Eigenstate Thermalization Hypothesis and Free Probability

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(Received 11 May 2022; revised 8 July 2022; accepted 23 September 2022; published 18 October 2022)

Quantum thermalization is well understood via the eigenstate thermalization hypothesis (ETH). The general form of ETH, describing all the relevant correlations of matrix elements, may be derived on the basis of a "typicality" argument of invariance with respect to local rotations involving nearby energy levels. In this Letter, we uncover the close relation between this perspective on ETH and free probability theory, as applied to a thermal ensemble or an energy shell. This mathematical framework allows one to reduce in a straightforward way higher-order correlation functions to a decomposition given by minimal blocks, identified as free cumulants, for which we give an explicit formula. This perspective naturally incorporates the consistency property that local functions of ETH operators also satisfy ETH. The present results uncover a direct connection between the eigenstate thermalization hypothesis and the structure of free probability, widening considerably the latter's scope and highlighting its relevance to quantum thermalization.

DOI: 10.1103/PhysRevLett.129.170603

Introduction.—The current framework for understanding the emergence of thermal equilibrium in isolated quantum systems goes under the name of the eigenstate thermalization hypothesis (ETH). Early works of Berry [1], Deutsch [2], and Srednicki [3] recognized the importance of understanding the eigenstates of chaotic systems a pseudorandom vectors that encode microcanonical ensembles. Inspired by random matrix theory (RMT), ETH was then fully established by Srednicki in Ref. [4], incorporating some additional structure required to account for nontrivial temperature or time dependences. See Ref. [5] for a review. According to ETH, the matrix elements of local observables A in the energy eigenbasis $H|E_i\rangle = E_i|E_i\rangle$ are pseudorandom numbers, whose statistical properties are smooth thermodynamic functions. In the original formulation, the average and variance read

$$\overline{A_{ii}} = \mathcal{A}(E_i), \qquad \overline{A_{ij}A_{ji}} = F^{(2)}_{E^+_{ij}}(\omega_{ij})e^{-S(E^+_{ij})} \quad \text{for } i \neq j, \tag{1}$$

where $E^+_{ij} = (E_i + E_j)/2$, $\omega_{ij} = E_i - E_j$, and $S(E)$ is the thermodynamic entropy at energy E. While $\mathcal{A}$ represents the microcanonical expectation value of A, $F^{(2)}_E(\omega)$ depends implicitly on the observable A ($|f_A(E,\omega)|^2$ with the standard notations [5]) and it is associated to correlations on energy shell. In this Letter, we will refer to them as on-shell correlations. The ETH assumptions (1) allow one to fully describe the local relaxation of observables to thermal equilibrium as well as to characterize two-time dynamical correlation functions [5,6]. Since its formulation, ETH has motivated a considerable body of numerical [7–9] and analytical work [9–12], also in relation to quantum entanglement [13–15]. Despite this progress, the precise relation between ETH and RMT is currently the focus of a large debate.

With the recent explosion in activity imported from the string theory community that revolutionized the field of quantum chaos [16], the question of how ETH applies to multipoint correlations (as the out-of-time order correlators OTOC) came to the forefront. Higher order correlators are important to several areas of many-body physics: from quantum information scrambling (through OTOCs) [17], to dynamic-heterogeneity effects (through the fluctuation of correlations) [18] or in pump-probe experiments (through three point functions) [19]. It became clear that a more general version of ETH had to be introduced, encompassing correlations between matrix elements hitherto neglected in the standard framework. In order to compute such correlations of $q > 2$ times, Ref. [20] formulated an extension of Eq. (1) based on *typicality* arguments [21–25], as applied to small rotations of nearby energy levels. The existence of matrix elements correlations on top of Eq. (1), recently confirmed numerically [26,27], has motivated discussions on the finer structure of the ETH beyond Gaussian RMT [28–32].

This perspective offered some understanding of the finite contribution of different matrix elements functions from a diagrammatic approach, although it did not provide an efficient calculational tool for the various terms, leaving unknown the general structure of multitime correlation functions.

In this Letter, we characterize this structure by identifying its intimate relation between the general form of the ETH and free probability theory. The latter can be thought of as the generalization of classical probability to noncommutative random variables, where the concept of "freeness" extends the one of "independence." Introduced by Voiculescu in connection to the theory of operator algebras [33], free probability theory turned out to have important links with several branches of mathematics and physics [34–36], such as RMT [37] and combinatorics. We are here interested in the combinatorial aspects of free probability, based on free cumulants and noncrossing partitions [38].

Using these tools, we show that the higher-order correlation functions of generic physical systems are determined by basic quantities: the thermal free cumulants, thus providing a sort of generalized Wick theorem. Our methodology is to use the properties of the ETH matrix elements and their diagrammatic description to link them with the free probability mathematical structure. We will first recall the derivation of the general form of ETH, based on invariance with respect to local rotations of nearby energy levels. By discussing the ETH diagrams relevant to correlation functions, we will show that they are in one-to-one correspondence to noncrossing partitions. Our main result is an explicit expression for the thermal free cumulants in terms of sums of the matrix elements over nonrepeated indices: simple loops in the diagrammatic language discussed in [20]. The thermal free cumulants are hence linked to the Fourier transform of the ETH on-shell correlation of order q. As a byproduct, free probability allows us to deduce bounds on the behavior of on-shell correlations in the frequency domain.

General ETH.—The ETH in its enlarged formulation was discussed in Ref. [20] to compute correlation functions of order q depending on $\vec{t} = (t_1, t_2, \ldots, t_{q-1})$ times. The latter can be written in terms of the product of q matrix elements. The ETH amounts in the following ansatz: the average of products with distinct indices $\{i_i, \ldots, i_q\}$ reads

$$\overline{A_{i_1 i_2} A_{i_2 i_3} \ldots A_{i_q i_1}} = e^{-(q-1)S(E^+)} F_{E^+}^{(q)}(\omega_{i_1 i_2}, \ldots, \omega_{i_{q-1} i_q}) \tag{2}$$

and with repeated indices it factorizes in the large N limit

$$\begin{aligned} &\overline{A_{i_1 i_2} \ldots A_{i_{k-1} i_1} A_{i_1 i_{k+1}} \ldots A_{i_q i_1}} \\ &= \overline{A_{i_1 i_2} \ldots A_{i_{k-1} i_1}}\; \overline{A_{i_1 i_{k+1}} \ldots A_{i_q i_1}}. \end{aligned} \tag{3}$$

In Eq. (2), $E^+ = (E_{i_1} + \cdots + E_{i_q})/q$ is the average energy, $\vec{\omega} = (\omega_{i_1 i_2}, \ldots, \omega_{i_{q-1} i_q})$ with $\omega_{ij} = E_i - E_j$ are $q-1$ energy differences and $F_{E^+}^{(q)}(\vec{\omega})$ is a smooth function of the energy density E^+/N and $\vec{\omega}$. Thanks to the explicit entropic factor, $F_E^{(q)}(\vec{\omega})$ is of order one and thus contains Eq. (1) as a particular case for $q = 1, 2$. We will refer to $F_E^{(q)}(\vec{\omega})$ as the on-energy shell correlations of order q. This generalization of ETH, which is necessary if matrix elements are considered to be not independent, implies that correlation functions at order q contain new information that is not in principle encoded in lower moments.

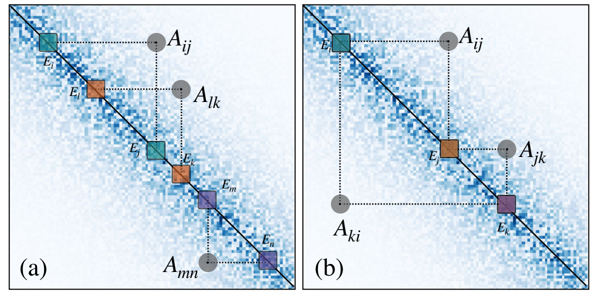

FIG. 1. Impact of the local rotational invariance of A_{ij} on the correlations between three matrix elements. The operator A in the energy eigenbasis is depicted as a random matrix with a band structure. To each matrix element A_{ij} is associated with a "small" U (box on the diagonal) which acts as a pseudorandom unitary matrix. Matrix elements with different indices (a) are characterized by different U and their average vanishes. When the indices are repeated on a loop (b) the U appear in pairs and yield a finite result.

The ETH ansatz in Eq. (2) can be derived using typicality arguments. The central idea is to use local invariance of the A_{ij}, stemming from small rotations that involve only nearby energy levels. The matrix elements are evaluated by substituting the operator with a "locally" rotated one (see Fig. 1) $A^u = UAU^\dagger$, i.e., $A_{ij} = \sum_{\tilde{i}\tilde{j}} U_{i\tilde{i}} A_{\tilde{i}\tilde{j}} U^*_{\tilde{j}j}$, with $U_{i\tilde{i}} = \langle E_i | E_{\tilde{i}} \rangle$ and $|E_{\tilde{i}}\rangle$ are the eigenstates of a slightly perturbed Hamiltonian [2]. By looking at a sufficiently small energy range around E_i, the $U_{i\tilde{i}}$ can be thought of as a pseudorandom unitary matrix. This is in analogy to Berry's conjecture, stating that the overlaps of chaotic eigenstates with a generic basis can be thought of as random Gaussian numbers. The size of this matrix has to be "small" to keep intact the energy band structure of A_{ij}, but it contains many level spacings. Hence the matrix elements are treated as belonging to an ensemble of local rotational invariances. By averaging over U, one can immediately deduce the finite contributions to any product of matrix elements. Averages are nonzero only if the matrices U appear at least twice. For example, for $\overline{A_{ij} A_{lk}}$ the only finite contribution comes from $\overline{A_{ij} A_{ji}}$ leading to Eq. (1). In the same way, finite products of q matrix elements necessarily have to be in a loop (see Fig. 1 for the pictorial example with $q = 3$). When the indices are different, this leads to Eq. (2). This scenario, complemented with some entropic arguments, results also in the factorization of Eq. (3), see Ref. [20].

As a consequence, we remark that this approach also accounts for the validity of the ETH ansatz between

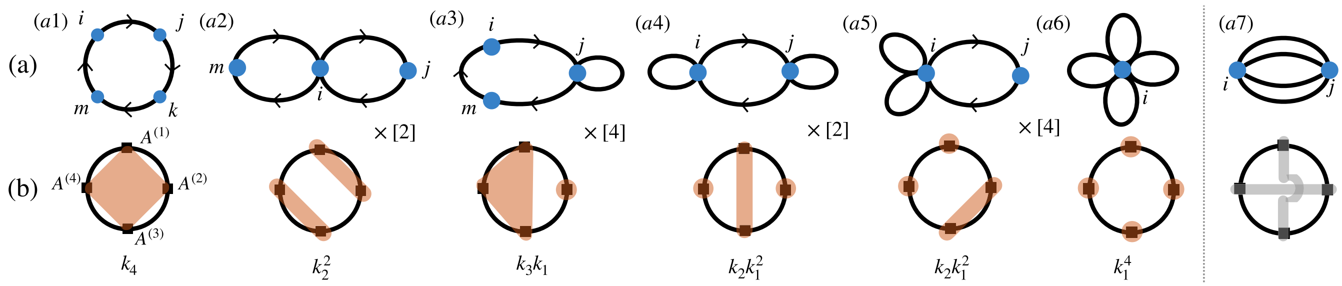

FIG. 2. ETH diagrams (a) and noncrossing partitions (b) for $q = 4$. (a1)–(a6) Loop and cactus diagrams that contribute to ETH correlators. The arrow indicates the presence of a time dependence. With $\times[n]$ we indicate that there are n cyclic permutations. (a7) Noncactus diagram. (b1)–(b6) Noncrossing partitions for $q = 4$. Each of the blocks contributes with a free cumulant k_n, where n is the number of points in the block. For completeness, we also represent the crossing partition after the dashed line.

different operators, e.g., $\overline{A_{ij}B_{ji}} = F^{(2)}_{E^+,AB}(\omega)e^{-S(E^+)}$, where we make explicit the dependence on the operators. Clearly, the ensemble defined by U is the same for A and B, since they come only from changes in H. Hence, the above argument can be applied to any set of local observables.

Computing expectations: diagrammatic expansion.—The ETH ansatz in Eq. (2) allows one to compute multitime thermal correlation functions of the form

$$S^{(q)}_\beta(\vec{t}) = \langle A(t_1)A(t_2)\ldots A(t_{q-1})A(0)\rangle_\beta, \tag{4}$$

where $\langle\bullet\rangle_\beta = \mathrm{Tr}(\rho\bullet)$ and $\rho = e^{-\beta H}/Z$ with $Z = \mathrm{Tr}(e^{-\beta H})$. Here, $A(t) = e^{iHt}Ae^{-iHt}$ is the observable at time t in the Heisenberg representation ($\hbar = 1$).

As introduced in Ref. [20], one can determine the contribution of $\overline{A_{i_1i_2}\ldots A_{i_qi_1}}$ to the thermal correlation $S^{(q)}_\beta(\vec{t})$ in a diagrammatic fashion, see Fig. 2(a) for $q = 4$. Let us briefly illustrate how to understand pictorially such ETH diagrams. Products as $A_{i_1i_2}\ldots A_{i_qi_1}$ are represented on a loop with q points. The matrix elements A_{ij} live on the oriented edge connecting two points i and j. The arrows keep track of nontrivial time dependencies. The different diagrams correspond to all the different ways one can contract the indices, i.e., identify two points. Such diagrams are classified as follows: (i) *Loops.*—all distinct vertices lie on a single closed circle [e.g., $A_{ij}A_{jk}A_{km}A_{mi}$ in Fig. 2(a1)]. Each loop with n vertices contributes with $\propto F^{(n)}e^{-(n-1)S}$; (ii) *Cactus diagrams.*—Trees of loops are joined to one another at single vertex [e.g., $A_{ij}A_{ji}A_{im}A_{mi}$ in Fig. 2(a2)]. Cactus with p leaves contribute with p products of the associated F. The example of a two-leaf cactus in Fig. 2(a2) contributes with $\propto (F^{(2)}e^{-S})^2$.

The *noncactus* diagrams [e.g., $A_{ij}A_{ji}A_{ij}A_{ji}$ in Fig. 2(a7)] have a further constraint to indices with respect to the other diagrams; their contribution is subleading for the correlation functions, as we now argue.

The thermal correlation $S^{(q)}_\beta(\vec{t})$ in Eq. (4) is given by the sum over all indices (and correspondingly all the diagrams) with the proper Boltzmann weight $e^{-\beta E_i}/Z$. The ETH ansatz (2) and (3) results in two main outcomes: (a) all summations of elements with repeated indices (cactus diagrams) factorize on results computed at the thermal energy $E_\beta = \langle H\rangle_\beta$. This means that leaves may be severed. (b) The contribution of noncactus diagrams is exponentially small with respect to the other terms. These properties follow from the smoothness of the ETH functions and the proper entropic counting. As an explicit example of (a), one can compute the diagram (a6) of Fig. 2, i.e.,

$$\frac{1}{Z}\sum_i e^{-\beta E_i}\overline{A_{ii}}^4 = \frac{1}{Z}\int dE e^{-\beta E + S(E)}\mathcal{A}^4(E) = (S^{(1)}_\beta)^4, \tag{5}$$

where we have substituted the ETH ansatz (1), summations with integrals $\sum_i = \int dE e^{S(E)}$ and performed the integral in E via saddle point technique, which fixes the energy by the thermodynamic condition $\beta = S'(E_\beta)$ and yields $\mathcal{A}^4(E_\beta) = (S^{(1)}_\beta)^4$. On the other hand, performing the same steps on the noncactus diagram Fig. 2(a7) and expanding the entropies [5] leads to

$$\frac{1}{Z}\sum_{i\neq j} e^{-\beta E_i}\overline{|A_{ij}|^2}\,\overline{|A_{ij}|^2} = \frac{1}{Z}\int dE^+ d\omega e^{-\beta E^+}e^{-\beta\omega/2}[F^{(2)}_{E^+}(\omega)]^2 \sim \mathcal{O}(e^{-N}). \tag{6}$$

In this Letter, we rationalize that cactus diagrams correspond in fact to the noncrossing partitions that play a role in free probability theory.

Hints of free probability theory.—We are interested in the combinatorial aspects of free probability, which are based on noncrossing partitions and free cumulants, as developed by Speicher [38]. A partition of a set $\{1,\ldots,q\}$ is a decomposition in blocks that do not overlap and whose union is the whole set. Partitions in which blocks do not "cross" are called noncrossing partitions. The set of all noncrossing partitions of $\{1,\ldots,q\}$ is denoted $NC(q)$. See the example in Fig. 2(b) for the partitions with $q = 4$, with $\times[n]$ we denote the n cycling permutations.

There are 14 noncrossing partitions of $q = 4$ elements, and only one is crossing, i.e., Fig. 2(b)(a7).

Noncrossing partitions appear in the definition of free cumulants. Consider some normalized ϕ [i.e., $\phi(1) = 1$], for example $\phi(\bullet) = \mathrm{Tr}(\bullet)/D$ for large $D \times D$ random matrices. The free cumulants k_q are defined implicitly from the moment-cumulant formula, stating that the moments of variables $A^{(i)}$ read

$$\phi(A^{(1)}, \ldots, A^{(q)}) = \sum_{\pi \in NC(q)} k_\pi(A^{(1)}, \ldots, A^{(q)}), \qquad (7)$$

where π is a noncrossing partition. Here, k_π is a product of cumulants, one term for each block of π. For instance, for $A^{(i)} = A \;\forall\; i$ and $k_n(A, \ldots, A) = k_n$, one has $\phi(A) = k_1$, $\phi(A^2) = k_2 + k_1^2$, $\phi(A^3) = k_3 + 3k_1k_2 + k_1^3$, and $\phi(A^4) = k_4 + 2k_2^2 + 4k_3k_1 + 6k_2k_1^2 + k_1^4$, i.e., Fig. 2(b). Classical cumulants are defined by a similar formula, where there is a sum over all the possible partitions and not only on the noncrossing ones. Notably, the relation between moments and cumulants differs in classical and free probability only for $n \geq 4$. Another interesting property is that the third and higher-order free cumulants of a Gaussian random matrix A vanish, i.e., $k_{q\geq 3}(A, \ldots, A) = 0$ (as for the classical cumulants of standard Gaussian random variables). At this level, Eq. (7) is a simple implicit definition of free cumulants in terms of moments, e.g., $k_1 = \phi(A)$, $k_2 = \phi(A^2) - \phi(A)^2$, etc.

ETH in these words.—The relation between ETH and free cumulants is thus clear: cactus diagrams correspond to noncrossing partitions, on the set identified by the matrices $A(t_i)$. Likewise, noncactus diagrams, not being associated with any noncrossing partition, do not count and they are related to crossing partitions [39]. As we argued, they do not contribute to the thermal moments. Analogously to Eq. (7), we introduce an implicit definition of the thermal free cumulants

$$S_\beta^{(q)}(\vec{t}) = \sum_{\pi \in NC(q)} k_\pi^\beta(A(t_1)A(t_2), \ldots, A(0)). \qquad (8)$$

Also here, k_π^β is a product of free cumulants, one for each block of the partition π, as k_n^β associated to n operators. Note that the cumulant $k_n^\beta(\vec{t})$ depends on the order in which we consider different operators, and we make this implicit in its time dependence. The ETH ansatz (2) is then the precise statement that the thermal free cumulants, may be substituted, for the purposes of computing time correlations, by sums as

$$k_q^\beta(\vec{t}) = k_q^{\mathrm{ETH}}(\vec{t}) = \frac{1}{Z} \sum_{i_1 \neq i_2 \neq i_q} e^{-\beta E_{i_1}} A(t_1)_{i_1 i_2} A(t_2)_{i_2 i_3} \ldots A(0)_{i_q i_1}, \qquad (9)$$

where all indices are different. In other words, free cumulants are simply given by the loop diagrams. This follows from two properties of the ETH ansatz discussed above: (a) that cactus diagrams factorize and (b) that only cactus diagrams (noncrossing partitions) matter. This first point is almost trivial for $q = 2$, where it is well known that via ETH one can compute

$$k_2^\beta(t) \equiv S_\beta^{(2)}(t) - [S_\beta^{(1)}]^2 = \langle A(t)A(0)\rangle_\beta - \langle A\rangle_\beta^2 \qquad (10)$$

$$= \frac{1}{Z} \sum_{i\neq j} e^{-\beta E_i} |A_{ij}|^2 e^{i(E_i - E_j)t} = k_2^{\mathrm{ETH}}(t), \qquad (11)$$

where one uses that the diagonal ETH matrix element is a smooth function of energy and therefore $\sum_i e^{-\beta E_i} A_{ii}^2 \simeq \langle A\rangle_\beta^2$ by saddle point integral, as in Eq. (5). One can show that this factorization holds at all orders. For instance, for $q = 4$ fixing $k_1(A) = \langle A\rangle_\beta = 0$, we obtain

$$\langle A(t_1)A(t_2)A(t_3)A(0)\rangle_\beta = k_4^\beta(t_1, t_2, t_3) + k_2^\beta(t_1 - t_2)k_2^\beta(t_3) + k_2^\beta(t_2 - t_3)k_2^\beta(t_1), \qquad (12)$$

where k_4^β is the term coming from the simple loop in Fig. 2(a1) and encodes all the correlations beyond Gaussian [26,31]. This expression now immediately follows from free probability expression [the diagrams (b1)–(b2) of Fig. 2], while it would require in principle a lengthy calculation [39].

Also in rotationally invariant random matrix ensembles, free cumulants are associated with diagrams with all distinct indices [42] and one can show that only the cactus diagrams matter.

Let us see how the structure of free probability leads to further results in the ETH context. First of all, it incorporates the consistency condition that products of operators obeying ETH shall obey ETH [4,40]. This can be checked directly from the free probability noncrossing partitions, see Ref. [39]. Second, we are led to ask questions such as the value of $k_q^{\mathrm{ETH}}(0) = (1/Z) \sum_{i_1 \neq i_2 \neq i_q} e^{-\beta E_{i_1}} A_{i_1 i_2} A_{i_2 i_3} \ldots A_{i_q i_1}$. free probability offers powerful computational tools to study such correlations via the generating functions [43]. Given the Stieltjes transform $G_\beta(z) = \mathrm{Tr}\{\rho[1/(z - A)]\}$, related to the generating function of thermal moments, one can study the so-called R transform

$$R_\beta(w) \equiv G_\beta^{-1}(w) - \frac{1}{w} = \sum_{q=1} k_q^\beta(0) w^{q-1}, \qquad (13)$$

that is always the generating function of equal-times free cumulants. In the case of ETH, $k_q^\beta(0) = k_q^{\mathrm{ETH}}(0)$ and it generalizes the result of fully rotational invariant random matrices [42]. Finally, free probability offers the tools—via the free cumulants—to pinpoint and characterize the non-Gaussian aspects of ETH [26–32].

Free cumulants on shell.—The thermal free cumulants defined in Eq. (9) admit an extremely appealing expression in terms of the ETH ansatz (2). By standard manipulations [39], one shows that

$$k_q^\beta(\vec{t}) = \int d\vec{\omega} F_{E_\beta}^{(q)}(\vec{\omega}) e^{i\vec{\omega}\cdot\vec{t} - \beta\vec{\omega}\cdot\vec{\ell}_q}, \tag{14}$$

where we defined the thermal shift $\vec{\ell}_q = \{[(q-1)/q], [(q-2)/q], \ldots, (1/q)\}$. This equation gives an important property: ETH qth on-shell correlations are related to the Fourier transform of the thermal free cumulants k_q^β

$$k_q^\beta(\vec{\omega}) = F_{E_\beta}^{(q)}(\vec{\omega}) e^{-\beta\vec{\omega}\cdot\vec{\ell}_q}. \tag{15}$$

This is familiar for $q = 2$, for which $k_2^\beta(\omega) = F_{E_\beta}^{(2)}(\omega) e^{-\beta\omega/2}$, which is the standard Kubo-Martin Schwinger (KMS) relation, leading to the fluctuation-dissipation theorem. The presence of this thermal shift—which only depends on temperature and on the correlation function order q—shall be interpreted as a generalized KMS condition, see Ref. [43]. Equation (15) naturally leads us to inspect the free cumulant expansion of the shifted correlator $\bar{S}_\beta^{(q)}(\vec{t}) \equiv S_\beta^{(q)}(\vec{t} - i\beta\vec{\ell}_q)$ given by

$$\bar{S}_\beta^{(q)}(\vec{t}) = \mathrm{Tr}[\rho^{1/q} A(t_1) \rho^{1/q} \ldots A(t_{q-1}) \rho^{1/q} A(0)], \tag{16}$$

which corresponds to a regularized version of S_β. One can look at the following connected correlation part of $\bar{S}_\beta^{(q)}$, i.e.,

$$\bar{k}_q(\vec{t}) = \frac{1}{Z} \sum_{i_1 \neq \ldots \neq i_q} e^{-\frac{\beta}{q}(E_{i_1} + \ldots E_{i_q})} A(t_1)_{i_1 i_2} \times A(t_2)_{i_2 i_3} \ldots A(t_q)_{i_q i_1}. \tag{17}$$

Diagrammatically, it is associated with the loop with q operators where the thermal weight $\rho^{1/q}$ is equally split. Nicely, its Fourier transform coincides with on shell correlations at the energy E_β, i.e., [20] $\bar{k}_q^\beta(\vec{\omega}) = F_{E_\beta}^{(q)}(\vec{\omega})$. This allows accessing such correlations directly from the time-dependent correlation functions in time and by taking their Fourier transform.

We now recall that the correlation between matrix elements with large energy differences should be small. This means that ETH correlation functions are usually expected to decay fast at large frequencies $\omega \gg 1$ as

$$F_E^{(q)}(\omega) \sim e^{-|\omega|/\omega_{\max}^{(q)}}. \tag{18}$$

The relations between free cumulants and $F_E^{(q)}(\vec{\omega})$ (15) allow one to infer relevant properties of the latter. Using Eq. (14) and the fact that free cumulants at equal times shall be well defined, in [39] we prove that on-shell correlations must decay at large frequencies in all directions at least as

$$F_{E_\beta}^{(q)}(\vec{\omega}) \sim \exp\left(-\beta \frac{q-1}{q} |\omega_i|\right) \tag{19}$$

$\forall\ i = 1, \ldots, q-1$. This gives the bound $\omega_{\max}^{(q-1)} \leq [(q-1)/q\beta]$, which generalizes the result for $q = 2$, 4 of Ref. [31]. These kinds of constraints have been related to operator growth or to timescales of multitime correlation functions (such as out-of-time order correlators) [31,44–47], which have been proven to obey strict bounds [16,48,49].

Conclusions.—We have found that the ETH, when generalized to all multipoint correlations in the spirit as Berry, Deutsch, and Srednicki, leads us directly to place it in the realm of free probability. This is a branch of mathematics where many developments have been made, and for which one may now turn to look for connections and analogies.

There is, however, a fundamental new element: the ensembles of matrices are not homogeneously full, but rather have a band structure, and a large, slowly varying diagonal. This structure exists on a specific basis, the one where the Hamiltonian is diagonal and its eigenvalues are ordered. The results are, likewise, always related to a specific energy shell, and not the matrix as a whole. This is a distinguishing feature of using free probability within ETH to respect to standard RMT results. The moments that define equilibrium correlation functions are then more complicated objects than those of a standard rotationally-invariant matrix model. Nonetheless, many results from these appear to generalize to the ETH setting and call for a rigorous understanding.

The ETH is at its most interesting when it fails, and integrability or many-body localization phenomena emerge. A more global understanding of ETH may then lead to a finer understanding of these effects.

We thank A. Polkovnikov for useful suggestions. S. P. thanks C. Malvenuto for discussions on noncrossing partitions. S. P. and J. K. are supported by the Simons Foundation Grant No. 454943. This work is supported by “Investissements d’Avenir” LabEx PALM (ANR-10-LABX-0039-PALM) (EquiDystant project, L. Foini). S. P. has received funding from the European Union’s Horizon Europe program under the Marie Sklodowska Curie Action VERMOUTH (Grant No. 101059865).

Note added.—This Letter has been submitted simultaneously with Ref. [50], which discusses the appearance of free cumulants in a stochastic transport model. The occurrence of free probability in both problems has a similar origin: the coarse-graining at microscopic either spatial or energy scales, and the unitary invariance at these microscopic scales. Thus the use of free probability tools promises to be ubiquitous in chaotic or noisy many-body quantum systems.

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[1] M. V. Berry, J. Phys. A **10**, 2083 (1977).
[2] J. M. Deutsch, Phys. Rev. A **43**, 2046 (1991).
[3] M. Srednicki, Phys. Rev. E **50**, 888 (1994).
[4] M. Srednicki, J. Phys. A **32**, 1163 (1999).
[5] L. D'Alessio, Y. Kafri, A. Polkovnikov, and M. Rigol, Adv. Phys. **65**, 239 (2016).
[6] E. Khatami, G. Pupillo, M. Srednicki, and M. Rigol, Phys. Rev. Lett. **111**, 050403 (2013).
[7] M. Rigol, V. Dunjko, and M. Olshanii, Nature (London) **452**, 854 (2008).
[8] M. Rigol and L. F. Santos, Phys. Rev. A **82**, 011604(R) (2010).
[9] A. Polkovnikov, K. Sengupta, A. Silva, and M. Vengalattore, Rev. Mod. Phys. **83**, 863 (2011).
[10] F. Anza, C. Gogolin, and M. Huber, Phys. Rev. Lett. **120**, 150603 (2018).
[11] T. Jin, SciPost Phys. **9**, 004 (2020).
[12] M. Bauer, D. Bernard, and T. Jin, Phys. Rev. E **101**, 012115 (2020).
[13] L. Vidmar and M. Rigol, Phys. Rev. Lett. **119**, 220603 (2017).
[14] C. Murthy and M. Srednicki, Phys. Rev. E **100**, 022131 (2019).
[15] M. Brenes, S. Pappalardi, J. Goold, and A. Silva, Phys. Rev. Lett. **124**, 040605 (2020).
[16] J. Maldacena, S. H. Shenker, and D. Stanford, J. High Energy Phys. 08 (2016) 106 .
[17] P. Hosur, X.-L. Qi, D. A. Roberts, and B. Yoshida, J. High Energy Phys. 02 (2016) 004.
[18] L. Berthier, G. Biroli, J.-P. Bouchaud, L. Cipelletti, and W. van Saarloos, *Dynamical Heterogeneities in Glasses, Colloids, and Granular Media* (OUP, Oxford, 2011), Vol. 150.
[19] J. Lloyd-Hughes, P. M. Oppeneer, T. P. Dos Santos, A. Schleife, S. Meng, M. A. Sentef, M. Ruggenthaler, A. Rubio, I. Radu, M. Murnane *et al.*, J. Phys. Condens. Matter **33**, 353001 (2021).
[20] L. Foini and J. Kurchan, Phys. Rev. E **99**, 042139 (2019).
[21] S. Goldstein, J. L. Lebowitz, R. Tumulka, and N. Zanghì, Eur. Phys. J. H **35**, 173 (2010).
[22] S. Goldstein, J. L. Lebowitz, C. Mastrodonato, R. Tumulka, and N. Zanghì, Proc. R. Soc. A **466**, 3203 (2010).
[23] S. Goldstein, J. L. Lebowitz, R. Tumulka, and N. Zanghì, Phys. Rev. Lett. **96**, 050403 (2006).
[24] P. Reimann, New J. Phys. **12**, 055027 (2010).
[25] P. Reimann, Nat. Commun. **7**, 10821 (2016).
[26] M. Brenes, S. Pappalardi, M. T. Mitchison, J. Goold, and A. Silva, Phys. Rev. E **104**, 034120 (2021).
[27] J. Wang, M. H. Lamann, J. Richter, R. Steinigeweg, A. Dymarsky, and J. Gemmer, Phys. Rev. Lett. **128**, 180601 (2022).
[28] T. Prosen, Ann. Phys. (N.Y.) **235**, 115 (1994).
[29] A. Dymarsky, arXiv:1804.08626.
[30] A. Chan, A. D. Luca, and J. T. Chalker, Phys. Rev. Lett. **122**, 220601 (2019).
[31] C. Murthy and M. Srednicki, Phys. Rev. Lett. **123**, 230606 (2019).
[32] J. Richter, A. Dymarsky, R. Steinigeweg, and J. Gemmer, Phys. Rev. E **102**, 042127 (2020).
[33] D. Voiculescu, in *Operator Algebras and Their Connections with Topology and Ergodic Theory* (Springer, New York, 1985), pp. 556–588.
[34] K. Ebrahimi-Fard and F. Patras, Front. Phys. **11**, 1 (2016).
[35] S. C. Morampudi and C. R. Laumann, Phys. Rev. B **100**, 245152 (2019).
[36] M. Bellitti, S. Morampudi, and C. R. Laumann, Phys. Rev. B **100**, 184201 (2019).
[37] J. A. Mingo and R. Speicher, *Free Probability and Random Matrices* (Springer, New York, 2017), Vol. 35.
[38] R. Speicher, Sémin. Lothar. Comb. **39**, B39c (1997), http://dml.mathdoc.fr/item/01127126/.
[39] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.129.170603 for additional details on the derivations, which includes Refs. [40,41].
[40] M. Srednicki, J. Phys. A **29**, L75 (1996).
[41] R. Simion, Discrete Math. **217**, 367 (2000).
[42] A. Maillard, L. Foini, A. L. Castellanos, F. Krzakala, M. Mézard, and L. Zdeborová, J. Stat. Mech. (2019) 113301.
[43] J. Bun, J.-P. Bouchaud, and M. Potters, Phys. Rep. **666**, 1 (2017).
[44] D. E. Parker, X. Cao, A. Avdoshkin, T. Scaffidi, and E. Altman, Phys. Rev. X **9**, 041017 (2019).
[45] Y. Gu, A. Kitaev, and P. Zhang, J. High Energy Phys. 03 (2022) 133.
[46] A. Avdoshkin and A. Dymarsky, Phys. Rev. Res. **2**, 043234 (2020).
[47] X. Cao, J. Phys. A **54**, 144001 (2021).
[48] N. Tsuji, T. Shitara, and M. Ueda, Phys. Rev. E **98**, 012216 (2018).
[49] S. Pappalardi, L. Foini, and J. Kurchan, SciPost Phys. **12** (2022).
[50] L. Hruza and D. Bernard, arXiv:2204.11680.