Simulability of High-Dimensional Quantum Measurements

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We investigate the compression of quantum information with respect to a given set $\mathcal{M}$ of high-dimensional measurements. This leads to a notion of simulability, where we demand that the statistics obtained from $\mathcal{M}$ and an arbitrary quantum state ρ are recovered exactly by first compressing ρ into a lower-dimensional space, followed by some quantum measurements. A full quantum compression is possible, i.e., leaving only classical information, if and only if the set $\mathcal{M}$ is jointly measurable. Our notion of simulability can thus be seen as a quantification of measurement incompatibility in terms of dimension. After defining these concepts, we provide an illustrative example involving mutually unbiased bases, and develop a method based on semidefinite programming for constructing simulation models. In turn we analytically construct optimal simulation models for all projective measurements subjected to white noise or losses. Finally, we discuss how our approach connects with other concepts introduced in the context of quantum channels and quantum correlations.

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Introduction.—Quantum measurements play a fundamental role in quantum theory and its applications, notably in quantum information processing and metrology. Indeed measurements represent the bridge between a quantum system and an external observer; hence essentially any quantum experiment relies on a quantum measurement process. More recently, the role of quantum measurements as a resource was clarified in the context of quantum information processing tasks; see Refs. [1,2] for a review. At the formal level, the concept of joint measurability [3] provides a framework for the characterization and quantification of the incompatibility of quantum measurements, which can be connected to their usefulness in, e.g., state-discrimination problems [4–9] and quantum steering [10–13].

A natural question is whether quantum measurement incompatibility can also be quantified in terms of dimension. Intuitively, a set of quantum measurements defined on a high-dimensional Hilbert space may feature a stronger form of incompatibility than what is possible for lower dimensions.

In this Letter we address this question and propose a notion of dimensionality for measurement incompatibility. This notion can be understood, and naturally motivated, in a scenario involving the compression of quantum information. Loosely speaking, we ask whether the statistics of a set of d-dimensional positive operator-valued measures (POVMs) $\mathcal{M}$ (considering any possible quantum state ρ) can be exactly recovered from first projecting ρ onto an n-dimensional space (with $1 \leqslant n < d$) and then performing POVMs in this lower-dimensional space. If such a compression is possible, we say the set $\mathcal{M}$ is n-dimensional simulable. Note that the case $n = 1$ exactly corresponds to the notion of joint measurability; indeed, in this case, the full quantum information can be compressed to classical information; see also Refs. [14,15]. However, as we will see below, there exist sets of POVMs $\mathcal{M}$ that are incompatible, but yet simulable with n-dimensional measurements with $1 < n < d$. The notion of n-dimensional simulability can thus be seen as an extension of the concept of joint measurability, providing a quantification of the incompatibility of quantum measurements in terms of dimension.

After introducing these ideas more formally, we provide illustrative examples based on sets of mutually unbiased measurements. Then, we present a method based on semidefinite programming (SDP) to show that a set of measurements is n-dimensional simulable. In turn we consider the continuous sets of all projective measurements subjected to noise or losses, and construct optimal n-simulation models. Finally, we establish a link to partially entanglement-breaking channels [16], and discuss connections to other concepts as well as open questions.

Scenario and definition.—Consider the following task: a sender (Alice) is located on the moon and wants to transmit an (arbitrary) d-dimensional quantum state ρ to a receiver (Bob) located on earth. Upon receiving ρ, Bob will perform a set of measurements $\mathcal{M} = \{M_{a|x}\}_{a,x}$, where $\{M_{a|x}\}_a$ denotes a POVM, i.e., $M_{a|x} \geqslant 0$ and $\sum_a M_{a|x} = 1\ \forall\ a,\ x$. This leads to the following statistics termed the target data: $p(a|x,\rho) = \mathrm{Tr}(M_{a|x}\rho)$.

So far, we assumed that the channel between Alice and Bob is ideal, i.e., a d-dimensional identity channel. In the following, however, we will consider the dimensionality of the quantum channel as a resource, which we will aim to minimize. Note that a classical channel (of arbitrary capacity) is always available for free.

We now ask if the target data could be reproduced by using a lower-dimensional quantum channel. That is, we look for a quantum instrument $\{\mathcal{E}_\lambda\}_\lambda$, where each $\mathcal{E}_\lambda : B(\mathbb{C}^d) \to B(\mathbb{C}^n)$ is a completely positive (CP) map, λ a classical outcome, and the map $\sum_\lambda \mathcal{E}_\lambda$ is trace preserving. Alice would then send the compressed n-dimensional state $\mathcal{E}_\lambda(\rho)$ to Bob, together with the classical outcome λ. Bob finally chooses from a set of measurements $\mathcal{N} = \{N_{a|x,\lambda}\}_{a,x,\lambda}$. The protocol is successful if we recover the target data, i.e., if

$$\sum_\lambda \mathrm{Tr}[N_{a|x,\lambda}\mathcal{E}_\lambda(\rho)] = \mathrm{Tr}(M_{a|x}\rho) \quad \forall\, \rho, \tag{1}$$

or equivalently if $M_{a|x} = \sum_\lambda \mathcal{E}^*_\lambda(N_{a|x,\lambda})$, where $\mathcal{E}^*_\lambda$ is the adjoint map to $\mathcal{E}_\lambda$ (for continuous instruments one replaces the sum with an integral; see examples below). In this case, we say that $\mathcal{M}$ is n-dimensional simulable, as illustrated in Fig. 1. Clearly, in the case $n = 1$, the instrument corresponds to a POVM, and the measurements afterward form classical postprocessings of the classical output. This coincides with the concept of joint measurability: $M_{a|x} = \sum_\lambda G_\lambda p(a|x,\lambda)$, where $\{G_\lambda\}$ represents the joint (or parent) POVM [1,2].

Illustrative example.—To illustrate the concept, we present an example involving mutually unbiased bases (MUBs). Recall that two bases $\{|\varphi^1_i\rangle\}_i$ and $\{|\varphi^2_j\rangle\}_j$ are termed mutually unbiased if $|\langle\varphi^1_i|\varphi^2_j\rangle|^2 = 1/d$ for all i and j. There are at most $d+1$ MUBs in dimension d, and a construction for the complete set of $d+1$ MUBs is only known when d is a power of a prime number [17].

Here we consider a set $\mathcal{M}$ consisting of m measurements in MUBs subject to white noise. That is, projection-valued measures (PVMs) composed of projectors $P_{a|x} = |\varphi^x_a\rangle\langle\varphi^x_a|$ preceded by a white noise channel. The resulting POVMs read as

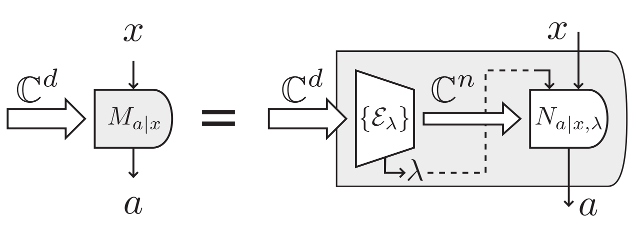

FIG. 1. A measurement assemblage $\{M_{a|x}\}$ in dimension d is said to be n-dimensional simulable if it can be replicated by first compressing the measured system down to dimension $n < d$ with a "parent" instrument $\{\mathcal{E}_\lambda\}$ independent of the setting x, and then performing some measurements $\{N_{a|x,\lambda}\}$ on the n-dimensional system.

$$M^\eta_{a|x} = \eta P_{a|x} + (1-\eta)\frac{\mathbb{1}}{d}, \tag{2}$$

where $x \in \{1,\ldots,m\}$ and $a \in \{1,\ldots,d\}$. Note that we use the standard construction of MUBs [17] with $\{|\varphi^1_i\rangle\}_i$ being the computational basis.

We now show that, depending on the amount of noise $1-\eta$, the set $\mathcal{M}$ becomes n-dimensional simulable. Let us first construct an appropriate map, implementing the compression from dimension d to n. For a given basis $\{|\psi_i\rangle\}_i$ we consider the set of $\binom{d}{n}$ projectors Π_λ onto an n-dimensional subspace of $\mathbb{C}^d$ spanned by the vectors of the basis.

This defines an instrument $\{\mathcal{E}_\lambda\}$, with each $\mathcal{E}_\lambda$ given by a Kraus operator $K_\lambda = \sqrt{\binom{d}{n}^{-1}(d/n)}\Pi_\lambda$, compressing from dimension d to n. Here $\binom{d}{n}$ is simply the number of projectors Π_λ with n ones on the diagonal, and the term $\sqrt{(d/n)}$ is due to normalization. Next, for any POVM P_a acting on $\mathbb{C}^d$, define its restriction to the subspace labeled by λ as

$$N_{a|\lambda} = \Pi_\lambda P_a \Pi_\lambda, \tag{3}$$

which is a POVM on $\mathbb{C}^n$ since $N_{a|\lambda} \geqslant 0$ and $\sum_a N_{a|\lambda} = \Pi_\lambda \mathbb{1}_d \Pi_\lambda = \mathbb{1}_n$. We can now compute

$$\sum_\lambda \mathcal{E}^*_\lambda(N_{a|\lambda}) = \frac{n-1}{d-1}P_a + \left(1 - \frac{n-1}{d-1}\right)\mathcal{T}_{\{|\psi_i\rangle\}}[P_a] \tag{4}$$

where $\mathcal{T}_{\{|\psi_i\rangle\}}[P_a] = \sum_i |\psi_i\rangle\langle\psi_i|P_a|\psi_i\rangle\langle\psi_i|$ is the twirling map in the basis $\{|\psi_i\rangle\}$ used to define the instrument. In particular, if the eigenbasis of P_a and $\{|\psi_i\rangle\}$ are mutually unbiased, one obtains $\mathcal{T}_{\{|\psi_i\rangle\}}[P_a] = (\mathbb{1}/d)$, while if they coincide, one trivially gets $\mathcal{T}_{\{|\psi_i\rangle\}}[P_a] = P_a$.

If the set $\mathcal{M}$ is composed of m MUBs, we construct an instrument that chooses one of the bases $y = 1,\ldots,m$ randomly and performs $\{\mathcal{E}_{\lambda|y}\}$ as defined above. Then, for the setting x Bob does the measurement $N_{a|\lambda,x,y}$ as defined in Eq. (3). This construction results in

$$\sum_{\lambda,y}\frac{1}{m}\mathcal{E}^*_{\lambda|y}(N_{a|\lambda,x,y}) = \frac{1}{m}P_{a|x} + \frac{m-1}{m}\left[\frac{n-1}{d-1}P_{a|x} + \left(1-\frac{n-1}{d-1}\right)\frac{\mathbb{1}}{d}\right], \tag{5}$$

which equals $M^\eta_{a|x}$ for $1-\eta = [(m-1)/m]\{1-[(n-1)/(d-1)]\}$, and implies the following observation.

Result 1. The set $\mathcal{M}$ in Eq. (2) of m noisy projective measurements in MUBs on $\mathbb{C}^d$ is n-dimensional simulable if

$$\eta \leqslant 1 - \frac{m-1}{m}\,\frac{d-n}{d-1}. \tag{6}$$

Let us first discuss the case of a pair of MUBs, i.e., $m = 2$. It is known that the pair is jointly measurable (i.e., one-dimensional simulable) if and only if $\eta \leqslant \eta^* = \frac{1}{2}\{1 + [1/(1+\sqrt{d})]\}$ [18,19]. Hence it follows that for a noise parameter satisfying $1 - \frac{1}{2}[(d-n)/(d-1)] \geqslant \eta > \eta^*$ (which is possible for all $d > 2$) we get a set of measurements that is incompatible but nevertheless n-dimensional simulable (for some $n > 1$). Alternatively, consider the full set of $m = 4$ MUBs in $d = 3$. This set is incompatible if and only if $\eta > (1 + 3\sqrt{5}) \approx 0.4818$ [19] while it is two-dimensional simulable for $\eta \leqslant \frac{5}{8} = 0.625$.

Finally, note that the above construction for n-dimensional simulability uses a heuristic choice of bases for the compression instrument. We verify below that this choice is suboptimal.

SDP method.—In order to explore more general schemes, involving arbitrary bases for the compression, we now present a numerical approach based on SDP. For the example of MUBs, this shows that better schemes are indeed possible.

Consider a set of measurements $M_{a|x}$, to which we add noise. For clarity we focus here on the case of white noise, but the technique applies in general. Formally, we consider sets of POVMs of the form

$$M^{\eta}_{a|x} = \eta M_{a|x} + (1-\eta)\mathrm{Tr}[M_{a|x}]\frac{\mathbb{1}_d}{d}.$$

Our goal now is to derive a lower bound on the critical value η^*, below which the set is n-dimensional simulable. In particular, if the lower bound is found to be trivial, i.e., $\eta \geqslant 1$, we conclude that the original measurements $M_{a|x}$ are n-dimensional simulable.

We first choose a compression map, i.e., an instrument $\{\mathcal{E}_\lambda\}$ consisting of a finite number of CP maps $\mathcal{E}_\lambda$. For a fixed compression map $\{\mathcal{E}_\lambda\}$ and measurements $M^{\eta}_{a|x}$ we can now find the maximal value of η (i.e., minimize the noise) while ensuring n-dimensional simulability, optimizing over Bob's final measurements, via the following SDP:

$$\begin{aligned} &\max_{\eta,\{\tilde{N}_{a|x,\lambda}\}} \eta \\ &\quad \text{such that} \sum_\lambda \mathcal{E}_\lambda \tilde{N}_{a|x,\lambda}\mathcal{E}^\dagger_\lambda = M^{\eta}_{a|x} \quad \forall\, a,x \\ &\quad \sum_a \tilde{N}_{a|x,\lambda} \leqslant \mathbb{1}_n \quad \forall\, x,\lambda \\ &\quad \tilde{N}_{a|x,\lambda} \geqslant 0 \quad \forall\, a,x,\lambda, \end{aligned} \tag{7}$$

where $\{\tilde{N}_{a|x,\lambda}\}_{a,x,\lambda}$ is the set of subnormalized measurements in dimension n. To illustrate the relevance of this method, let us consider again the case of a pair of MUBs in dimension $d = 3$, i.e., setting $M_{a|x} = |\varphi^x_a\rangle\langle\varphi^x_a|$, and compressing to $n = 2$. Here we choose a simple compression channel: we first choose a basis (via a unitary U_μ) and perform a projection onto the $\binom{d}{n}$ n-dimensional subspaces, denoted by projectors Π_i. Hence we set $\mathcal{E}_\lambda = U_\mu \Pi_i$ with $\mu \in 0,\ldots,|\mu|-1$, $i \in \{0,\ldots,\binom{d}{n}-1\}$ and hence $\lambda \in \{0,\ldots,|\mu|\binom{d}{n}-1\}$. Optimizing over choices of the $|\mu| = 2$ and $|\mu| = 3$ basis, we find noise thresholds of $\eta \approx 0.7803$ and $\eta \approx 0.8281$, respectively, hence clearly improving upon the bound of $\eta = 5/8$ we got in the analytical construction; see Eq. (6). Moreover, when allowing for more basis (up to $|\mu| = 5$), we could find no improvement. The MATLAB code can be found in the Supplemental Material [20].

The case of all projective measurements.—So far we analyzed sets $\mathcal{M}$ with finitely many measurements. Now we turn our attention to continuous sets of measurements. Precisely, we consider assemblages $\mathcal{M}^{\eta}_{\mathrm{PVM}}$ made of all rank-1 projective measurements subjected to white noise. Note that this automatically extends to all projective measurements which can be obtained from rank-1 projective measurements by postprocessing. Assuming an isotropic noise, our set is made of all POVMs

$$\mathcal{M}^{\eta}_{\mathrm{PVM}} = \{M^{\eta}_{a|U}\}_U \quad \text{with} \quad M^{\eta}_{a|U} = U^\dagger M^{\eta}_a U, \tag{8}$$

where U runs through all unitary operators on $\mathbb{C}^d$ and M^{η}_a is the noisy (or lossy) measurement in the computational basis. On the one hand, the continuous case may seem more complicated as the infinite number of possible measurements cannot be tackled with the SDP of Eq. (7). On the other hand, the symmetry of the set helps simplifying the optimal compression scheme as we shall see now.

The set $\mathcal{M}^{\eta}_{\mathrm{PVM}}$ is a particular case of what we call an *invariant assemblage*. This is a set $\mathcal{M}$ such that $M_a \in \mathcal{M}$ implies that $U^\dagger M_a U$ is also in $\mathcal{M}$ for all unitaries U. For such an invariant assemblage the choice of the compression bases does not play any role, which leads to the following observation.

Result 2. An n-dimensional simulable invariant assemblage $\mathcal{M}$ can be compressed with the continuous instrument $\{\mathcal{E}_V\}_V$ with CP map density $\mathcal{E}_V$, such that $\mathcal{E}_V(\rho) = K_V \rho K^\dagger_V$ with

$$K_V = \sqrt{\frac{d}{n}}\Pi_n V, \tag{9}$$

with V running through all unitary operators on $\mathbb{C}^d$, and Π_n is a projection onto a fixed n-dimensional subspace of $\mathbb{C}^d$. In addition, if there exists a measurement $M_a \in \mathcal{M}$ such that $U^\dagger M_a U$ generates all of $\mathcal{M}$ (i.e., the action is *transitive*), then it suffices to find $N_{a|V}$ satisfying $M_a = \int dV\, \mathcal{E}^*_V(N_{a|V})$, as then $U^\dagger M_a U = \int dV\, \mathcal{E}^*_V(N_{a|VU^\dagger})$.

Here and below $dV = d\mu(V)$ is the Haar measure. If there are multiple orbits, that is, a family of measurements $\mathcal{N}$ such that $U^\dagger \mathcal{N} U$ generates all of $\mathcal{M}$, then it suffices to find a measurement $N_{a|V}$ as above for each measurement in $\mathcal{N}$. A detailed proof of the result can be found in the Supplemental Material [20], and we explain only the intuition here. First, because of the invariance of $\mathcal{M}$ one can freely apply any (random) unitary V to the state before the compression. Thus if the set is compressible with some instrument $\{\mathcal{E}_\lambda\}$ it is also compressible with an instrument where a random V is applied before $\{\mathcal{E}_\lambda\}$. Second, one shows that any such instrument can be obtained by postprocessing of $\{\mathcal{E}_V\}$ defined in the statement of the result. Finally, the relation between the compressed POVMs $N_{a|V}$ and $N_{a|VU^\dagger}$ is a simple consequence of Haar measure invariance.

All noisy projective measurements.—We now consider the set of all projective measurements $\mathcal{M}^\eta_{\mathrm{PVM}}$ in Eq. (8) subject to white noise

$$M^\eta_a = \eta|a\rangle\langle a| + \frac{1-\eta}{d}\mathbb{1}, \tag{10}$$

where $|a\rangle$ denotes the d vectors of the computational basis. We now want to find the $N_{a|V}$ such that $\int dV\, \mathcal{E}^*_V(N_{a|V})$ equals M^η_a for the highest value of η possible. In fact, the optimal compressed POVM here is given by

$$N_{a|V} = \underset{\tilde{N}_a:\substack{\mathrm{POVM}\\ \mathrm{on}\,\mathbb{C}^n}}{\operatorname{argmax}} \sum_{a=1}^{d} \langle a|V^\dagger \tilde{N}_a V|a\rangle \tag{11}$$

with $\tilde{N}_a$ embedded in $\mathbb{C}^d$, which can be seen in two steps; cf. Supplemental Material [20] for details. First, this choice does result in an operator of the desired form $M'_a = \eta(x)|a\rangle\langle a| + \{[1-\eta(x)]/d\}\mathbb{1}$ with $\eta(x) = [(dx-1)/(d-1)]$ and

$$x = \frac{1}{n}\int dV \max_{\tilde{N}_a:\substack{\mathrm{POVM}\\ \mathrm{on}\,\mathbb{C}^n}} \sum_{a=1}^{d} \langle a|V^\dagger \tilde{N}_a V|a\rangle. \tag{12}$$

Second, by construction with the max inside the integral this yields the highest value of x, and thus η, possible (This is precisely the intuition behind the definition [Eq. (11)].}. This leads to the following full characterization of n-dimensional simulability of noisy PVMs in any finite dimensional system.

Result 3. The set of all noisy (white noise) projective measurements $\mathcal{M}^\eta_{\mathrm{PVM}}$ in dimension d is n-dimensional simulable if and only if

$$\eta \leqslant \eta_{d\mapsto n} = \frac{dx-1}{d-1} \tag{13}$$

with x given in Eq. (12) where dV is the Haar measure on the unitary operators V on $\mathbb{C}^d$, $\{|a\rangle\}_{a=1}^d$ is the computational basis of $\mathbb{C}^d$, and $\mathbb{C}^n$ is any n-dimensional subspace of $\mathbb{C}^d$.

TABLE I. Some values of the white noise threshold $\eta_{d\mapsto n}$ for the n-dimensional simulability of $\mathcal{M}^\eta_{\mathrm{PVM}}$—the set of all projective measurements in dimension d. We computed the values with Wolfram *Mathematica* using the build-in function CircularUnitaryMatrixDistribution to sample from the Haar measure (5000 points in each case), and SemidefiniteOptimization to solve the optimal POVMs $\tilde{N}_a$ in Eq. (12); the code can be found in the Supplemental Material [20]. The values in italics give the known white noise threshold for the incompatibility of all PVMs and equal to $\eta_{d\mapsto 1} = [\sum_{k=1}^d (1/k) - 1]/(d-1)$, cf. [11,21].

$n\backslash d$	2	3	4	5	6
1	*0.5*	*0.42*	*0.36*	*0.32*	*0.29*
2		0.70	0.56	0.48	0.42
3			0.77	0.64	0.56
4				0.81	0.70
5					0.84

The POVM maximization [Eq. (12)] is a simple SDP, so the threshold $\eta_{d\mapsto n}$ can be computed numerically by sampling from the Haar measure (or integrating numerically) and solving the SDP for each V. We report some values in Table I.

An upper bound on the threshold can be obtained by applying the Cauchy-Schwarz inequality to the sum in Eq. (12), and leads to $\eta_{d\mapsto n} \leqslant \{[d\sqrt{[(n+1)/(d+1)]} - 1]/[d-1]\}$, as we show in the Supplemental Material [20] by using results from Refs. [22,23]. There we also derive a lower bound $\eta_{d\mapsto(d-1)} \geqslant (\{d^2 - d[1 + \sum_{k=1}^d (1/k)] + 1\}/(d-1)^2)$ for the case $n = d-1$.

For $n = 1$ our considerations reduce to the joint measurability of noisy projective measurements, where the white noise threshold is known $\eta_{d\mapsto 1} = [\sum_{k=1}^d (1/k) - 1]/(d-1)$ [21]. In this case the subspace $\mathbb{C}^n$ contains a single state $|\Psi\rangle = V|1\rangle$. Accordingly to Eq. (12) $\eta_{d\mapsto 1}$ can be computed from $x = \int \mathrm{d}\Psi \max_a |\langle a|\Psi\rangle|^2 = (1/d)\sum_{k=1}^d (1/k)$, where d$\Psi$ is the uniform measure over states in $\mathbb{C}^d$ (invariant under unitraries), which is equivalent to the derivation in Ref. [21].

All lossy projective measurements.—We now briefly analyze the set of all projective measurements $\mathcal{M}^\eta_{\mathrm{PVM}}$ in Eq. (8) subject to loss:

$$M^\eta_a = \eta|a\rangle\langle a|; \qquad M^\eta_\emptyset = (1-\eta)\mathbb{1} \tag{14}$$

This set describes measurements with a limited efficiency—the additional element $M^\eta_\emptyset$ corresponds to the "no-click" outcome. The required rank-1 form of the operators $M^\eta_a \propto |a\rangle\langle a|$ implies that for $a \neq \emptyset$ the POVM element $\hat{N}_{a|V}$ can only be nonzero if $\mathcal{E}^*_V(\hat{N}_{a|V}) \propto |a\rangle\langle a|$. In other words, the vector $|a\rangle$ has to be in the n-dimensional subspace selected by the instrument, i.e.,

$$|a\rangle \in \mathrm{span}\{V|1\rangle, \ldots, V|n\rangle\}. \tag{15}$$

This condition is only fulfilled for a set of V of measure zero, hence $M'_a \propto |a\rangle\langle a| \Rightarrow M'_a = 0$. We can thus conclude that

Result 4. For any positive efficiency $\eta > 0$ the set of all lossy projective measurements $\mathcal{M}^{\eta}_{\mathrm{PVM}}$ is not n-dimensional simulable for any $n < d$.

An equivalent definition.—The notion of joint measurability has a direct connection to entanglement-breaking properties of quantum channels, cf. Refs. [13] and [24,25]. As a final point we now discuss how this connection naturally extends to n-dimensional simulability.

A quantum channel Λ is said to be n-partially entanglement breaking (n-PEB) if, for all states ρ, the Schmidt number of the state $(\Lambda \otimes \mathbb{1})[\rho]$ is less than or equal to n [16]. In the finite-dimensional setting, this is equivalent to the existence of Kraus operators $\{K_\lambda\}$ of Λ each with $\mathrm{rank}(K_\lambda) \leqslant n$ [16]. This notion provides an alternative way to define n-dimensional simulability:

Claim 5. A measurement assemblage $M_{a|x}$ is n-dimensional simulable if and only if there exists a quantum channel Λ that is n-PEB and a measurement assemblage $N_{a|x}$ such that

$$M_{a|x} = \Lambda^*(N_{a|x}). \tag{16}$$

For the if direction one uses the singular value decomposition of the rank-n Kraus operators $\{K_\lambda\}$ and $N_{a|x}$ to define the instrument $\{\mathcal{E}_\lambda\}$ with n-dimensional output and the subsequent measurement $N_{a|x,\lambda}$. For the only if direction one can simply view any compression instrument $\{\mathcal{E}_\lambda\}$ as a channel Λ that outputs a quantum system of dimension n and a classical register encoding λ. This channel is manifestly n-PEB. The details of the proof can be found in the Supplemental Material [20]. Note that we prove the result for a finite amount of classical communication λ, respectively a channel Λ with a finite dimensional output, and conjecture it to be true in general.

Related concepts.—We now compare the idea of n-dimensional simulability with some previously introduced notions. First, in Ref. [26] a related notion of "n-compressibility" of a set of quantum measurements has been proposed. Similarly to our definition a set of measurements is said to be n-compressible if $M_{a|x} = \sum_\lambda \mathcal{E}^*_\lambda(N_{a|x,\lambda})$ [Notably the authors of Ref. [26] also consider the possibility to bound the amount of classical communication via the number of different instrument outcomes λ (the dimension of the classical register output by the instrument).], but in sharp contrast the compressed measurement has to take the form $N_{a|x,\lambda} = \tilde{\mathcal{E}}^*_{|\lambda}(M_{a|x})$. Here, $\tilde{\mathcal{E}}_{|\lambda}$ is a set of completely positive and trace preserving maps decompressing the system back into dimension d, and the same measurement $M_{a|x}$ has to be used after the decompression. Clearly, a set of n-compressible measurements is n-dimensional simulable by construction, but not the other way around. In particular, it is known that single measurements can have a larger compression dimension than one [26].

Other works have investigated measurement (in)compatibility in subspaces [27–29], while Ref. [30] defined a concept of n-compatibility considering a scenario where a set of measurements is performed on n copies of a state. As far as we can say, these concepts are unrelated to n-dimensional simulability.

Finally, the problem of implementing POVMs with given resources has been investigated [31–33], for example the implementation of POVMs using only projective measurements of the same dimension [34–36]. This question has been connected to the compression of quantum information [31].

Conclusion.—We have introduced the concept of n-dimensional simulability of a set of measurements, motivated by a scenario of compression of quantum information. When full compression is possible (i.e., $n = 1$), our notion corresponds to joint measurability. Hence our approach can be used as a quantification of measurement incompatibility in terms of dimension. We discussed a number of examples, providing analytical constructions as well as numerical methods.

More generally, the concept of n-dimensional simulability turns out to be connected to several other relevant notions of quantum information theory. First, as we showed above, n-dimensional simulability relates to partially entanglement-breaking channels. Second, there is a direct connection between n-dimensional simulability and the notion of genuine high-dimensional steering [37], which will be presented in detail in a companion article [38]. These links generalize the well-known connection between joint measurability, steering, and quantum channels [10–12]. They also open new questions, for example, whether the dimensionality of a quantum channel could be tested in a partially device-independent manner.

Beyond quantum compression, our approach also has implications for device-independent quantum information processing. For example, it is clear that a set of measurements that is n-dimensional simulable is of limited use for randomness certification, as it can lead to (at most) $2\log(n)$ bits of randomness in a black-box setting.

Finally, an intriguing question is whether our approach could be generalized to the case of continuous variable measurements. It turns out that, in the infinite-dimensional case, the Kraus operators' ranks and entanglement-breaking properties of a channel are not tied together anymore [39,40], opening different possibilities to extend our concept.

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[20] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.129.190401 for the proof of Result 2 (Sec. A), the proof of Result 3 (Sec. B), an upper bound on the n-simmulability of all noisy PVMs (Sec. C), a lower bound for the simmulability of noisy PVMs of $n = d - 1$ (Sec. D), and the proof of Result 5 (Sec. E). DiscreteSDP.zip contains the MATLAB code used to obtain the numerical results discussed after Eq. (7). Table 1. nb containes the *Mathematica* code that was used to produce the results of Table 1.

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