Vanishing and Nonvanishing Persistent Currents of Various Conserved Quantities

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For every conserved quantity written as a sum of local terms, there exists a corresponding current operator that satisfies the continuity equation. The expectation values of current operators at equilibrium define the persistent currents that characterize spontaneous flows in the system. In this Letter, we consider quantum many-body systems on a finite one-dimensional lattice and discuss the scaling of the persistent currents as a function of the system size. We show that, when the conserved quantities are given as the Noether charges associated with internal symmetries or the Hamiltonian itself, the corresponding persistent currents can be bounded by a correlation function of two operators at a distance proportional to the system size, implying that they decay at least algebraically as the system size increases. In contrast, the persistent currents of accidentally conserved quantities can be nonzero even in the thermodynamic limit and even in the presence of the time-reversal symmetry. We discuss "the current of energy current" in $S = 1/2$ XXZ spin chain as an example and obtain an analytic expression of the persistent current.

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Introduction.—The Noether theorem predicts the presence of a conserved quantity for every global continuous symmetry [1–3]. This fundamental theorem underlies the conservation of many important quantities such as the energy and the momentum in uniform stationary systems and the U(1) charges in many-body systems. There can also be other types of conserved quantities that commute with the Hamiltonian without apparent symmetry reasons. Such quantities are the key behind the integrability of exactly solvable models. They also affect the thermalization of the system [4–6].

For each conserved quantity, one can define a current operator that satisfies the continuity equation [Eq. (1) below]. We call the expectation value of current operators at equilibrium "persistent currents." Persistent currents can flow in systems that do not have any ends, such as the one-dimensional ring illustrated in Fig. 1(a). Based on a variational argument that uses the so-called "twist operator," Bloch showed that the persistent U(1) current vanishes in the limit of large system size in (quasi) one-dimensional systems [7–11]. Recently, Kapustin and Spodyneiko proved a corresponding statement for the persistent energy current via a new method focusing on the response toward deformations of the Hamiltonian [12].

Then natural questions arise: do the persistent currents of other conserved quantities vanish in the thermodynamic limit, just like the persistent current of the U(1) charge and the energy? If so, how do we prove it? What is their scaling as a function of the system size? In this Letter, we answer these questions one by one. To our surprise, we find that the persistent currents can, in general, be nonzero even in the thermodynamic limit and even in the presence of the time-reversal symmetry. Given this finding, we derive sufficient conditions for persistent currents to vanish in the thermodynamic limit. It is known that current operators have ambiguities in their definitions. To address these questions in a meaningful manner, we should first show that the persistent current is independent of such ambiguities.

Our analysis also provides an alternative proof of the absence of persistent energy current in the thermodynamic limit. Our argument is advantageous in two ways compared to Ref. [12]: (i) the finite-size scaling is accessible and (ii) the absence of a finite-temperature phase transition is not assumed. Moreover, our argument improves Bloch's original bound $O(L^{-1})$ for the persistent U(1) current to an exponential decay at a finite temperature when the system size is large enough [7–11].

Setting.—We consider a quantum many-body system defined on a one-dimensional lattice $\Lambda \equiv \{1, 2, \ldots, L\}$. The boundary condition is set to be periodic and $x + nL$ ($n \in \mathbb{N}$) is identified with $x \in \Lambda$. The distance between $x, y \in \Lambda$ is given by $d(x, y) \equiv \min_n |x - y + nL|$. For example, $x = L$ is right next to $x = 1$ because $d(L, 1) = 1$ [Fig. 1(a)]. The Hamiltonian $\hat{H} \equiv \sum_{x=1}^{L} \hat{h}_x$ is given as the

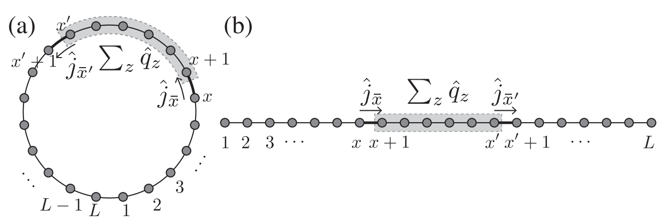

FIG. 1. Lattice Λ with (a) periodic boundary condition and (b) open boundary condition. The link between x and $x + 1$ is denoted by $\bar{x}$.

sum of local terms $\hat{h}_x$ supported around x. Ranges of $\hat{h}_x$'s are bounded by a constant $r_h \in \mathbb{N}$. The system is not necessarily translation invariant.

Suppose there exists a Hermitian operator $\hat{Q} \equiv \sum_{x=1}^{L} \hat{q}_x$ that commutes with $\hat{H}$. Ranges of $\hat{q}_x$'s are also bounded by a constant $r_q \in \mathbb{N}$. By definition, $[\hat{h}_x, \hat{q}_y] = 0$ if $d(x,y) > r$ with $r \equiv r_h + r_q$. The system size $L \in \mathbb{N}$ is assumed to be much bigger than r. When $\hat{q}_x$ generates a compact Lie group acting exclusively at x (i.e., $r_q = 0$), $\hat{q}_x$ is integer-valued in some unit, which we set 1 by proper normalization. In this case, the twist operator $\hat{U} \equiv e^{(2\pi i/L)\sum_{x=1}^{L} x\hat{q}_x}$ is well-defined and Bloch's variational argument is applicable [7–11]. We proceed without assuming such properties of $\hat{q}_x$.

Given $\hat{H}$ and $\hat{Q}$, we introduce the current operator associated with the link $\bar{x}$ between x and $x+1$ through the continuity equation [Fig. 1(a)]

$$i\left[\hat{H}, \sum_{z=x+1}^{x'} \hat{q}_z\right] = \hat{j}_{\bar{x}} - \hat{j}_{\bar{x}'} \quad (x' > x). \tag{1}$$

We assume that $\hat{j}_{\bar{x}}$ is localized around the link $\bar{x}$ with a finite support. When $d(x,x') \gg r$, the supports of $\hat{j}_{\bar{x}}$ and $\hat{j}_{\bar{x}'}$ do not overlap and $\hat{j}_{\bar{x}}$ can be uniquely singled out, for the given $\hat{h}_x$ and $\hat{q}_x$ (up to a temperature independent constant that we set 0). Note that the persistent current $\langle \hat{j}_{\bar{x}} \rangle$ is independent of the link $\bar{x}$. This is because, for any operator $\hat{o}$,

$$\langle [\hat{H}, \hat{o}] \rangle = \langle \hat{H}\hat{o} \rangle - \langle \hat{o}\hat{H} \rangle = 0, \tag{2}$$

when the expectation value is computed using the Gibbs state or the ground state (or any eigenstate) of $\hat{H}$. We obtain $\langle \hat{j}_{\bar{x}} \rangle = \langle \hat{j}_{\bar{x}'} \rangle$ by applying Eq. (2) to Eq. (1) [10].

The decomposition of $\hat{H}$ and $\hat{Q}$ into local terms $\hat{h}_x$ and $\hat{q}_x$ is not unique [13]. Let $\hat{H} = \sum_{x=1}^{L} \hat{h}'_x$ and $\hat{Q} = \sum_{x=1}^{L} \hat{q}'_x$ be an alternative decomposition, and let $\hat{j}'_{\bar{x}}$ be the current operator corresponding to this choice. Owing to the assumed locality, we can write $\sum_{z=x+1}^{x'} \hat{q}'_z = \sum_{z=x+1}^{x'} \hat{q}_z + \delta\hat{q}_{\bar{x}} - \delta\hat{q}_{\bar{x}'}$. Substituting this into Eq. (1), we find $\hat{j}'_{\bar{x}} - \hat{j}_{\bar{x}} = i[\hat{H}, \delta\hat{q}_{\bar{x}}]$. Again applying Eq. (2), we conclude that $\langle \hat{j}_{\bar{x}} \rangle$ is independent of the choice of local operators despite the fact that $\hat{j}_{\bar{x}}$ itself may be ambiguous. Similar conclusion can be found in Refs. [14,15], but our discussion is slightly more general in that we assumed only the locality of $\hat{q}_x$.

Tight-binding model.—Before further presenting abstract arguments, let us first discuss illustrative examples. We first consider a single-band tight-binding model with hopping parameters t_d:

$$\hat{H} = \sum_{x=1}^{L} \hat{h}_x, \qquad \hat{h}_x = \sum_{d=0}^{r_h} t_d \hat{c}^\dagger_{x+d}\hat{c}_x + \text{H.c.}, \tag{3}$$

where $\hat{c}_x$ is the annihilation operator of fermions at $x \in \Lambda$. Introducing the Fourier transformation $\hat{c}^\dagger_k \equiv L^{-1/2}\sum_{x=1}^{L} \hat{c}^\dagger_x e^{ikx}$ for $k = 2\pi j/L$, we obtain the diagonalized form $\hat{H} = \sum_k \varepsilon_k \hat{c}^\dagger_k \hat{c}_k$. The band dispersion $\varepsilon_k \equiv \sum_{d=0}^{r_h} t_d e^{-ikd} + \text{c.c.}$ defines the group velocity $v_k \equiv \partial_k \varepsilon_k = -i\sum_{d=0}^{r_h} d t_d e^{-ikd} + \text{c.c.}$ The ground state in a N fermion system is given by the Slater determinant of the N-lowest energy states. We fix the filling $\nu = N/L$ in the canonical ensemble, while N will be automatically chosen by fully occupying states with $\varepsilon_k < 0$ in the grand canonical ensemble (the chemical potential μ is included in ε_k via $t_0 = \mu/2$). For brevity, we assume that states with momentum k in the range $k_- \le k \le k_+$ are occupied and those outside are unoccupied in the ground state [Fig. 2(a)].

Let us consider a Hermitian operator of the form

$$\hat{Q} = \sum_{x=1}^{L} \hat{q}_x, \qquad \hat{q}_x = \sum_{d'=0}^{r_q} q_{d'} \hat{c}^\dagger_{x+d'}\hat{c}_x + \text{H.c.}, \tag{4}$$

which commutes with $\hat{H}$ as it is diagonal in the Fourier space: $\hat{Q} = \sum_k q_k \hat{c}^\dagger_k \hat{c}_k$ with $q_k \equiv \sum_{d'=0}^{r_q} q_{d'} e^{-ikd'} + \text{c.c.}$ For example, $q_k = 1$ for the U(1) charge $\hat{q}_x = \hat{c}^\dagger_x \hat{c}_x$ and $q_k = \varepsilon_k$ for the energy $\hat{q}_x = \hat{h}_x$. Using Eq. (1), we identify the current operator $\hat{j}_{\bar{x}} = \sum_{d=1}^{r_h}\sum_{d''=1}^{d} \hat{\sigma}^d_{x-d''+1}$, where

$$\hat{\sigma}^d_x \equiv \sum_{d'=0}^{r_q} i(t^*_d q^*_{d'} \hat{c}^\dagger_x \hat{c}_{x+d+d'} + t^*_d q_{d'} \hat{c}^\dagger_{x+d'} \hat{c}_{x+d}) + \text{H.c.} \tag{5}$$

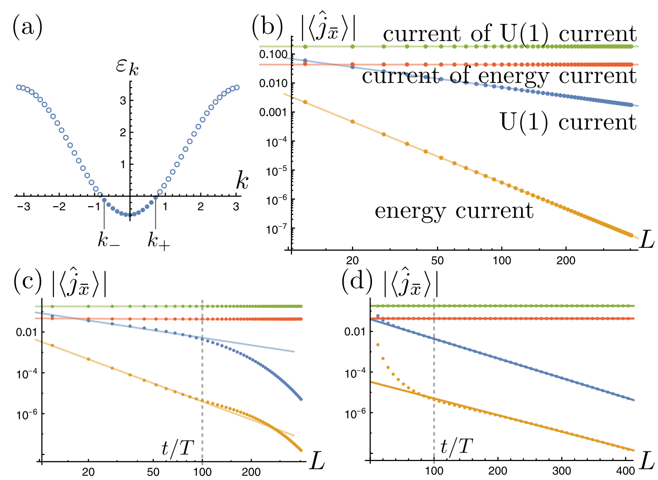

FIG. 2. Persistent currents in the tight-binding model with $t_1 = -te^{-i\theta/L}$ and $t_0 = \mu/2$. We set $t = 1$, $\mu = \sqrt{2}$ (corresponding to the quarter filling), $\theta = \pi/2$, and $L = 8n - 4$ ($n \in \mathbb{N}$). (a) The band dispersion for $L = 52$. (b) Log-log plot of persistent currents at $T = 0$. Lines are obtained by fitting. The slopes for the U(1) current and the energy current correspond to L^{-1} and L^{-3} decay. (c),(d) Log-log and linear-log plot of persistent currents at $T = t/100$. We use the same color labels as in (b).

See Sec. I of Supplemental Material (SM) [16] for the derivation. The averaged current operator $\hat{\bar{j}} \equiv L^{-1}\sum_{x=1}^{L}\hat{j}_{\bar{x}}$ takes an intuitive form $\hat{\bar{j}} = L^{-1}\sum_k q_k v_k \hat{c}_k^\dagger \hat{c}_k$ in the Fourier space, which is simply the charge q_k multiplied by the group velocity $v_k = \partial_k \varepsilon_k$. The persistent current $\langle \hat{j}_{\bar{x}}\rangle = L^{-1}\sum_{k=k_-}^{k_+} q_k v_k$ at $T = 0$ can be evaluated by the Euler-Maclaurin formula

$$\langle \hat{j}_{\bar{x}}\rangle = \int_{k_-}^{k_+} \frac{q_k v_k}{2\pi}dk + \frac{q_{k_+}v_{k_+} + q_{k_-}v_{k_-}}{2L} + O(L^{-2}). \qquad (6)$$

Now we show that $\langle \hat{j}_{\bar{x}}\rangle$ vanishes in the thermodynamic limit when q_k is a function of ε_k but not a function of v_k. This is the case when $\hat{Q}$ is the U(1) charge and the Hamiltonian itself. If we write $q_k = \partial_{\varepsilon_k} f(\varepsilon_k)$ [$f(\varepsilon_k) = \varepsilon_k$ for the U(1) charge and $f(\varepsilon_k) = \varepsilon_k^2/2$ for the energy], the first term in Eq. (6) can be written as $[f(\varepsilon_{k_+}) - f(\varepsilon_{k_-})]/(2\pi)$, which is small because $|\varepsilon_{k_+} - \varepsilon_{k_-}| = O(L^{-1})$ in the ground state. In the canonical ensemble, $\varepsilon_{k_\pm} = O(1)$ and $\langle \hat{j}_{\bar{x}}\rangle \propto L^{-1}$ in general. In the grand canonical ensemble at $T = 0$, in which all single particle levels with $\varepsilon_k < 0$ are occupied, the Fermi levels $\varepsilon_{k_\pm}$ themselves are $O(L^{-1})$ and the persistent energy current decays faster.

At $T > 0$, the persistent current for $q_k = \partial_{\varepsilon_k} f(\varepsilon_k)$ decays exponentially (Sec. III of SM). We show the numerical result for $T = t/100$ in Figs. 2(c) and 2(d), which demonstrates a crossover between the algebraic and the exponential decay around $L \sim t/T$.

On the other hand, in more general cases, q_k may not take the above form. For example, we can reuse the above current operator $\hat{j}_{\bar{x}}$ as an example of $\hat{q}_x$ in Eq. (4). Then the corresponding q_k is v_k for the U(1) current and $\varepsilon_k v_k$ for the energy current. In this case, the first term in Eq. (6) does not vanish in general as demonstrated in Fig. 2(b).

XXZ spin chain.—The above discussion heavily relies on the simplicity of the noninteracting model. However, the key conclusion remains valid even in the presence of interactions. As an example, let us consider $S = 1/2$ XXZ spin chain. The Hamiltonian is $\hat{H} = \sum_{x=1}^{L}\hat{h}_x$ with $\hat{h}_x = J(\hat{s}_{x+1}^x\hat{s}_x^x + \hat{s}_{x+1}^y\hat{s}_x^y + \Delta\hat{s}_{x+1}^z\hat{s}_x^z)$, where $\hat{s}_x^{x,y,z}$ is the spin-1/2 operator at $x \in \Lambda$. The energy current operator for $\hat{q}_x = \hat{h}_x$ is given by $\hat{j}_{\bar{x}}^E = i[\hat{h}_x, \hat{h}_{x+1}]$ [17]. The total energy current $\hat{Q}^{EC} = \sum_{x=1}^{L}\hat{j}_{\bar{x}}^E$ commutes with $\hat{H}$ [17], allowing us to discuss "the current of the energy current" $\hat{j}_{\bar{x}}^{EC}$ [18–20]. We append the concrete expressions of these operators in Sec. II of SM. Since the energy current $\hat{j}_{\bar{x}}^E$ are odd under the time-reversal symmetry, "the current of energy current" $\hat{j}_{\bar{x}}^{EC}$ is even. Thus, it can flow without breaking the time-reversal symmetry. Note that both $\hat{j}_{\bar{x}}^E$ and $\hat{j}_{\bar{x}}^{EC}$ are traceless, implying that their expectation values vanish in the infinite temperature limit. Unless $\Delta = 0$, neither the total U(1) current corresponding to $\hat{q}_x = \hat{s}_{\bar{x}}^z$ nor "the total current of energy current" $\sum_{x=1}^{L}\hat{j}_{\bar{x}}^{EC}$ commute with $\hat{H}$.

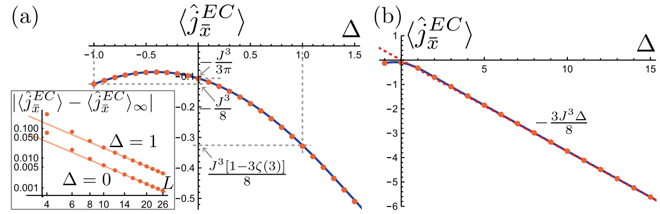

FIG. 3. The expectation value of "the current of the energy current" $\hat{j}_{\bar{x}}^{EC}$ in XXZ spin chain with $J = 1$. Panels (a) and (b) display different ranges of Δ. Orange points are obtained by exact diagonalization for $L = 26$ chain. Blue solid curves are the analytic expression in Eq. (7) in the thermodynamic limit. Red dashed line in (b) represents the asymptotic behavior $\Delta = -3J^3\Delta/8$. The inset in (a) checks the convergence as L increases for $\Delta = 0$ and 1. Fitting lines have the slope corresponding to the L^{-2} correction.

The $\Delta = 0$ point reduces the tight-binding model with $t_1 = -J/2$ and $k_\pm = \pm\pi/2$ and Eq. (6) gives $\langle \hat{j}_{\bar{x}}^{EC}\rangle = -J^3/(3\pi)$ in the thermodynamic limit. For $-1 < \Delta \leq 1$, we find the following analytic expressions using the results of Ref. [21] (Sec. II of SM):

$$\langle \hat{j}_{\bar{x}}^{EC}\rangle_\infty = J^3\frac{\pi\Delta - 2(1-\Delta^2)^{3/2}\zeta_\eta(3)}{8\pi}, \qquad (7)$$

$$\zeta_\eta(3) \equiv \int_{-\infty}^{+\infty}\frac{1}{\sinh(x - i\frac{\pi}{2})}\frac{\cosh[\eta(x - i\frac{\pi}{2})]}{\sinh^3[\eta(x - i\frac{\pi}{2})]}dx, \qquad (8)$$

where η $(0 \leq \eta < 1)$ parametrizes $\Delta = \cos(\pi\eta)$. For example, $\langle \hat{j}_{\bar{x}}^{EC}\rangle_\infty = J^3[1 - 3\zeta(3)]/8$ at $\Delta = 1$ [$\zeta(z)$ is the Riemann zeta function] and $\langle \hat{j}_{\bar{x}}^{EC}\rangle_\infty \to -J^3/8$ as $\Delta \to -1$. The expressions in Eqs. (7) and (8) can be extended to $\Delta > 1$ by setting $\eta = i\eta'$ with $\eta' > 0$. For example, $\langle \hat{j}_{\bar{x}}^{EC}\rangle_\infty \simeq -3J^3\Delta/8$ when $\Delta \gg 1$. Our numerical results up to 26 spins are presented in Fig. 3.

Vanishing persistent current under open boundary condition.—We have seen through examples that not all persistent currents vanish in the thermodynamic limit. To have a better understanding, here we temporarily consider the open boundary condition (OBC) and prove that the persistent current vanishes for any conserved quantity under OBC. This consideration serves as the reference point when evaluating the persistent current under periodic boundary condition (PBC) later.

The crucial difference between PBC and OBC lies in the definition of the distance. Under OBC, the distance between $x, y \in \Lambda$ is simply given by $\tilde{d}(x, y) \equiv |x - y|$. Thus, $x = L$ and $x = 1$ are at a long distance [Fig. 1(b)]. Let $\hat{\tilde{H}} \equiv \sum_{x=1}^{L}\hat{\tilde{h}}_x$ be the Hamiltonian under OBC, in which all interactions across the "seam" between $x = L$ and $x = 1$ are switched off. We demand that the local Hamiltonians remain unchanged in the bulk, i.e., $\hat{\hat{h}}_x = \hat{h}_x$ when $r_h < x < L - r_h + 1$. Near boundaries, $\hat{\hat{h}}_x$'s are arbitrary as long as $\hat{\hat{h}}_x$ is supported around x and its range is bounded by r_h.

Another important assumption on $\hat{\hat{H}}$ is that there exists a conserved charge $\hat{\hat{Q}} \equiv \sum_{x=1}^{L} \hat{\hat{q}}_x$ that may differ from $\hat{Q}$ only near the boundaries. For example, in the case of internal symmetries, one can set $\hat{\hat{Q}} = \hat{Q}$ by symmetrizing $\hat{\hat{H}}$ using $e^{i\theta\hat{Q}}$. In contrast, accidentally conserved quantities of $\hat{H}$ may not have a correspondence in $\hat{\hat{H}}$. For example, the total energy current (more generally, a conserved quantity under PBC in which the highest order term contains an odd number of spin operators) in the XXZ model is not conserved under OBC [22].

We assume that the current operator $\hat{\hat{j}}_{\bar{x}}$, satisfying the continuity equation

$$i\left[\hat{\hat{H}}, \sum_{z=x+1}^{x'} \hat{\hat{q}}_z\right] = \hat{\hat{j}}_{\bar{x}} - \hat{\hat{j}}_{\bar{x}'} \qquad (x' > x), \tag{9}$$

remains localized around the link $\bar{x}$ with a finite support. If we set $x = 0$ and $x' = L$ in Eq. (9), we find $\hat{\hat{j}}_{\bar{L}} - \hat{\hat{j}}_{\bar{0}} = -i[\hat{\hat{H}}, \hat{\hat{Q}}] = 0$. Because of the assumed locality of $\hat{\hat{j}}_{\bar{0}}$ and $\hat{\hat{j}}_{\bar{L}}$, this is equivalent with $\hat{\hat{j}}_{\bar{0}} = \hat{\hat{j}}_{\bar{L}} = 0$. This is reasonable since nothing can flow into or flow out of the system under OBC. Then, again from Eq. (9), we find a compact expression $\hat{\hat{j}}_{\bar{x}} = \sum_{z=x+1}^{L} i[\hat{\hat{H}}, \hat{\hat{q}}_z]$, which takes the form of $[\hat{\hat{H}}, \hat{o}]$. Thus, the persistent current, computed using the Gibbs state or the ground state of $\hat{\hat{H}}$, is precisely zero under OBC without the large L limit.

Interpolating Hamiltonian.—Now we return to PBC in which the distance is measured by $d(x, y)$. The Hamiltonian $\hat{\hat{H}}$ for OBC can be regarded as the Hamiltonian under PBC, since it satisfies the locality condition. In contrast, the Hamiltonian $\hat{H}$ for PBC cannot be used under OBC in general, since it may contain interactions between the two boundaries $x = L$ and 1 that are regarded as long-ranged with respect to $\tilde{d}(x, y)$ of OBC.

We introduce a one-parameter family of Hamiltonians $\hat{H}(s) = \sum_{x=1}^{L} \hat{h}_x(s) \equiv s\hat{H} + (1 - s)\hat{\hat{H}}$ for $s \in [0, 1]$, which linearly interpolates our original Hamiltonian $\hat{H}$ and the reference Hamiltonian $\hat{\hat{H}}$. By construction of $\hat{\hat{H}}$, the local Hamiltonians $\hat{h}_x(s)$ in the bulk region do not depend on s, i.e.,

$$\hat{h}_x(s) = \hat{h}_x, \qquad r_h < x < L - r_h + 1. \tag{10}$$

In the following, we denote by $\langle \hat{o} \rangle_s$ the expectation value with respect to the Gibbs state $\hat{\rho}(s) \equiv e^{-\hat{H}(s)/T}/Z(s)$ $[Z(s) \equiv \mathrm{tr} e^{-\hat{H}(s)/T}]$ at a finite T or the ground state of $\hat{H}(s)$ at $T = 0$.

We assume that, for any $s \in [0, 1]$, the system has a conserved charge $\hat{Q}(s) \equiv \sum_{x=1}^{L} \hat{q}_x(s)$ with $\hat{Q}(1) = \hat{Q}$ and $\hat{Q}(0) = \hat{\hat{Q}}$, in which $\hat{q}_x(s)$ is independent of s at least when x is away from 1 and L

$$\hat{q}_x(s) = \hat{q}_x \quad \text{if } 1 \ll x \ll L. \tag{11}$$

These assumptions are automatically fulfilled for the case when $\hat{Q}(s)$ is the Hamiltonian $\hat{H}(s)$ itself. Also, for internal symmetries, we can simply set $\hat{q}_x(s) = \hat{q}_x$ for any $x \in \Lambda$ and $s \in [0, 1]$. Substituting Eqs. (10) and (11) into the continuity equation

$$i\left[\hat{H}(s), \sum_{z=x+1}^{x'} \hat{q}_z(s)\right] + \hat{j}_{\bar{x}'}(s) - \hat{j}_{\bar{x}}(s) = 0 \ (x' > x), \tag{12}$$

we find that the current operator $\hat{j}_{\bar{x}}(s)$ is also independent of s when x is away from 1 and L.

Bound for persistent current.—With these preparations, let us evaluate $\langle \hat{j}_{\bar{x}} \rangle$ for the original Hamiltonian $\hat{H}$. Let $x_0 \in \Lambda$ be a site away from 1 and L. We have

$$\langle \hat{j}_{\bar{x}} \rangle = \langle \hat{j}_{\bar{x}_0} \rangle_{s=1} = \int_0^1 ds \partial_s \langle \hat{j}_{\bar{x}_0} \rangle_s, \tag{13}$$

for any $x \in \Lambda$, where we used the fact that $\langle \hat{j}_{\bar{x}} \rangle$ is independent of $\bar{x}$ and that $\hat{j}_{\bar{x}_0}(s) = \hat{j}_{\bar{x}_0}$ is independent of s as long as $1 \ll x_0 \ll L$. The last equality used $\langle \hat{\hat{j}}_{\bar{x}_0} \rangle_{s=0} = 0$ under OBC as discussed above.

According to the linear response theory, the derivative $\partial_s \langle \hat{j}_{\bar{x}_0} \rangle_s$ is given by a correlation function $\partial_s \langle \hat{j}_{\bar{x}_0} \rangle_s = -g_s(\hat{j}_{\bar{x}_0}, \hat{u}_{\bar{L}})$, where $\hat{u}_{\bar{L}} \equiv \partial_s \hat{H}(s) = \sum_{x=1}^{L} (\hat{h}_x - \hat{\hat{h}}_x)$ is localized around the link $\bar{L}$ between $x = L$ and $x = 1$. At a finite T, $g_s(\hat{o}, \hat{o}') = T^{-1} \langle\langle \hat{o}, \hat{o}' \rangle\rangle_s$ is given by is the canonical correlation

$$\langle\langle \hat{o}, \hat{o}' \rangle\rangle_s \equiv T \int_0^{T^{-1}} d\alpha \langle e^{\alpha\hat{H}(s)} \hat{o} e^{-\alpha\hat{H}(s)} \hat{o}' \rangle_s - \langle \hat{o} \rangle_s \langle \hat{o}' \rangle_s, \tag{14}$$

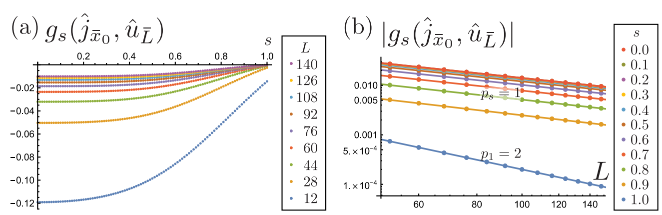

FIG. 4. $g_s(\hat{j}_{\bar{x}_0}, \hat{u}_{\bar{L}})$ for the U(1) current in the tight-binding model in Fig. 2. We set $t = 1$ and $\theta = \pi/2$. The lines in the panel (b) are obtained by fitting. The power p_s is 1 for $0 \leq s < 1$ except $p_1 = 2$ for $s = 1$.

and, at $T = 0$,

$$g_s(\hat{o}, \hat{o}') = \left\langle \hat{o} \frac{1 - \hat{P}(s)}{\hat{H}(s) - E_0(s)} \hat{o}' \right\rangle_s + \text{c.c.}, \quad (15)$$

where $\hat{P}(s)$ is the projection onto the ground state of $\hat{H}(s)$ and $E_0(s)$ is the ground state energy. Because of the property $g_s([\hat{H}(s), \hat{o}], \hat{o}') = \langle [\hat{o}', \hat{o}] \rangle_s$, $g_s(\hat{j}_{\bar{x}_0}, \hat{u}_{\bar{L}})$ is independent of x_0 as long as $1 \ll x_0 \ll L$. Up to this point, all expressions are exact.

Now, recall that $\hat{j}_{\bar{x}_0}$ and $\hat{u}_{\bar{L}}$ are respectively localized around the link $\bar{x}_0$ and $\bar{L}$. If x_0 is set to be $L/2$ when L is even and $(L+1)/2$ when L is odd, the distance between the supports of these two operators can be approximated by $L/2$. Hence, even in gapless systems, $g_s(\hat{j}_{\bar{x}_0}, \hat{u}_{\bar{L}})$ should decay as L increases. Let us postulate the power-law decay, i.e., $|g_s(\hat{j}_{\bar{x}_0}, \hat{u}_{\bar{L}})| < c_s L^{-p_s}$ first. For example, Fig. 4 illustrates the case for the above tight-biding model at $T = 0$. In this case, the persistent current can be bounded by Eq. (13) as

$$|\langle \hat{j}_{\bar{x}} \rangle| \leq \int_0^1 ds |g_s(\hat{j}_{\bar{x}_0}, \hat{u}_{\bar{L}})| \leq c L^{-p} \quad (16)$$

with $c \equiv \max_s c_s$ and $p \equiv \min_s p_s$. In contrast, when the correlation function decays exponentially, i.e., $|g_s(\hat{j}_{\bar{x}_0}, \hat{u}_{\bar{L}})| \leq c'_s e^{-L/\xi_s}$, we instead have

$$|\langle \hat{j}_{\bar{x}} \rangle| \leq \int_0^1 ds |g_s(\hat{j}_{\bar{x}_0}, \hat{u}_{\bar{L}})| \leq c' e^{-L/\xi} \quad (17)$$

with $c' \equiv \max_s c'_s$ and $\xi \equiv \max_s \xi_s$. In gapless systems at $T > 0$, if the gapless mode has the velocity v, a crossover from the algebraic decay ($L \lesssim v/T$) to the exponential decay ($L \gtrsim v/T$) is expected in general [23,24], as we have seen in Figs. 2(c) and 2(d).

Conclusions.—In this Letter, we considered a process in which all interactions across the seam between $x = L$ and $x = 1$ are gradually switched off. When the quantity $\hat{Q}$ remains conserved during this process, the persistent current can be bounded by Eqs. (16) and (17). This is the case for internal symmetries and the Hamiltonian itself. In contrast, when $\hat{Q}$ fails to be conserved during the process, this argument is not applicable and the persistent current can be nonzero even in the large L limit.

The fundamental difference between Noether charges and accidentally conserved quantities lies in their action on local operators. Let $\hat{o}_x$ be an operator supported only at $x \in \Lambda$. When $\hat{Q}$ is a Noether charge of an internal symmetry, the support of $\hat{o}_x(\theta) \equiv e^{i\theta\hat{Q}} \hat{o}_x e^{-i\theta\hat{Q}}$ remains localized to x. In contrast, when $\hat{Q}$ is an accidentally conserved quantity, the support of $\hat{o}_x(\theta)$ is spread over the entire space even if $\hat{Q}$ is given as a sum of local terms. (The speed of the spread obeys the Lieb-Robinson bound [25].) This is why $\hat{Q}$ cannot be interpreted as a generator of symmetries and generically cannot remain conserved in the deformation process from PBC to OBC.

The mechanism for nonvanishing persistent currents here is different from the one proposed recently [26,27]. Although our discussion was limited to one dimension, several implications on higher dimensions can be derived in the same way as in Ref. [10].

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