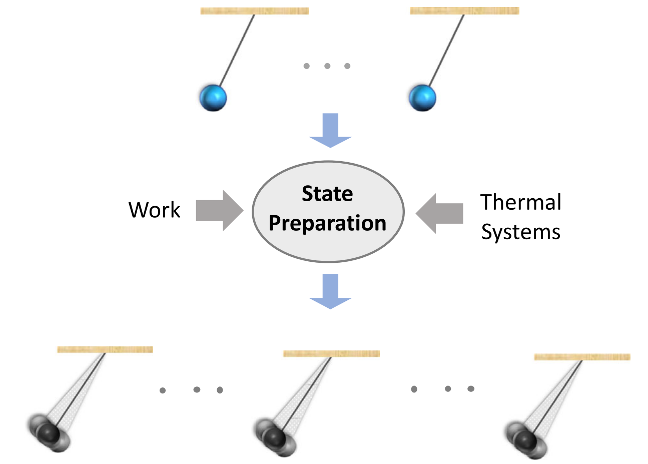

FIG. 1. Preparing a quantum system in a general state requires consumption of both work and coherence. Here, we study the coherence cost of preparing state, when there are no limitations on work consumption. Equivalently, we characterize the minimum rate of consumption of quantum clocks that is needed to prepare a general state, when one does not have access to the standard reference clock.

under TI operations. It is worth noting that going beyond these operations makes coherence a free resource: using any non-TI operation it is possible to generate energetic coherence from incoherent states, albeit this may require correlation between the input of the operation and an auxiliary system [6].

TI operations and the notion of coherence cost also arise in the study of quantum clocks. While coherent states and non-TI operations should be defined relative to a background reference clock, Eq. (1) means that TI operations can be defined and implemented without access to such clocks [6,24,25]. Suppose one does not have access to the reference clock, but is given quantum clocks that are synchronized with it. What is the minimum rate of consumption of quantum clocks in pure states, that is needed to prepare copies of a desired system (see Fig. 1)? Again, we find that the answer is given by the QFI of the system about the time parameter.

Pure states in the IID regime.—We study systems with finite-dimensional Hilbert spaces. Each system is specified by its Hamiltonian H and density operator ρ. We assume the systems under consideration have periodic dynamics with a fixed but arbitrary period τ such that $\tau = \inf\{t > 0 : e^{-iHt}\rho e^{iHt} = \rho\}$. Under TI operations, a system with period τ can only be converted to systems with period τ/k, for an integer k. In the following, we consider n copies of a system with Hamiltonian H and state ρ, which means their joint state is $\rho^{\otimes n}$ and their total Hamiltonian is $\sum_{j=0}^{n-1} I^{\otimes j} \otimes H \otimes I^{\otimes(n-j-1)}$.

Consider many copies of a system with Hamiltonian H_1, pure state ψ_1, and period τ. Is it possible to convert these systems to many copies of another system with the same period τ, in pure state ψ_2 and Hamiltonian H_2, using TI operations? Since exact conversions are often impossible and physically intractable, as usual we allow a vanishing error quantified, e.g., in terms of the trace distance $D(\rho,\sigma) = \|\rho - \sigma\|_1/2$ (or, equivalently, one minus fidelity [27–29]). In the following, $V_H(\psi) = \langle\psi|H^2|\psi\rangle - \langle\psi|H|\psi\rangle^2$ denotes the energy variance of pure state ψ with respect to Hamiltonian H. In Supplemental Material (SM) [30], we prove our first main result:

Theorem 1: Consider a pair of systems with pure states ψ_1 and ψ_2 and Hamiltonians H_1 and H_2, with equal periods. Using TI operations the state conversion

$$|\psi_1\rangle^{\otimes n} \overset{TI}{\longrightarrow} \overset{\epsilon_n}{\approx} |\psi_2\rangle^{\otimes\lceil Rn\rceil} \quad \text{as } n \to \infty, \qquad \epsilon_n \to 0,$$

with vanishing error ϵ_n in trace distance is possible if rate $R \leq V_{H_1}(\psi_1)/V_{H_2}(\psi_2)$ and is impossible if $R > V_{H_1}(\psi_1)/V_{H_2}(\psi_2)$.

Hence, in the IID regime oscillators in pure states with the same frequencies are equivalent resources, in the sense that by adding or absorbing sufficient amount of energy their coherence content, or equivalently, their information content about time, can be converted from one form to another. Note that the maximal achievable rate from system 1 to 2, namely $V_{H_1}(\psi_1)/V_{H_2}(\psi_2)$, is the inverse of the maximal rate from system 2 to 1. In this sense the process is reversible. Consequently, in this regime the usefulness of a clock can be quantified by a single number, namely its energy variance. In other words, we can pick a standard *clock bit* (coherence bit) or cbit with period τ and quantify the amount of resource of a general state relative to this standard. A convenient choice is a two-level system with Hamiltonian $H_{\text{cbit}} = \pi\sigma_z/\tau$ and state $|\Theta\rangle_{\text{cbit}} = (|0\rangle + |1\rangle)/\sqrt{2}$, with the energy variance π^2/τ^2.

Theorem 1 strengthens and generalizes a previously known result [25,31,32] in multiple ways. The common intuition behind all these results, first discussed in [31,32], is based on the central limit theorem, which implies that the total energy distribution of many copies of a state converges to a Gaussian distribution, and hence is characterized by its variance and mean, which are both additive. Then, as the mean energy can be changed arbitrarily by TI operations, the conversion rate is determined by the ratio of variances.

One aspect of this theorem that makes it stronger than the previous result is the requirement of convergence in the trace distance, whose significance arises from Helstrom's theorem [16,28,29]. According to this theorem states with vanishing trace distance are indistinguishable and therefore equivalent resources. To establish such convergence, in addition to the standard results in the resource theory of asymmetry [23,25,26,33], we also apply local limit theorems in probability theory [34–37], which imply that in the IID regime the energy distribution converges to a translated Poisson distribution. Another new aspect of the above result is the rigorous upper bound on the achievable rate R.

Since variance is additive for uncorrelated systems and is nonincreasing in exact state conversions under TI operations, it is straightforward to show that the rate $R > V_{H_1}(\psi_1)/V_{H_2}(\psi_2)$ is not achievable in exact state conversions [25]. However, this argument fails in the presence of error ϵ_n: for a pair of output states with trace distance ϵ_n, the energy variances can differ by order $\epsilon_n\lceil Rn\rceil^2\|H\|^2$. Hence, the variance per copy can differ by order $\epsilon_n\lceil Rn\rceil\|H\|^2$, which does not necessarily vanish, even if $\epsilon_n \to 0$ in the limit $n \to \infty$. We overcome this complication and show that for $R > V_{H_1}(\psi_1)/V_{H_2}(\psi_2)$, error cannot vanish as $n \to \infty$ [see Eq. (4) below for the general result].

Theorem 1 only applies to pure states. In the rest of this Letter, we consider a variant of this scenario where the outputs are mixed. But, first we discuss the interpretation of the energy variance in this theorem.

Quantum Fisher information (QFI).—Consider the family of states $\{e^{-iHt}\rho e^{iHt}\}_t$ corresponding to the time-evolved versions of a system in the initial state ρ and Hamiltonian H. The QFI relative to the time parameter t for this family of state is

$$F_H(\rho) = 2\sum_{j,k}\frac{(p_j - p_k)^2}{p_j + p_k}|\langle\phi_j|H|\phi_k\rangle|^2, \qquad (2)$$

where $\rho = \sum_j p_j|\phi_j\rangle\langle\phi_j|$ is the spectral decomposition of ρ. Equivalently, QFI can be expressed as the second derivative of the fidelity of states ρ and $e^{-iHt}\rho e^{iHt}$ with respect to the parameter t [38]. According to the standard interpretation of this quantity in quantum estimation, $F_H(\rho)$ determines how well one can estimate the unknown parameter t, by measuring $n \gg 1$ copies of state $e^{-iHt}\rho e^{iHt}$: the mean squared error $\langle\delta t^2\rangle$ for any unbiased estimator satisfies the Cramér-Rao bound $\langle\delta t^2\rangle \geq [nF_H(\rho)]^{-1}$, which is attainable in the asymptotic regime [16–18,39]. QFI has found extensive applications beyond quantum metrology (see, e.g., Refs. [13,40–50]). In particular, it has been studied as an example of measures of asymmetry and (unspeakable) coherence [33,51–53] (skew information [23,54–56] and the relative entropy of asymmetry [57,58] are two other well-known examples). However, prior to this Letter, the operational interpretation of QFI as the coherence cost, which distinguishes this measure of coherence from the others, was not known.

QFI has various nice properties, including the following. (i) Faithfulness: it is zero if, and only if, state is incoherent. (ii) Monotonicity: it is nonincreasing under any TI operation $\mathcal{E}_{\text{TI}}$, i.e., $F_H[\mathcal{E}_{\text{TI}}(\rho)] \leq F_H(\rho)$. In particular, it remains invariant under energy-conserving unitaries. (iii) Additivity: for a composite noninteracting system with the total Hamiltonian $H_{\text{tot}} = H_1 \otimes I_2 + I_1 \otimes H_2$, QFI is additive for uncorrelated states, i.e., $F_{H_{\text{tot}}}(\rho_1 \otimes \rho_2) = F_{H_1}(\rho_1) + F_{H_2}(\rho_2)$. (iv) Convexity: for any $p \in [0,1]$ and states ρ and σ, $F_H(p\rho + (1-p)\sigma) \leq pF_H(\rho) + (1-p)F_H(\sigma)$.

For pure states, QFI reduces to the energy variance, namely $F_H(\psi) = 4V_H(\psi)$. Therefore, Theorem 1 means that in the IID regime, the maximal rate of conversion between pure states is determined by the ratio of their QFI's. This interpretation suggests that to generalize the result to mixed states, the role of variance should be replaced by QFI. As we show below, this conjecture is partially correct, namely when the output states are mixed but the inputs are still pure. On the other hand, [6] shows that this conjecture fails for generic mixed input states. It is also worth noting that the state conversion described in Theorem 1 requires coherent interactions between the input and output: unless ψ_2 is an energy eigenstate, it is not possible to achieve a positive rate $R > 0$ with a vanishing error, using measure-and-prepare (i.e., entanglement-breaking) TI operations [6]. This again suggests that the operational interpretation of QFI in the context of parameter estimation cannot fully explain the special role of variance in Theorem 1.

Coherence cost.—Consider a system with state ρ and Hamiltonian H with period τ. We define the coherence cost $C_c^{\text{TI}}(\rho)$ of this system as the minimal rate at which cbits with period τ (i.e., two-level systems with state $|\Theta\rangle_{\text{cbit}} = (|0\rangle + |1\rangle)/\sqrt{2}$ and Hamiltonian $H_{\text{cbit}} = \pi\sigma_z/\tau$) have to be consumed for preparing copies of this system in the IID regime, i.e.,

$$C_c^{\text{TI}}(\rho) = \inf R\colon \Theta_{\text{cbit}}^{\otimes\lceil Rn\rceil} \overset{TI}{\rightarrow} \overset{\epsilon_n}{\approx} \rho^{\otimes n} \quad \text{as } n \to \infty, \qquad \epsilon_n \to 0,$$

where the vanishing error ϵ_n is quantified in the trace distance. This quantity can be thought of as the counterpart of the entanglement cost in entanglement theory [59]. (Note that a different notion of coherence cost for speakable coherence is previously studied in [15,60].) Our second main result is

Theorem 2 (Operational interpretation of QFI): The coherence cost of a system with Hamiltonian H, state ρ, and period τ is proportional to its QFI about the time parameter. That is

$$C_c^{\text{TI}}(\rho) = \frac{F_H(\rho)}{F_{\text{cbit}}} = \left(\frac{\tau}{2\pi}\right)^2 \times F_H(\rho). \qquad (3)$$

The lower bound $C_c^{\text{TI}}(\rho) \geq F_H(\rho)/F_{\text{cbit}}$ is a special case of a more general result, which is of independent interest: Consider a pair of systems with states ρ_1 and ρ_2 and Hamiltonians H_1 and H_2. If there exists a sequence of TI operations converting copies of system 1 to 2 with rate $R(\rho_1 \to \rho_2)$ and with a vanishing error in the trace distance (in the sense defined above), then

$$R(\rho_1 \to \rho_2) \leq \frac{F_{H_1}(\rho_1)}{F_{H_2}(\rho_2)}. \qquad (4)$$

Although this might be expected from the monotonicity and additivity of QFI, as we discussed in the case of variance, in the presence of a nonzero vanishing error these properties do not necessarily imply Eq. (4). In SM [30], we prove this bound using the connection between QFI and Bures distance. At the end of this Letter we sketch the proof of the other side of Theorem 2. But, first we discuss how QFI appears in the single-copy regime.

QFI in the single-copy regime.—A natural way to quantify the coherence content of a mixed state ρ is to find the minimum QFI of a purification of ρ. More precisely, consider an auxiliary system A with Hamiltonian H_A and let $|\Phi_\rho\rangle_{SA}$ be a pure joint state of SA, with the reduced state $\mathrm{Tr}_A(|\Phi_\rho\rangle\langle\Phi_\rho|_{SA}) = \rho$. What is the minimum possible energy variance, or, equivalently the QFI of such pure states with respect to the total Hamiltonian of systems S and A?

Theorem 3: QFI of system S with state ρ and Hamiltonian H_S, is four times the minimum energy variance of all purifications of ρ with auxiliary systems not interacting with S, i.e.,

$$F_{H_S}(\rho) = \min_{\Phi_\rho, H_A} F_{H_{\mathrm{tot}}}(\Phi_\rho) = 4 \times \min_{\Phi_\rho, H_A} V_{H_{\mathrm{tot}}}(\Phi_\rho), \quad (5)$$

where $H_{\mathrm{tot}} = H_S \otimes I_A + I_S \otimes H_A$, and the minimization is over all pure states $|\Phi_\rho\rangle_{SA}$ satisfying $\mathrm{Tr}_A(|\Phi_\rho\rangle\langle\Phi_\rho|_{SA}) = \rho$, and all Hamiltonians H_A of system A.

This is closely related to the result of [61,62] in the context of metrology (see SM [30] for further discussion). SM presents two different proofs of Theorem 3; one is based on the Uhlmann's theorem [28,29] and the connection between fidelity and QFI (which is similar to the argument of [61]) whereas the second proof is via direct minimization. The latter approach implies that for purification $|\Phi_\rho\rangle_{SA} = \sum_j \sqrt{p_j}|\phi_j\rangle_S|\phi_j\rangle_A$ of state $\rho = \sum_j p_j|\phi_j\rangle\langle\phi_j|$ the minimum in Eq. (5) is achieved for Hamiltonian

$$H_A = -2\sum_{j,k} \frac{\sqrt{p_j p_k}}{p_j + p_k} |\phi_j\rangle\langle\phi_k|H_S|\phi_j\rangle\langle\phi_k|. \quad (6)$$

For this Hamiltonian $F_{H_S}(\rho) = 4[V_{H_S}(\rho) - V_{H_A}(\rho)]$ and the QFI of A is nonzero, provided that the QFI of S is nonzero and ρ is full rank. This has a remarkable implication: even though A carries a nonzero QFI, by discarding this subsystem one does not loose QFI.

Does this theorem determine the coherence cost of ρ? From Theorem 1 one may expect that purification Φ_ρ can be obtained by consuming cbits at rate $(\tau/2\pi)^2 F_{H_{\mathrm{tot}}}(\Phi_\rho)$, which in turn would imply ρ can be obtained with this coherence cost. And the above theorem implies that $F_{H_{\mathrm{tot}}}(\Phi_\rho)$ can be as low as $F_{H_S}(\rho)$. However, there is an issue with this argument: Theorem 1 applies to periodic systems, whereas in general, the dynamics of Φ_ρ under Hamiltonian H_{tot} is not periodic. Imposing the requirement of periodicity, in general increases the minimum variance of purification. For instance, suppose for the same purification Φ_ρ instead of Hamiltonian in Eq. (6) one chooses $H_A = -H_S^*$, that is the complex conjugate of $-H_S$ in the basis $\{|\phi_j\rangle\}$. Then, the period of the joint system will be generally τ. But, now the energy variance is equal to $2W_{H_S}(\rho) \geq F_{H_S}(\rho)$, where $W_{H_S}(\rho) = -\mathrm{Tr}([\sqrt{\rho}, H_S]^2)/2$ is another quantifier of coherence and asymmetry, named skew information [23,54–56].

To overcome this issue, we use a different approach for preparing ρ: we consider ensemble of pure states with density operator ρ. Interestingly, there exists an optimal ensemble whose average QFI is equal to the QFI of ρ.

Theorem 4: QFI is four times the convex roof of variance, i.e.,

$$F_H(\rho) = \min_{\{q_k, \eta_k\}} \sum_k q_k F_H(\eta_k) = 4 \times \min_{\{q_k, \eta_k\}} \sum_k q_k V_H(\eta_k), \quad (7)$$

where the minimization is over ensembles of pure states $\{q_k, \eta_k\}$ satisfying $\sum_k q_k|\eta_k\rangle\langle\eta_k| = \rho$. Furthermore, assuming the dynamics of ρ under H is periodic, the optimal ensemble can be chosen such that each η_k is either an eigenstate of Hamiltonian H or its period under H is an integer fraction of the period of ρ under H.

In analogy with the entanglement theory, the right-hand side of Eq. (7) can be called coherence of formation [63]. The first part of this theorem was originally conjectured by Toth and Petz [64] and was later proven by Yu [65]. Since then this result has found various applications in quantum metrology (see, e.g., Ref. [66]). Note that the convexity of F_H implies that if $\sum_k q_k|\eta_k\rangle\langle\eta_k| = \rho$ then $F_H(\rho) \leq \sum_k q_k F_H(\eta_k)$. Achievability of this bound was proved in [65].

Our resource-theoretic approach reveals a direct connection between this property of QFI and its property studied in Theorem 3, which results in a simple proof of Theorem 4: Let $|\Phi_\rho\rangle_{SA}$ and H_A be, respectively, an optimal purification of ρ, and the corresponding Hamiltonian of the auxiliary system A satisfying Eq. (5). Let $\{|E_k\rangle\}$ be an eigenbasis of Hamiltonian H_A. By measuring A in this basis, one obtains the average joint state $\sigma_{SA} = \sum_k q_k|\eta_k\rangle\langle\eta_k|_S \otimes |E_k\rangle\langle E_k|_A$, where q_k is the probability of observing $|E_k\rangle$ and $|\eta_k\rangle_S = \langle E_k|\Phi\rangle_{SA}/\sqrt{q_k}$ is the corresponding state of S. Then,

$$F_{H_S}(\rho) \leq F_{H_{\mathrm{tot}}}(\sigma_{SA}) \leq F_{H_{\mathrm{tot}}}(\Phi_\rho). \quad (8)$$

Here, both bounds follow from the monotonicity of QFI under TI operations. State ρ of system S can be obtained from σ_{SA} by discarding system A, and σ_{SA} is obtained from Φ_ρ, by measuring A in the energy eigenbasis; both operations are clearly TI. Then, the fact that $F_{H_{\mathrm{tot}}}(\Phi_\rho) = F_{H_S}(\rho)$, implies that both bounds hold as equality. Finally, since energy eigenstates $\{|E_k\rangle\}$ have zero QFI and are orthogonal, QFI of σ_{SA} is equal to the expected QFI of the ensemble $\{q_k, |\eta_k\rangle\}$, i.e., $\sum_k q_k F_{H_S}(\eta_k) = F_{H_{\mathrm{tot}}}(\sigma_{SA}) = F_{H_S}(\rho)$. Thus, Eq. (7) holds with $|\eta_k\rangle = (\sum_j U_{kj}\sqrt{p_j}|\phi_j\rangle)/\sqrt{q_k}$, and probability $q_k = \langle E_k|\rho|E_k\rangle = \sum_j p_j |U_{kj}|^2$, where $U_{kj} = \langle E_k|\phi_j\rangle$ are the matrix elements of the unitary that diagonalizes H_A in Eq. (6) in the eigenbasis of ρ. In summary, the fact that QFI is the minimum variance of purifications (Theorem 3) implies that QFI is also the convex roof of variance (Theorem 4). The second part of Theorem 4 is shown in SM [30].

Sketch of proof of Theorem 2.—By combining Theorems 1 and 4 with the standard typicality arguments (e.g., in [15,67]), we show that the coherence cost of any state is determined by its QFI. Let $(q_k, |\eta_k\rangle) : k \in \mathbb{S}$ be the optimal ensemble satisfying Eq. (7). As we saw in the above proof, $\mathbb{S}$ is a finite set. Then, $\rho^{\otimes m} = \sum_{\mathbf{k}} q_{\mathbf{k}} |\eta_{\mathbf{k}}\rangle\langle\eta_{\mathbf{k}}|$, where $\mathbf{k} = k_1 \cdots k_m$, $q_{\mathbf{k}} = q_{k_1} \cdots q_{k_m}$ and $|\eta_{\mathbf{k}}\rangle = |\eta_{k_1}\rangle \cdots |\eta_{k_m}\rangle$. For any $k \in \mathbb{S}$ let $n_l(\mathbf{k})$ be the number of occurrence of state $|\eta_l\rangle$ in $|\eta_{\mathbf{k}}\rangle$. Then, for $\delta > 0$ define typical strings as those for which the relative frequency of any $l \in \mathbb{S}$ is between $q_l - \delta$ and $q_l + \delta$, i.e., $\{\mathbf{k} = k_1 \cdots k_m | \ \forall \ l \in \mathbb{S}: |(n_l(\mathbf{k})/m) - q_l| \leq \delta\}$. Then,

$$\rho^{\otimes m} = \sum_{\mathbf{k}\in\text{typical}} q_{\mathbf{k}} |\eta_{\mathbf{k}}\rangle\langle\eta_{\mathbf{k}}| + \sum_{\mathbf{k}\notin\text{typical}} q_{\mathbf{k}} |\eta_{\mathbf{k}}\rangle\langle\eta_{\mathbf{k}}|. \quad (9)$$

Now we define a sequence of TI operations that prepare $\rho^{\otimes m}$ with a vanishing error: sample string $\mathbf{k}$ with probability $q_{\mathbf{k}}$. If $\mathbf{k}$ is not a typical string, prepare a fixed incoherent state, which does not consume any cbits. By the law of large numbers, as $m \to \infty$ the probability of such events goes to zero and therefore the corresponding error vanishes. For typical $\mathbf{k}$, up to a permutation, $|\eta_{\mathbf{k}}\rangle$ can be written as $\bigotimes_l |\eta_l\rangle^{\otimes n_l(\mathbf{k})}$, and typicality implies $n_l(\mathbf{k}) \leq m(q_l + \delta)$. Therefore, $|\eta_{\mathbf{k}}\rangle$ can be obtained from $\bigotimes_l |\eta_l\rangle^{\otimes \lceil m(q_l+\delta)\rceil}$, which has the energy variance $\sum_l \lceil m(q_l + \delta)\rceil V_H(\eta_l)$. Using the second part of Theorem 4, one can show that the period of this state is equal to τ, the period of ρ. Then, using a simple variant of Theorem 1 we show that as $m \to \infty$, by consuming $(\tau/\pi)^2 \sum_l \lceil m(q_l + \delta)\rceil V_H(\eta_l)$ cbits, we can prepare state $|\eta_{\mathbf{k}}\rangle$ with a vanishing error (note that the energy variance of cbit is π^2/τ^2). Using the facts that $\sum_l q_l V_H(\eta_l) = F_H(\rho)/4$ and $V_H(\eta_l) \leq \|H\|^2$, where $\|H\|$ is the operator norm, we conclude that for any $\delta > 0$, by consuming cbits at rate $(\tau/2\pi)^2 \times (F_H(\rho) + 4\delta\|H\|^2)$ per copy, one can prepare copies of the desired system with vanishing error. This proves one direction of Theorem 2. See SM [30] for details and the proof of the other direction.

Conclusion.—Preparing a general state requires consumption of both work and energetic coherence. When coherence is a free resource, the work cost is determined by the free energy and when work is free the coherence cost is determined by QFI. In a more complete picture both of these resources should be taken into account. Understanding the possible tradeoff between these resource costs remains an open question. Also, generalizing the present results to the case of non-Abelian groups, such as SO(3) will be interesting (see, e.g., Refs. [68,69] for progress in this direction). Our resource-theoretic approach enabled us to clarify a previously unnoticed relation between fundamental properties of QFI, which is arguably the most studied quantity in quantum metrology and estimation theory. As QFI has found extensive applications in different areas of physics, exploring further implications of Theorems 2 and 3 will be interesting.

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