Solvable 3D Kondo Lattice Exhibiting Pair Density Wave, Odd-Frequency Pairing, and Order Fractionalization

Piers Coleman[1,2] Aaditya Panigrahi[1] and Alexei Tsvelik[3]

[1]*Center for Materials Theory, Department of Physics and Astronomy, Rutgers University,*
136 Frelinghuysen Road, Piscataway, New Jersey 08854-8019, USA
[2]*Department of Physics, Royal Holloway, University of London, Egham, Surrey TW20 0EX, United Kingdom*
[3]*Division of Condensed Matter Physics and Materials Science, Brookhaven National Laboratory, Upton, New York 11973-5000, USA*

The Kondo lattice model plays a key role in our understanding of quantum materials, but a lack of small parameters has posed a long-standing problem. We present a three-dimensional $S = 1/2$ Kondo lattice model describing a spin liquid within an electron sea. Strong correlations in the spin liquid are treated exactly, enabling a controlled analytical approach. Like a Peierls or BCS phase, a logarithmically divergent susceptibility leads to an instability into a new phase at arbitrarily small Kondo coupling. Our solution captures a plethora of emergent phenomena, including odd-frequency pairing, pair density wave formation and order fractionalization. The ground-state state is a pair density wave with a fractionalized charge e, $S = 1/2$ order parameter, formed between electrons and Majorana fermions.

The rich physics of the Kondo lattice, describing an array of local moments interacting with an electron sea, plays a key role in our understanding of quantum materials, from heavy fermion compounds to twisted moiré lattices [1–4]. A key element of this model is the transformation of spins into heavy fermions, producing a large Fermi surface [5,6]. These interactions have to date been treated with approximate methods, such as the large-N expansion [7–11] and dynamical mean field theory [12].

Our work builds on a series of important developments in the theory of Kitaev models and their connection with the Kondo lattice [13–18]. Earlier variants of Kitaev-Kondo lattices include models that couple the original, spin-gapped Kitaev spin liquid to a conduction sea [15–17], and models that couple two-dimensional Yao-Lee spin liquid to a conduction sea via an octupolar coupling [18].

Here, we introduce a three-dimensional Kondo lattice model which couples a Z_2 spin liquid with a Fermi surface to a conduction sea, in which the Kondo lattice physics can be treated analytically to leading logarithmic accuracy. The spin fluid is a three-dimensional generalization of the Yao-Lee spin liquid [19], embedded on a hyperoctagonal lattice [13], chosen because its cubic, trivalent structure gives rise to an exactly solvable, gapless spin liquid whose gapless Majorana excitations lie on a Fermi surface. Like the Kitaev spin liquid, the Z_2 gauge fields associated with the fractionalized spins are static and can be treated exactly [19,20].

Recent work has hypothesized that hybridization between electrons and fractionalized excitations can give rise to a new kind of *fractionalized* order with half-integer quantum numbers [21,22]. Such order emerges as a result of condensation of bound states of electrons and fractionalized excitations of the spin liquid. Our Kondo lattice model provides a rigorous example of this phenomenon and gives a clear description of its manifestation in terms of physical observables such as the electron self-energy. In particular, at half filling, the judicious choice of the lattice guarantees a perfect nesting between spinon and electron Fermi surfaces and allows us to sum the leading logarithmic divergences, establishing an instability at infinitesimal Kondo coupling into a pair-density wave with a charge e, $S = 1/2$ order parameter, which induces odd-frequency pairing among the conduction electrons and also gives rise to a neutral, Majorana Fermi surface.

Our model Hamiltonian $H = H_C + H_{YL} + H_K$, where

$$H_C = -t\sum_{\langle ij \rangle}(c_{i\sigma}^\dagger c_{j\sigma} + \text{H.c.}) - \mu\sum_j c_{j\sigma}^\dagger c_{j\sigma},$$

$$H_{YL} = (K/2)\sum_{\langle ij \rangle}(\vec{\sigma}_i \cdot \vec{\sigma}_j)\lambda_i^{\alpha_{ij}}\lambda_j^{\alpha_{ij}},$$

$$H_K = J\sum_j \vec{S}_j \cdot (c_j^\dagger \vec{\sigma} c_j). \tag{1}$$

Here, $\langle i, j \rangle$ are neighboring sites on the hyperoctagonal lattice [13], a trivalent body centered cubic (bcc) crystal with four atoms per primitive unit cell, coiled around a helix to form alternating square and octagonal spirals (Fig. 1). H_C describes hopping electrons while H_{YL} describes a three-dimensional Yao-Lee (3DYL) spin liquid on the same lattice, with an orbital and spin degree of freedom at each site, denoted by Pauli operators λ_j^a ($a = 1$, 2, 3) and spins $\vec{S}_j = \vec{\sigma}_j/2$, respectively. The $\alpha_{ij} = x$, y, z label anisotropic xx, yy, and zz orbital interactions on
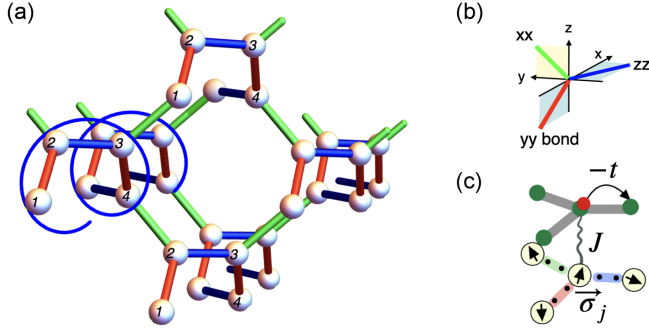
177601-1

FIG. 1.   (a) Hyperoctagonal lattice: a four-atom coil (1,2,3,4) on a bcc lattice gives rise to alternating square and octagonal spirals [13]. (b) Anisotropic coupling of orbital degrees of freedom according to the plane in which the bond lies. (c) Kondo coupling between the Yao-Lee spin liquid and the electron sea.

bonds that lie in the yz, zx, and xy planes, respectively. Finally, J is an antiferromagnetic Kondo coupling between the electrons and local moments.

Fermionization.—The Yao-Lee model [19] belongs to a family of Z_2 Kitaev spin liquids which can be solved exactly using fermionization. Following [19] we represent spin and orbital operators as products of Majorana fermions $\vec{\sigma}_j = -i\vec{\chi}_j \times \vec{\chi}_j$ and $\vec{\lambda}_j = -i\vec{b}_j \times \vec{b}_j$, where we use the normalization $\{\chi_i^a, \chi_j^b\} = \delta_{ij}\delta^{ab}$. The spin-orbital operator is given by $\vec{\sigma}_j \lambda_j^\alpha = -2iD_j\vec{\chi}_j b_j^\alpha$ where the constants $D_j = 8i\chi_j^1\chi_j^2\chi_j^3 b_j^1 b_j^2 b_j^3 = \pm 1$ commute with H. In the gauge $D_j = 1$,

$$H_{YL} = K\sum_{\langle i,j \rangle} u_{ij}(i\vec{\chi}_i \cdot \vec{\chi}_j), \qquad (2)$$

where $u_{ij} = -2ib_i^{\alpha_{ij}} b_j^{\alpha_{ij}} = \pm 1$ is a Z_2 gauge field that commutes with the Hamiltonian.

The 3DYL model describes free fermions moving in a static Z_2 gauge field. This model shares many of the properties of a 2D Kitaev spin liquid, most notably, the presence of gapped Z_2 flux excitations. On the hyperoctagonal lattice these are described by Wilson loops—products of the gauge fields $W = \prod u_{(i,j)} = \pm 1$ around closed tenfold and twelvefold loops [where (i, j) orders the sites i and j along xx, yy, and zz bonds so that the site furthest in the y, z, and x directions, respectively, is placed first [13]]. In the spin liquid ground state, all loops are trivial $W = 1$ [13]; flipping the sign of a Wilson loop creates a flux excitation (vison), with an energy determined as a fraction of K. Unlike 2D Kitaev spin liquids, the three-dimensional models undergo an Ising phase transition into a Higgs phase where gauge fluctuations (visons) are confined such that only short Wilson loops are allowed [23–25]. For the Kitaev model on the hyperoctagon lattice the transition at $T_{c1} \sim 0.012K$ [24,25] gives rise to a spin gap, but in the 3DYL, spin excitations occur without the

creation of visons, eliminating the spin gap; moreover, the three Majorana modes enhance T_{c1} by a factor of 3. Below T_{c1} the visons are confined and the Majorana fields describe coherent, fractionalized spin excitations.

In the ground state, choosing a gauge where $u_{(i,j)} = 1$ and taking into account that $\chi_{-\mathbf{k}} = \chi_{\mathbf{k}}^\dagger$,

$$H_{YL} = K\sum_{\mathbf{k}\in\square}\vec{\chi}_{\mathbf{k}\alpha}^\dagger h(\mathbf{k})_{\alpha\beta}\vec{\chi}_{\mathbf{k}\beta}, \qquad (3)$$

where $\alpha, \beta \in [1, 4]$ are site indices, while

$$h(\mathbf{k}) = \begin{pmatrix} 0 & i & ie^{-i\mathbf{k}\cdot\mathbf{a}_2} & ie^{-i\mathbf{k}\cdot\mathbf{a}_1} \\ -i & 0 & -i & ie^{-i\mathbf{k}\cdot\mathbf{a}_3} \\ -ie^{i\mathbf{k}\cdot\mathbf{a}_2} & i & 0 & -i \\ -ie^{i\mathbf{k}\cdot\mathbf{a}_1} & -ie^{i\mathbf{k}\cdot\mathbf{a}_3} & i & 0 \end{pmatrix}, \qquad (4)$$

and $\mathbf{a}_1 = (1,0,0)$; $\mathbf{a}_2 = \frac{1}{2}(1,1,-1)$; $\mathbf{a}_3 = \frac{1}{2}(1,1,1)$, are the primitive bcc lattice vectors. Since $\chi_{-\mathbf{k}} = \chi_{\mathbf{k}}^\dagger$, the momentum sum is restricted to half the Brillouin zone, corresponding to a cube ($\square$) of side length 2π centered at the P point at (π, π, π). The spectrum $E_{\mathbf{k}} \equiv K\epsilon(\mathbf{k})$, determined by $\det[\epsilon\underline{1} - h(\mathbf{k})] = 0$, or

$$\epsilon^4 - 6\epsilon^2 - 8\epsilon(s_x s_y s_z) + [9 - 4(s_x^2 + s_y^2 + s_z^2)] = 0 \quad (5)$$

[where $s_l \equiv \sin(k_l/2)$, $l = x, y, z$], contains a *single* Fermi surface centered at P [14] (Fig. 2).

Since the electrons and Majoranas move on the same lattice, at half filling their Fermi surfaces are perfectly nested and can be brought into coincidence by applying a gauge transformation to the electrons,

$$(c_1, c_2, c_3, c_4)_{\vec{R}} \to e^{-i\mathbf{Q}\cdot\mathbf{R}}(c_1, ic_2, c_3, -ic_4)_{\vec{R}}, \qquad (6)$$

where $\mathbf{Q} = (\pi, \pi, \pi)$ and $\mathbf{R} = n_1\mathbf{a}_1 + n_2\mathbf{a}_2 + n_3\mathbf{a}_3$ locates the unit cell. In this gauge,

$$H_c = \sum_{\mathbf{k}\in BZ} c_{\mathbf{k}\sigma\alpha}^\dagger [-t\,h(\mathbf{k}) - \mu\underline{1}]_{\alpha\beta} c_{\mathbf{k}\sigma\beta}, \qquad (7)$$
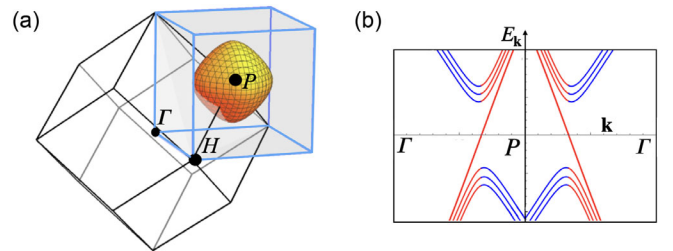

FIG. 2.   (a) Majorana Brillouin zone of the hyperoctagonal lattice showing coincident conduction and Majorana Fermi surfaces around the P point. (b) Hybridization of Majorana modes (in blue) with conduction band (in red) leaves one conduction Majorana band decoupled, forming a neutral Majorana surface.

and H_{YL}(3) have the same form. At low temperatures, where flux excitations can be ignored, we can rewrite the Kondo interaction in terms of the spin $\vec{S}_j = -(i/2)\vec{\chi}_j \times \vec{\chi}_j$ and decouple it using a Hubbard-Stratonovich transformation, in terms of a charge e spinor, $V_j = (V_{j\uparrow}, V_{j\downarrow})^T$,

$$H_K = \sum_j [(c_j^\dagger \vec{\sigma} V_j) \cdot \vec{\chi}_j + \text{H.c.}] + 2\frac{V_j^\dagger V_j}{J}. \quad (8)$$

The terms multiplying $\vec{\chi}_j$ must themselves be Majorana fermions, enabling us to rewrite H_K in the compact form

$$H_K = \sum_j \left[-iV_j(\vec{c}_j \cdot \vec{\chi}_j) + \frac{V_j^2}{J} \right], \quad (9)$$

where we have cast $V_j = (V_j/\sqrt{2})(z_{\uparrow j}, z_{\downarrow j})^T$ in terms of a normalized spinor and a real amplitude $V_j/\sqrt{2}$ to delineate the magnitude of the spinor V_j, (denoted by a roman V), and we have divided the electrons into four Majorana components, $(c_j^0, \vec{c}_j)$,

$$\begin{pmatrix} c_{j\uparrow} \\ c_{j\downarrow} \end{pmatrix} = \frac{1}{\sqrt{2}}(c_j^0 + i\vec{c}_j \cdot \vec{\sigma}) \begin{pmatrix} z_{j\uparrow} \\ z_{j\downarrow} \end{pmatrix}. \quad (10)$$

Thus in a coherent Kondo lattice, the vector components of the electron hybridize with the spinons and the scalar component decouples.

Now the field V_j is a fluctuating field inside the path integral, but the nesting between the Majorana and conduction Fermi surfaces ensures that it has a susceptibility to condense that is logarithmically divergent in temperature or chemical potential. Like a Cooper or Peierls instability, this ensures that instability into a condensed phase must occur for arbitrarily weak Kondo coupling.

We shall now focus on the stable, uniform condensate $V_j = V$. At half-filling ($\mu = 0$), vector and scalar electron components decouple, so that

$$H = \sum_{\mathbf{k} \in \square} \left[-tc_{\mathbf{k}}^{0\dagger} h_{\mathbf{k}} c_{\mathbf{k}}^0 + \vec{\psi}_{\mathbf{k}}^\dagger \begin{pmatrix} -th_{\mathbf{k}} & -iV \\ iV & K\hat{h}_{\mathbf{k}} \end{pmatrix} \vec{\psi}_{\mathbf{k}} \right] + \frac{NV^2}{J}, \quad (11)$$

where $\vec{\psi}_{\mathbf{k}} = (\vec{c}_{\mathbf{k}}, \vec{\chi}_{\mathbf{k}})^T$, N is the number of sites, and we denote $h_{\mathbf{k}} \equiv h(\mathbf{k})$.

As we now demonstrate, at a temperature T_{c2}, below the Ising phase transition T_{c1}, the system undergoes a second phase transition where the spinor order parameter V condenses [Fig. 3(a)]. To demonstrate the instability, we note that since only states close to the Fermi surface contribute at small $J \ll K, t$, we can project the Hamiltonian onto the band with a Fermi surface, with a dispersion for the band and the spin liquid fermions being
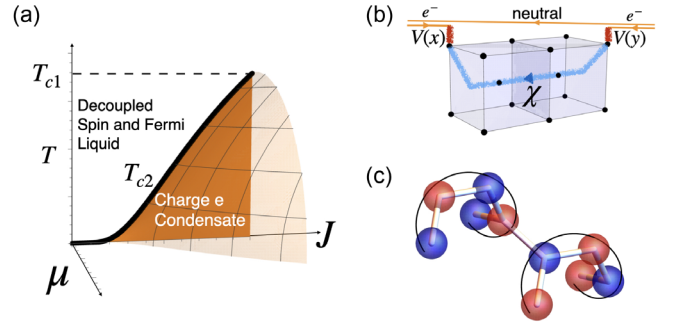

FIG. 3. (a) Schematic phase diagram for the 3D Coleman-Panigrahi-Tsvelik (*CPT*) model: below T_{c1}, the Majorana fermions in the spin liquid become phase coherent and at T_{c2} the charge e condensate develops. At particle-hole symmetry ($\mu = 0$) the charge e condensate forms for arbitrarily small Kondo coupling. (b) Development of coherent charge e condensate allows tunneling through the spin liquid over arbitrarily long distances, liberating a coherent, quasineutral conduction mode. (c) Staggered configuration of the gap function $\Delta(R) \sim e^{i2\mathbf{Q}\cdot\mathbf{R}}(\hat{\mathbf{d}}^1 + i\hat{\mathbf{d}}^2) \cdot \vec{\sigma}(1, -1, 1, -1)$ at the four sites of the unit cell, forming a pair-density wave.

equal to $-t\epsilon(k)$ and $K\epsilon(k)$, respectively. The calculations then can be done analytically; we will restrict ourselves to the simplest case $\mu = 0$. Summing the leading logarithmic ladder diagrams, we find the critical temperature T_{c2} is defined by the condition $J\chi_{1e}(T_{c2}) = 1$, where

$$\chi_{1e}(T) = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \frac{\tanh[\frac{\beta t\epsilon(\mathbf{k})}{2}] + \tanh[\frac{\beta K\epsilon(\mathbf{k})}{2}]}{2(K+t)\epsilon(\mathbf{k})}, \quad (12)$$

is the charge e pairing susceptibility [26]. To logarithmic accuracy, $\chi_{1e}(T) = [\rho/(1 + K/t)] \ln[W/T]$, where $\rho = 2\sqrt{3}/\pi^2 t$ is the conduction density of states, giving $T_{c2} = W \exp\{-[(1 + K/t)/\rho J]\}$, so a phase transition into the order fractionalized state will occur for arbitrarily small Kondo coupling. Of course, deviations from particle-hole symmetry at finite chemical potential destroy the nesting, so when $\mu \neq 0$, a transition takes place at finite $J > J_C$ from a fractionalized Fermi liquid (FL*) state [27] with a small Fermi surface, into the order-fractionalized state [Fig. 3(a)]. The important point, however, is that in the vicinity of particle-hole symmetry, the broken symmetry state is rigorously established.

Below T_{c2} the vector Majorana modes are gapped, leaving behind a single, coherent Majorana mode $c_{\mathbf{k}}^0$ of the conduction electrons. This feature is robust and is related to the mismatch between the quantum numbers of the itinerant Dirac fermions and Majorana triplet of the Yao-Lee spin liquid. The spectrum of the gapped fermions close to the Fermi surface is given by

$$E_\pm = (K - t)\epsilon(\mathbf{k})/2 \pm \sqrt{(K+t)^2\epsilon^2(\mathbf{k})/4 + |V|^2}. \quad (13)$$

To understand the nature of the fractionalized order, it is useful to consider the fractionalized order parameter $\hat{v}(\mathbf{x}_j) = -J(\vec{\sigma} \cdot \vec{\chi}_j)c_j$. This quantity carries a Z_2 charge, and by Elitzur's theorem, cannot develop long range order. On the other hand, we know that the χ_j field represents a physical degree of freedom at low temperatures, where Z_2 fluctuations have become massive. To reconcile this situation, we must consider the gauge invariant density matrix

$$\rho(x, y) = \langle \hat{v}(\mathbf{x})W(\mathbf{x}, \mathbf{y})\hat{v}^\dagger(\mathbf{y}) \rangle \xrightarrow{|\mathbf{x}-\mathbf{y}|\to\infty} V(\mathbf{x})V^\dagger(\mathbf{y}), \quad (14)$$

where $W(\mathbf{x}, \mathbf{y}) = \prod u_{(l+1,l)}$ is a Wilson line connecting the sites $\mathbf{x}$, $\mathbf{y}$. Once $T < T_{c1}$ the Wilson lines are not only constants of motion, but they are independent of the path between the two sites $\mathbf{x}$ and $\mathbf{y}$. We can calculate the gauge invariant quantity in the gauge where $u_{(i,j)} = 1$ so that $W = +1$, and in this way, we can be sure that the gauge invariant density matrix asymptotically factorizes into a product of well-defined spinor order [22]. In short, once $T < T_{c1}$ where the absence of visons guarantees that typical Wilson lines are equal to $+1$, this fractionalized long-range order is guaranteed to develop.

One of the physical manifestations of this fractionalized off-diagonal long-range order, is the development of long-range tunneling of the electrons through the spin liquid, which manifests through the development of odd-frequency triplet pairing and the emergence of a Majorana Fermi surface. The Z_2 gauge invariant self-energy for the electrons that describes the coherent tunneling through the spin liquid is given by [Fig. 3(b)]

$$\Sigma_{\alpha\beta}(x - x') = V^2[\sigma^a \mathcal{Z}(x)]_\alpha D(x - x')[\mathcal{Z}^\dagger(x')\sigma^a]_\beta, \quad (15)$$

where $\sigma^a \mathcal{Z}(x) = \sigma^a[z(x), i\sigma_y z^*(x)]^T$ is the spinorial hybridization of the vector Majorana components of the electron with the spin liquid at $x \equiv (\mathbf{x}, \tau)$, written in a Balian-Werthamer notation, and $D(x - x')$ is the space-time propagator of the Majorana fermions. Now the Fermi surface in the spin liquid means that $D(x) \sim e^{ik_F[\hat{\mathbf{x}}]|x|}/|x|$ decays as a power law at large distances, (where $k_F[\hat{\mathbf{x}}]$ is the Fermi wave vector on the patch of Fermi surface normal to $\hat{\mathbf{x}}$). This implies that the electron self-energy factorizes into a product of spinors at widely separated points in space-time, a key feature of order fractionalization [Fig. 3(b)].

In the ground state where V is uniform we can Fourier transform the self-energy to obtain

$$\Sigma_{\alpha\beta}(\mathbf{k}, \omega) = (1 - \mathcal{Z}\mathcal{Z}^\dagger)_{\alpha\beta} V^2 D(\mathbf{k}, \omega), \quad (16)$$

where $\mathcal{Z}\mathcal{Z}^\dagger$ projects out the unhybridized scalar component of the conduction sea and $D(\mathbf{k}, \omega) = [\omega - Kh(\mathbf{k})]^{-1}$ is the spin-liquid propagator in momentum space. Without the projector, this scattering would describe the resonant scattering of electrons in a Kondo insulator, but the elimination of the scalar component means that a tunneling electron that reemerges into the spin liquid loses all knowledge of its original charge, allowing it to emerge as a hole, producing resonant Andreev scattering. On the Fermi surface, where $h(\mathbf{k})$ has vanishing eigenvalues $\Sigma(\mathbf{k}, \omega) \sim 1/\omega$ describes odd frequency pairing that is infinitely retarded in time. One of the clearest manifestations of long-range order in the charge e order, is that the projective nature of the scattering self-energy remains coherent in momentum space, allowing the residual scalar Majorana conduction electron c_0 to propagate coherently over arbitrarily long distances.

To examine this resonant pairing in more detail, it is useful to construct the composite order parameter, formed from bilinears of the z field,

$$\hat{\mathbf{d}}^1 + i\hat{\mathbf{d}}^2 = z^T(-i\sigma_2)\boldsymbol{\sigma}z, \qquad \hat{\mathbf{d}}^3 = z^\dagger\boldsymbol{\sigma}z. \quad (17)$$

The triad $(\hat{\mathbf{d}}^1, \hat{\mathbf{d}}^2, \hat{\mathbf{d}}^3)$ describes coexisting magnetic and superconducting order. We can then divide the self-energy into normal and pairing components

$$\Sigma = \Sigma_N + \Delta(\mathbf{k}, \omega)\tau_+ + \Delta^\dagger(\mathbf{k}, \omega)\tau_-, \quad (18)$$

where

$$\Sigma_N(\mathbf{k}, \omega) = \frac{1}{4}[3 - (\hat{\mathbf{d}}^3 \cdot \boldsymbol{\sigma})\tau_3]\Sigma_0(\mathbf{k}, \omega),$$

$$\Delta(\mathbf{k}, \omega) = -\frac{1}{4}[(\hat{\mathbf{d}}^1 + i\hat{\mathbf{d}}^2) \cdot \boldsymbol{\sigma}]\Sigma_0(\mathbf{k}, \omega). \quad (19)$$

Σ_N describes a kind of odd-frequency magnetism (with no on-site magnetic polarization). The second term $\Delta(\mathbf{k}, \omega)$ in (19) describes a triplet gap function, with a complex d vector $\hat{\mathbf{d}}^1 + i\hat{\mathbf{d}}^2$ which breaks time-reversal symmetry.

However, hidden from immediate view, is the fact that the Andreev scattering $\Delta(\mathbf{k}, \omega)$ actually describes a pair density wave. To see this, let us now transform our solution back to the original electron gauge. Reversing the transformation (6), we see that in the original gauge $(V_1, V_2, V_3, V_4)_{\mathbf{R}_j} = \exp[i\mathbf{Q} \cdot \mathbf{R}_j](1, -i, 1, i)V_0$. Now the hyperoctagon lattice can be viewed as made of four-atom coils marked by 1,2,3,4 on Fig. 1, arranged on bcc lattice. From (6) it follows that the $\hat{\mathbf{d}}_{1,2}$ alternate along the coil and between the center and corners of the bcc lattice. In other words, the magnetic vector $\hat{\mathbf{d}}^3$ is uniform, but the superconducting d vector $\hat{\mathbf{d}}_j^1 + i\hat{\mathbf{d}}_j^2$ is staggered between neighboring sites, forming a pair density wave (PDW).

The Goldstone modes and topology.—We briefly touch on the topic of collective excitations and topology. As pointed out above, the gauge invariant quantities must connect two V fields at different points by a string of gauge fields. However, below the melting temperature T_{c1} we can

safely forget about Z_2 gauge fields and work in a fixed gauge $u_{(i,j)} = 1$. In this situation, the spinor V acquires the status of a true order parameter. Symmetry dictates that well below the transition where the amplitude fluctuations are weak the Ginzburg-Landau free energy density f for the low energy sector is given by

$$f[x] = \frac{\rho}{2}|(-i\vec{\nabla} + e\vec{A})z_\sigma|^2 + \frac{\vec{B}^2}{8\pi} - g\mu_B\vec{B} \cdot (z^\dagger\vec{\sigma}z), \quad (20)$$

where $\vec{B} = \nabla \times \vec{A}$ and as before, $V_{j\sigma} = (V/\sqrt{2})z_\sigma$. We have also included a Zeeman coupling.

Since the spinor z is defined by three Euler angles, transforming under the double group SU(2), small fluctuations of the order parameter consist of three Goldstone modes, one of them being Higgsed if the condensate is charged. The free energy (20) has been discussed in various contexts, in particular in connection with multiband superconductivity when the superconducting pairing takes place on Fermi surface consisting of multiple sheets [28,29]. The case of zero electric charge $e = 0$ emerges in connection with frustrated magnetism [30,31]. It is distinct from the conventional superconductivity due to the different topology of the order parameter manifold; here it is of the S^3 sphere. Since $\pi_1[\text{SU}(2)] = 0$, it forms a fragile superconductor with a Meissner effect, but *zero* critical current. The existence of the integer-valued topological invariant in three dimensions $\pi_3[\text{SU}(2)] = \mathbb{Z}$

$$Q = \frac{i}{24\pi^2} \int d^3x \epsilon_{\mu\nu\lambda}\text{Tr}(U^+\partial_\mu UU^+\partial_\nu UU^+\partial_\lambda U), \quad (21)$$

where

$$U = \begin{pmatrix} z_\uparrow & -z_\downarrow^* \\ z_\downarrow & z_\uparrow^* \end{pmatrix} \quad (22)$$

is an SU(2) matrix, suggests a possibility of nontrivial topological defects of the kind found in the Skyrme model of nuclear matter (for a review, see, for example, [32]). The latter model, however, contains a term with four derivatives whose presence is required to prevent the skyrmions from collapse. It has been argued by [28] and later by [29,30] that such terms are generated once one takes into account the fluctuating magnetic field. These authors suggested that in this case, the Ginzburg Landau free energy (20) admits additional knotted solitons, or Hopfion topological configurations [28–30]. However, numerical calculations performed in [33] indicate that the Hopfions are unstable leaving their existence an open question.

Conclusions.—We have presented a model of a three dimensional Kondo lattice which exhibits a remarkable range of properties associated with strong correlations. Some of these properties, such as pair density wave formation, have been observed experimentally, others, like odd-frequency pairing and the formation of a neutral Fermi surface, have been a matter of ongoing debate. The success of our approach is based on the fact that we are able to treat the strong correlations exactly in the asymptotic region of weak Kondo coupling.

There are three key aspects to our work. First, the condensate represents a new class of superconductivity, with a fractionalized order parameter that transforms under a double group. This feature leads to a number of robust consequences. In particular, the group topology determines the fragility of the superconducting order: although it displays diamagnetism (Meissner effect), the critical current is zero. Since the first homotopy $\pi_1[\text{SU}(2)] = 0$, there are no vortices, but a nontrivial third homotopy $\pi_3[\text{SU}(2)] = \mathbb{Z}$ suggests a possibility of such topological excitations as Hopfions or hedgehogs. The gauge invariant order parameter also breaks time reversal symmetry in a fashion that is protected by Kramers theorem and independent of crystal lattice, thus the transition into this state will not split under strain. Second, the proposed superconducting order forms a PDW that coexists with a novel form of magnetic order. This property is related to the fact that the nested Fermi surfaces of the conduction electrons and the Majoranas are centered at different points in the Brillouin zone, so that when an electron enters the spin liquid, it needs to borrow momentum from the condensate. Third, the low temperature excitations are described by a quasineutral Fermi surface whose existence is guaranteed by the mismatch between the quantum numbers of electrons and the Majorana spin excitations, and the long-range coherence of the charge e condensate. This guarantees that even in the situation of perfect nesting (half filled conduction band) there is a residual Majorana Fermi surface.

We end by noting that one of the key features of the current model is the stabilization of an underlying Z_2 spin liquid inside a Kondo lattice, by orbital degrees of freedom which decouple by forming a kind of valence bond solid. This is a situation that conceivably could occur in quantum materials, such as the topological Kondo insulator SmB_6, which under field, exhibits quantum oscillations reminiscent of a bulk Fermi surface [34,35]. This material is thought to involve a quartet spin state interacting with a conduction sea [36,37]. It is interesting to speculate that these orbital degrees of freedom may, under some circumstances, freeze into a valence bond-solid, stabilizing an underlying Majorana spin liquid within the Kondo insulator.

[1] Stefan Wirth and Frank Steglich, Exploring heavy fermions from macroscopic to microscopic length scales, Nat. Rev. Mater. **1**, 16051 (2016).

[2] Aline Ramires and Jose L. Lado, Emulating Heavy Fermions in Twisted Trilayer Graphene, Phys. Rev. Lett. **127**, 026401 (2021).

[3] Ajesh Kumar, Nai Chao Hu, Allan H. MacDonald, and Andrew C. Potter, Gate-tunable heavy fermion quantum criticality in a moiré Kondo lattice, Phys. Rev. B **106**, L041116 (2022).

[4] Zhi-Da Song and B. Andrei Bernevig, MATBG as topological heavy fermion: I. Exact mapping and correlated insulators, arXiv:2111.05865.

[5] Masaki Oshikawa, Topological Approach to Luttinger's Theorem and the Fermi Surface of a Kondo Lattice, Phys. Rev. Lett. **84**, 3370 (2000).

[6] Tamaghna Hazra and Piers Coleman, Luttinger sum rules and spin fractionalization in the $SU(N)$ Kondo lattice, Phys. Rev. Res. **3**, 033284 (2021).

[7] N. Read and D. M. Newns, On the solution of the Coqblin-Schrieffer Hamiltonian by the large-N expansion technique, J. Phys. C **16**, 3273 (1983).

[8] Piers Coleman, $1/N$ expansion for the Kondo lattice, Phys. Rev. B **28**, 5255 (1983).

[9] Assa Auerbach and K. Levin, Kondo Bosons and the Kondo Lattice: Microscopic Basis for the Heavy Fermi Liquid, Phys. Rev. Lett. **57**, 877 (1986).

[10] A. J. Millis and P. A. Lee, Large-orbital-degeneracy expansion for the lattice Anderson model, Phys. Rev. B **35**, 3394 (1987).

[11] Piers Coleman, Constrained Quasiparticles and Conduction in Heavy-Fermion Systems, Phys. Rev. Lett. **59**, 1026 (1987).

[12] Q. Si, S. Rabello, K. Ingersent, and J. L. Smith, Locally critical quantum phase transitions in strongly correlated metals, Nature (London) **413**, 804 (2001).

[13] M. Hermanns and S. Trebst, Quantum spin liquid with a Majorana Fermi surface on the three-dimensional hyperoctagon lattice, Phys. Rev. B **89**, 235102 (2014).

[14] Maria Hermanns, Simon Trebst, and Achim Rosch, Spin-Peierls Instability of Three-Dimensional Spin Liquids with Majorana Fermi Surfaces, Phys. Rev. Lett. **115**, 177205 (2015).

[15] Urban F. P. Seifert, Tobias Meng, and Matthias Vojta, Fractionalized Fermi liquids and exotic superconductivity in the Kitaev-Kondo lattice, Phys. Rev. B **97**, 085118 (2018).

[16] Wonjune Choi, Philipp W. Klein, Achim Rosch, and Yong Baek Kim, Topological superconductivity in the Kondo-Kitaev model, Phys. Rev. B **98**, 155123 (2018).

[17] Vanuildo S. de Carvalho, Rafael M. P. Teixeira, Hermann Freire, and Eduardo Miranda, Odd-frequency pair density wave in the Kitaev-Kondo lattice model, Phys. Rev. B **103**, 174512 (2021).

[18] Carlene S. de Farias, Vanuildo S. de Carvalho, Eduardo Miranda, and Rodrigo G. Pereira, Quadrupolar spin liquid, octupolar Kondo coupling, and odd-frequency superconductivity in an exactly solvable model, Phys. Rev. B **102**, 075110 (2020).

[19] Hong Yao and Dung-Hai Lee, Fermionic Magnons, Non-Abelian Spinons, and the Spin Quantum Hall Effect from an Exactly Solvable Spin-1/2 Kitaev Model with SU(2) Symmetry, Phys. Rev. Lett. **107**, 087205 (2011).

[20] Alexei Kitaev, Anyons in an exactly solved model and beyond, Ann. Phys. (Amsterdam) **321**, 2 (2006).

[21] Yashar Komijani, Anna Toth, Premala Chandra, and Piers Coleman, Order fractionalization, arXiv:1811.11115.

[22] Alexei M. Tsvelik and Piers Coleman, Order fractionalization in a Kitaev-Kondo model, Phys. Rev. B **106**, 125144 (2022).

[23] Urban F. P. Seifert, Xiao-Yu Dong, Sreejith Chulliparambil, Matthias Vojta, Hong-Hao Tu, and Lukas Janssen, Fractionalized Fermionic Quantum Criticality in Spin-Orbital Mott Insulators, Phys. Rev. Lett. **125**, 257202 (2020).

[24] Petr A. Mishchenko, Yasuyuki Kato, and Yukitoshi Motome, Finite-temperature phase transition to a Kitaev spin liquid phase on a hyperoctagon lattice: A large-scale quantum Monte Carlo study, Phys. Rev. B **96**, 125124 (2017).

[25] Tim Eschmann, Petr A. Mishchenko, Kevin O'Brien, Troels A. Bojesen, Yasuyuki Kato, Maria Hermanns, Yukitoshi Motome, and Simon Trebst, Thermodynamic classification of three-dimensional Kitaev spin liquids, Phys. Rev. B **102**, 075125 (2020).

[26] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.129.177601 for a calculation of the charge $1e$ pair susceptibility and the density of states of the half-filled hyperoctagonal lattice.

[27] T. Senthil, Subir Sachdev, and Matthias Vojta, Fractionalized Fermi Liquids, Phys. Rev. Lett. **90**, 216403 (2003).

[28] Egor Babaev, Ludvig D. Faddeev, and Antti J. Niemi, Hidden symmetry and knot solitons in a charged two-condensate Bose system, Phys. Rev. B **65**, 100512(R) (2002).

[29] Egor Babaev, Non-Meissner electrodynamics and knotted solitons in two-component superconductors, Phys. Rev. B **79**, 104506 (2009).

[30] Carlos Naya, Daniel Schubring, Mikhail Shifman, and Zhentao Wang, Skyrmions and hopfions in three-dimensional frustrated magnets, Phys. Rev. B **106**, 094434 (2022).

[31] Cristian D. Batista, Mikhail Shifman, Zhentao Wang, and Shang-Shun Zhang, Principal Chiral Model in Correlated Electron Systems, Phys. Rev. Lett. **121**, 227201 (2018).

[32] G. Holzwarth and B. Schwesinger, Baryons in the Skyrme model, Rep. Prog. Phys. **49**, 825 (1986).

[33] Juha Jäykkä, Jarmo Hietarinta, and Petri Salo, Topologically nontrivial configurations associated with Hopf charges investigated in the two-component Ginzburg-Landau model, Phys. Rev. B **77**, 094509 (2008).

[34] B. S. Tan, Y.-T. Hsu, B. Zeng, M. Ciomaga Hatnean, N. Harrison, Z. Zhu, M. Hartstein, M. Kiourlappou, A. Srivastava, M. D. Johannes, T. P. Murphy, J.-H. Park, L. Balicas, G. G. Lonzarich, G. Balakrishnan, and S. E. Sebastian, Unconventional fermi surface in an insulating state, Science **349**, 287 (2015).

[35] Onur Erten, Po-Yao Chang, Piers Coleman, and Alexei M. Tsvelik, Skyrme Insulators: Insulators at the Brink of Superconductivity, Phys. Rev. Lett. **119**, 057603 (2017).

[36] Victor Alexandrov, Maxim Dzero, and Piers Coleman, Cubic Topological Kondo Insulators, Phys. Rev. Lett. **111**, 226403 (2013).

[37] Markus Legner, Andreas Rüegg, and Manfred Sigrist, Surface-State Spin Textures and Mirror Chern Numbers in Topological Kondo Insulators, Phys. Rev. Lett. **115**, 156405 (2015).