Dynamics in Systems with Modulated Symmetries

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We extend the notions of multipole and subsystem symmetries to more general *spatially modulated* symmetries. We uncover two instances with exponential and (quasi)periodic modulations and provide simple microscopic models in one, two, and three dimensions. Seeking to understand their effect on the long-time dynamics, we numerically study a stochastic cellular automaton evolution that obeys such symmetries. We prove that, in one dimension, the periodically modulated symmetries lead to a diffusive scaling of correlations modulated by a finite microscopic momentum. In higher dimensions, these symmetries take the form of lines and surfaces of conserved momenta. These give rise to exotic forms of subdiffusive behavior with a rich spatial structure influenced by lattice-scale features. Exponential modulation, on the other hand, can lead to correlations that are infinitely long-lived at the boundary while decaying exponentially in the bulk.

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Introduction.—Unconventional symmetries can give rise to interesting phenomena, such as new equilibrium phases of matter with novel low-energy features [1–8] and unusual nonequilibrium properties such as subdiffusive transport [9–15] and Hilbert space fragmentation [16–18]. Two cases that have been investigated extensively in recent years are multipole moment and subsystem symmetries. The latter often exhibit "UV-IR" mixing [7,19–21]: Their long-wavelength properties are sensitive to lattice-scale features, leading to distinct field-theoretic and hydrodynamic descriptions. The corresponding conserved quantities are *spatially modulated*, $\mathcal{Q}_{\{\alpha_{\mathbf{r}}\}} = \sum_{\mathbf{r}} \alpha_{\mathbf{r}} q_{\mathbf{r}}$, where $q_{\mathbf{r}}$ is some local charge at location $\mathbf{r}$. As such, they fail to commute with spatial translations. For multipole conservation $\alpha_{\mathbf{r}}$ is a polynomial of the coordinates $r_i = x,\ y,\ z$, while for subsystem symmetries $\alpha_{\mathbf{r}}$ takes nonzero values only on a spatial submanifold.

In the present Letter, we extend this notion to more general cases of $\alpha_{\mathbf{r}}$. We show that such symmetries appear in some simple, locally interacting systems and give various examples in one (1D), two (2D), and three (3D) dimensions. The symmetries we identify come in two flavors. One type is (exponentially) localized at the boundaries of the system, leading to infinitely long-lived boundary correlations, resembling the physics of *strong zero modes* [22]. The other type corresponds to the conservation of certain momentum components of a local observable. Interestingly, in 2D and 3D we find various models where long-lived modes exist along some closed hypersurfaces in momentum space, resembling systems with Bose surfaces [7,23–26] and the UV-IR-mixing phenomenon [19–21]. We will discuss the effect of these symmetries on the system's dynamics and show that they lead to unusual features in correlation functions, such as long-lived spatial oscillations on microscopic scales.

Modulated symmetries.—We first consider a family of 1D models to introduce the notion of modulated symmetries. We consider stochastic cellular automaton dynamics, which allow for large-scale numerical simulations, although all of our models can be mapped to quantum models that realize the same set of symmetries. We consider a chain of classical discrete spins that take values $s_x \in \{-S, \ldots, S\}$ on each site $x = 0, \ldots, L-1$. The dynamics is generated by local gates G_x, acting in a finite neighborhood of site x. The effect of a gate of range $\ell + 1$ is described by a set of integers n_i such that, when applying $G_x = \{n_i\}$, the spins are updated as $s_{x+i} \to s_{x+i} \pm n_i$ with $i \in \{0, \ldots, \ell - 1\}$. The updates are applied probabilistically among those for which $|s_{x+i} \pm n_i| \leq S$, with symmetric transition rates: At each application, either (i) $s_{x+i} \to s_{x+i} + n_i$ is applied or (ii) its inverse, $s_{x+i} \to s_{x+i} - n_i$, or (iii) no update is made. The full evolution is given by a random sequence of these gates [27].

We consider a family of models labeled by integers $q \geq 1,\ p \geq 0$, defined by the gates

$$G_x^{(p,q)} = \{n_0, n_1, n_2\} = \{q, -p, q\}, \tag{1}$$

acting on a three-site block centered around x. We notice that, for $2q \neq p$, these models do not conserve the total charge $\mathcal{Q} = \sum_j s_j$ or any of its higher moments. Nevertheless, there still exist some global conserved quantities, which we now construct.

Consider the general ansatz $\mathcal{Q}_{\{\alpha_j\}} \equiv \sum_j \alpha_j s_j$. Then $\mathcal{Q}_{\{\alpha_j\}}$ is a conserved quantity for the evolution generated by $G^{(q,p)}$ if and only if $\{\alpha_j\}$ fulfills the recurrence relation

$$\alpha_{j+2} = \frac{p}{q}\alpha_{j+1} - \alpha_j. \quad (2)$$

As a linear recurrence, Eq. (2) admits a general solution in terms of the roots r_1 and r_2 of the associated characteristic equation

$$r^2 - \frac{p}{q}r + 1 = 0. \quad (3)$$

The solutions can be parametrized by the initial conditions α_0 and α_1 [32], which implies that the model Eq. (1) has at most two linearly independent conserved quantities of this kind. Note that if q divides p, then $\mathcal{Q}_{\{\alpha_j\}}$ has an integer spectrum and, thus, generates a representation of U(1); otherwise, the symmetry is a unitary representation of the additive group $\mathbb{R}$ [33]. As the second-order polynomial in Eq. (3) is palindromic [34], its two roots r_1 and r_2 are inverses of each other: $r_2 = 1/r_1$. Thus, three different scenarios can occur, depending on the ratio p/q.

(i) Dipole conservation. If $2q = p$, then $r_2 = r_1 = 1$, which leads to a general solution of the form $\alpha_j = a_0 + a_1 j$; this reproduces the conservation of charge and dipole moment. Although conserving higher m moments of the charge requires longer-range gates [12], one would find that $r = 1$ is an $(m+1)$-fold degenerate root, so that α_j includes the contribution $\sum_{n=0}^{m} a_n j^n$. In fact, $r = 1$ will be the only root for the shortest-range gates conserving the first m moments of the charge.

(ii) (Quasi)periodic modulation. If $2q > p$, then Eq. (3) has two complex solutions $e^{\pm ik^*}$ with $k^* = \arccos(p/(2q))$ [35]. A general solution of Eq. (2) then takes the form $\alpha_j = ae^{ik^*j} + be^{-ik^*j} = A\cos(k^*j + \phi)$ with constants a and b (equivalently, A and ϕ) fixed by α_0 and α_1. Thus, although the total charge is not conserved, some finite momentum component of it is. However, while the recursion relation can always be solved in a system with open boundary conditions (OBCs), the corresponding momentum mode might not exist in a finite system with periodic boundaries (PBCs). Indeed, we could search directly for a conserved quantity of the form $\mathcal{s}_k \equiv \sum_j e^{ikj} s_j$, by plugging the ansatz $\alpha_j = e^{ikj}$ into Eq. (2), which then becomes $\chi(k) \equiv \cos(k) - (p/2q) = 0$. We can distinguish two possibilities, depending on whether the solution $k = k^*$ is a rational multiple of π or not. According to Niven's theorem [37], the former is the case if and only if $p/(2q) \in \{0, \pm\frac{1}{2}, \pm 1\}$; in this case, the modulation k^* is commensurate, having a finite periodicity on the lattice, and the symmetry is exact for some finite system sizes that are integer multiples of its period. In the more general case, however, k^* is incommensurate, the modulation is *quasiperiodic*, and the conserved quantity does not exist for any finite system with PBCs. Nevertheless, for sufficiently large systems, there will be momentum modes that are almost conserved, and the symmetry reemerges in the thermodynamic limit.

(iii) Exponentially localized. For $2q < p$, the solutions $r_{1,2}$ are real, positive, and nondegenerate. This implies $r_1 > 1$ and $r_2 = 1/r_1 < 1$, leading to two conserved quantities exponentially localized at the two boundaries of the system. Thus, it is more appropriate to instead label solutions by the two end points α_0 and α_{L-1} [27] rather than α_0 and α_1. Note that, in this case, it is not possible to satisfy the recursion relation with PBCs.

These models can be combined to construct longer-range gates with the same conserved quantities [27]. Moreover, allowing for $p < 0$ translates into r_1, $r_2 < 0$ for dipole moment and exponentially localized symmetries, leading to an additional staggering of the associated densities. See Ref. [38] for an extended discussion.

Hydrodynamic description.—Continuous symmetries provide long-lived modes that dominate the dynamics at long times; this idea is at the base of hydrodynamics [39–41]. Recent works investigated how hydrodynamics changes in the presence of multipole and subsystem symmetries [9–14], leading to subdiffusive transport. However, not all models we have introduced conserved the total charge, and, thus, the spin density is not the relevant long-wavelength degree of freedom for which a hydrodynamic theory should be written. Nevertheless, these modes can be probed via the physically relevant and accessible "infinite-temperature" spin-spin correlations $C(\mathbf{r}, t) \equiv \overline{s_\mathbf{r}(t)s_\mathbf{0}(0)}$, where $\overline{(\cdots)}$ denotes averaging over all randomly chosen initial spin configurations and circuit realizations. This approach captures the behavior of the three types of models we introduced and has the advantage of being easily generalized to higher dimensions.

Consider now periodically modulated symmetries corresponding to conserved momentum components of the total spin. These can be identified by the vanishing of some characteristic function, $\chi(\mathbf{k}) = 0$. To understand the dynamical consequences, we will assume a description in the spirit of linear hydrodynamics, which provides a closed linear equation of motion for the relevant slow degrees of freedom. In momentum space, this can be written as $\partial_t \mathcal{s}_\mathbf{k}(t) = -\omega(\mathbf{k})\mathcal{s}_\mathbf{k}(t)$. The key difference from more usual hydrodynamic descriptions is that we cannot simply expand the "imaginary frequency" $\omega(\mathbf{k})$ near $\mathbf{k} \approx \mathbf{0}$. Instead, we have to take into account the slow modes at finite momenta originating from the modulated symmetries.

To obtain $\omega(\mathbf{k})$, we require that (i) $\omega(\mathbf{k}) \geq 0$, leading to physically meaningful solutions, (ii) $\omega(\mathbf{k}) = 0 \Leftrightarrow \chi(\mathbf{k}) = 0$ to exactly capture the conserved momenta modes corresponding to the relevant slow degrees of freedom, and (iii) $\omega(\mathbf{k})$ is analytic around these points. This latter condition rules out $\omega(\mathbf{k})$ being an odd power of $|\chi(\mathbf{k})|$. A natural approximation that satisfies all these requirements and should correctly capture the leading-order behavior in the regimes where $\omega(\mathbf{k}) \approx 0$ is [42] $\omega(\mathbf{k}) \sim |\chi(\mathbf{k})|^2$. We emphasize that this relation holds only for momenta $\mathbf{k}$ close to the conserved modes at which $\chi(\mathbf{k}) = 0$. One can check that this approximation correctly captures the known behavior in a variety of models, including those with dipole conservation and subsystem symmetries [27].

Within linear response, the spin-spin correlator behaves as the Green's function of this equation of motion [39], $C(\mathbf{r}, t) = \int d^d k e^{i\mathbf{k}\cdot\mathbf{r} - \omega(\mathbf{k})t}$. We can rewrite this as

$$C(\mathbf{r} = \mathbf{0}, t) = \int_0^\infty d\omega \rho(\omega) e^{-\omega t}. \tag{4}$$

The long-time decay is, therefore, determined by the density of states (DOS) $\rho(\omega)$ near $\omega \approx 0$ [43].

Consider $G^{(p,q)}$ with $2q > p$. Near the conserved momentum k^*, we have $\omega(k^* + \delta k) \sim \delta k^2$. This gives rise to a DOS $\rho(\omega) \sim \omega^{-1/2}$; inserting this into Eq. (4) yields a diffusive scaling: $C(0, t) \sim t^{-1/2}$. This is consistent with our numerical results for $(p, q) = (3, 2)$, shown in Fig. 1(a), and for longer-range gates [27]. Nevertheless, as this is not an exact symmetry for PBCs, $C(0, t)$ is expected to decay exponentially at sufficiently late times [see data for $L = 30$ in Fig. 1(a)]. The situation changes for dipole conservation

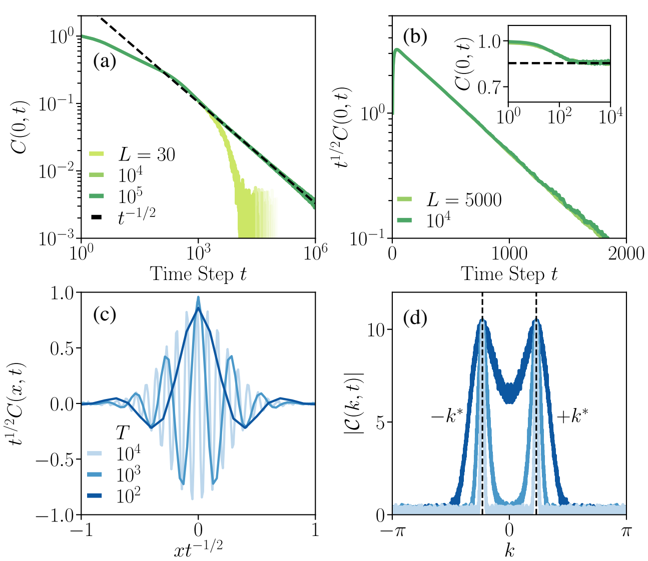

FIG. 1. Dynamics in 1D. Evolution of the spin-spin correlator $C(x, t)$ for the 1D models in Eq. (1). (a) $C(0, t)$ for quasiperiodic symmetries with $S = 5$ and $(p, q) = (3, 2)$. (b) Exponentially localized symmetries with $S = 10$ and $(p, q) = (3, 1)$. The inset shows the boundary correlation which is lower bounded by Mazur's bound (black dashed line). (c) and (d) Spatial correlations for the model in (a): "dressed" scaling collapse of $C(x, t)$ (c) and its spatial Fourier transform $\mathcal{C}(k, t)$, which becomes increasingly peaked at $k = \pm k^*$ (d).

($2q = p$). In this case, $k^* = 0$, so the leading contribution vanishes, and we instead have $\omega(k) \sim k^4$, recovering the known subdiffusive scaling $C(0, t) \sim t^{-1/4}$ [9–12,44].

The role of finite-momenta modes becomes much more apparent when we consider the spatial structure of the correlations. Taking into account the slow modes around $k \approx \pm k^*$, we obtain $C(x, t) \sim t^{-1/2} \mathcal{N}(x/\sqrt{t}) \cos(k^* x)$ with $\mathcal{N}$ denoting a Gaussian function, i.e., diffusive behavior modulated by a factor that oscillates at the microscopic scale $1/k^*$. This behavior is numerically verified in Figs. 1(c) and 1(d). Additional details on the numerical implementation can be found in Ref. [27]. The influence of finite (lattice-scale) momentum components in the Brillouin zone (BZ) on long-time and large-distance correlations can be seen as an infinite-temperature manifestation of UV-IR mixing in these models [19–21].

We can also apply our approximation to models with exponentially localized symmetries. In this case, $\omega(k) \sim |\chi(k)|^2$ is finite everywhere, which indicates an exponential decay of correlations. Nevertheless, there can be a correction coming from the large density of states near the minimum of $\omega(k)$. To see this, consider again the model Eq. (1), but this time with $2q < p$. The dispersion has a minimum at $k = 0$, and expanding around it we find $\omega(k) \approx (\frac{1}{2} k^2 - k_0^2)^2$, with $k_0^2 \equiv (2q - p)/(2q)$. Integrating over k, we find the long-time asymptotic form $C(0, t) \sim e^{-k_0^4 t}/\sqrt{t}$. The numerical results shown in Fig. 1(b) are consistent with this scaling for bulk correlations. However, the exponentially localized symmetries have a strong effect on the dynamics near the boundary, leading to infinitely long-lived correlations, as one can prove using Mazur's inequality [27,45,46] [dashed line in the inset in Fig. 1(b)].

Generalization to higher dimensions.—We now generalize our discussion to 2D systems. We begin by constructing a model which features the quasiperiodically modulated symmetries discussed above. However, in this case, the conserved momentum components will not only lie at isolated points in the BZ, but extend along continuous lines.

In our microscopic model, local gates $G_{x,y}$ act on a 4×4 block of a 2D square lattice in the vicinity of the site with coordinates x, y. The gate is again specified by a set of integers, such that $G_{x,y}: s_{x+i,y+j} \to s_{x+i,y+j} \pm n_{i,j}$, with

$$\begin{aligned} G &= \{n_{0,0}, n_{0,3}, n_{3,0}, n_{3,3}, n_{1,1}, n_{1,2}, n_{2,1}, n_{2,2}\} \\ &= \{1, 1, 1, 1, -1, -1, -1, -1\}, \end{aligned} \tag{5}$$

whose action is illustrated in Fig. 2(a).

This model has many U(1) symmetries: It conserves the total charge, its dipole moment, and the $\mathcal{Q}_{xy}^{(2)}$ and $\mathcal{Q}_{x^2-y^2}^{(2)}$ components of the quadratic moment (however, it does not conserve $\mathcal{Q}_{x^2+y^2}^{(2)}$). Moreover, it conserves the *staggered magnetization* along all rows and columns: $\mathcal{S}_{x_0} = \sum_y (-1)^y s_{x_0,y}$ and $\mathcal{S}_{y_0} = \sum_x (-1)^x s_{x,y_0}$. However, these

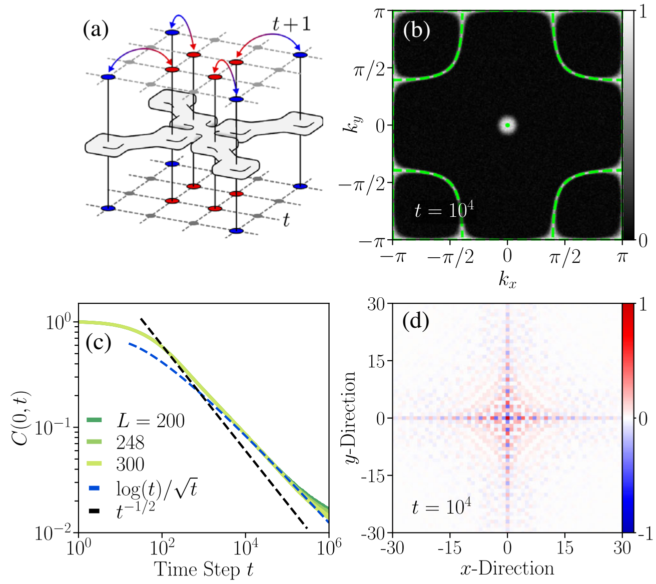

FIG. 2. Dynamics in 2D. (a) Schematics of a local gate corresponding to Eq. (5). (b) Correlation function $\mathcal{C}(\mathbf{k},t)$ within the Brillouin zone at $t = 10^4$. The solution of Eq. (7) is shown in a green (dashed) line. (c) The autocorrelation decays as $C(0,t) \sim \log(t)/\sqrt{t}$. (d) Correlations $C(\mathbf{x},t)$ are concentrated along the two axes and show oscillations on lattice scales that survive for long times.

do not exhaust the set of modulated symmetries of the model. To detect additional modulated conserved quantities $\mathcal{Q}_{\{\alpha_\mathbf{r}\}} = \sum_\mathbf{r} \alpha_\mathbf{r} s_\mathbf{r}$, we can look for nontrivial solutions of the associated two-dimensional recurrence relation

$$\sum_{i,j} n_{i,j}\alpha_{x+i,y+j} = 0. \qquad (6)$$

Although a complete analytical solution of two-dimensional recurrences might be feasible (e.g., via generating functions or rewriting it as a matrix equation), the procedure can be quite involved and not leading to a closed-form expression. Instead, we use the ansatz $\alpha_\mathbf{r} = e^{i\mathbf{k}\cdot\mathbf{r}}$ and focus on periodically modulated symmetries. Equation (6) then reduces to

$$\chi(\mathbf{k}) \equiv \sum_{a,b} n_{a,b} e^{i(ak_x+bk_y)} \propto \cos(k_x/2)\cos(k_y/2) \times [\cos(k_x) + \cos(k_y) - 2\cos(k_x)\cos(k_y)] = 0. \qquad (7)$$

The solutions of $\chi(\mathbf{k}) = 0$ are highlighted in green in Fig. 2(b). The conservation of total charge corresponds to the mode $\mathcal{s}_\mathbf{0}$ at $\mathbf{k} = (0,0)$, while the staggered subsystem symmetries $\mathcal{S}_x$ and $\mathcal{S}_y$ show up as the lines $k_y = \pi$ and $k_x = \pi$, respectively. Moreover, we find a set of contour lines (forming a closed loop in the Brillouin zone) along which the second line in Eq. (7) vanishes. As in 1D, each of these corresponds to an exact symmetry for OBCs, whose total number scales with the linear system size $\mathcal{O}(L)$ [27]. However, most points along these lines are not realized exactly in a finite system with PBCs but become exact symmetries only in the thermodynamic limit, leading to an (infinite-dimensional) emergent symmetry group. While straight lines in the BZ correspond, upon inverse Fourier transformation, to symmetry operators that act along columns or rows on the lattice, linear combinations of symmetries lying along these contours do not lead to subsystem symmetries and, instead, are delocalized on the whole system [27].

As we saw, the asymptotic decay of $C(\mathbf{0},t)$ is governed by the DOS near $\omega \approx 0$. For the 2D model in Eq. (5), $\rho(\omega)$ picks up contributions from various parts of the BZ [see Fig. 2(b)] [27]. We find that the leading contributions arise from the five points where multiple lines of conserved momenta meet. Around all of these points, $\rho(\omega) \sim \omega^{-1/2}\log(\omega)$ and, consequently, $C(\mathbf{0},t) \sim t^{-1/2}\log(t)$: subdiffusion with a logarithmic correction, similarly to the case of U(1) subsystem symmetries [13] [the remaining parts of the BZ provide a subleading $\rho(\omega) \sim \omega^{-1/2}$ contribution]. We confirm this numerically in Fig. 2(c). The finite momentum contributions also lead to spatial oscillations of $C(\mathbf{x},t)$ at short scales [see Fig. 2(d)], which can be clearly identified in its Fourier transform $\mathcal{C}(\mathbf{k},t)$ shown in Fig. 2(b), concentrated along the solutions of $\chi(\mathbf{k}) = 0$. Another consequence is that $C(\mathbf{r},t)$ does not have full rotational invariance but instead concentrates around the two coordinate axes.

Equation (6) has additional solutions of the form $\alpha_\mathbf{r} = e^{\kappa_x x}e^{ik_y y}, e^{ik_x x}e^{\kappa_y y}$ [47] for open boundary conditions. We expect them to have no effect on bulk correlations at infinite temperature analogous to the 1D models we studied above. Their effect on boundary dynamics will be explored in an upcoming work.

Many other models which exhibit conserved momenta along various shapes in the BZ exist, including ones that do not conserve the total charge $\mathcal{Q} = \sum_\mathbf{r} s_\mathbf{r}$. An example is shown in Fig. 3(a) [27]. The construction can also be easily extended to higher dimensions. For example, in 3D, one can find exact conserved quantities lying in intersecting 2D manifolds in momentum space [27] as we show in Fig. 3(b). These symmetries lead to different scalings for the correlations, depending on the details of $|\chi(\mathbf{k})|$. The decay is at least as slow as $C(\mathbf{0},t) \sim t^{-1/2}$, coming from the fact that expanding the dispersion along a codimension 1 hypersurface (a line in 2D or a surface in 3D) is formally similar to an expansion in one dimension. However, the actual behavior can be much slower than this. For example, in 2D, points where many conserved lines intersect [see Fig. 3(a)] or ones where lines touch (rather than cross) can lead to strongly subdiffusive dynamics [27].

A natural question is how to construct a complete hydrodynamic description, which could be used to predict the long-time behavior of the spin-spin correlations.

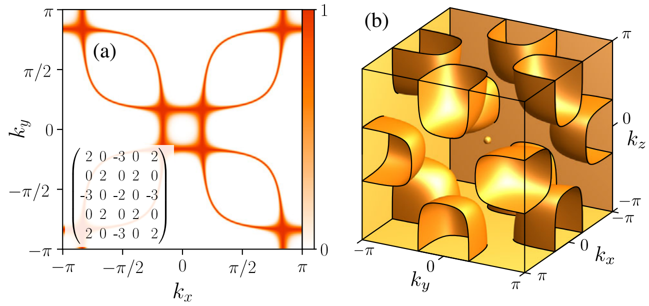

FIG. 3. Generalizations with (quasi)periodic symmetries. (a) Example of a 2D system which does not conserve total charge and for which $C(\mathbf{0},t)\sim \log(t)/\sqrt{t}$. The figure shows $e^{-|\chi(\mathbf{k})|^2}$. (b) 3D generalization of the 2D model in Eq. (5). The figure shows the solutions of $\chi(\mathbf{k})=0$.

One could decompose the spin density as $s(\mathbf{r})\sim \sum_{\{\mathbf{k}|\chi(\mathbf{k})=0\}} e^{i\mathbf{k}\cdot\mathbf{r}}\mathfrak{s}_{\mathbf{k}}(\mathbf{r})$ and construct a hydrodynamic theory for the relevant long-wavelength degrees of freedom $\mathfrak{s}_{\mathbf{k}}(\mathbf{r})$'s, which could be further constrained by the conservation of higher moments of the charge. Although addressing a different question, this approach is similar to the expansion of the UV boson (fermion) in terms of low-energy degrees of freedom lying close to the Bose (Fermi) surface (see, e.g., Ref. [26]). Moreover, the situation becomes rather complicated in two and higher spatial dimensions, where infinitely many momenta lie along different intricate shapes in the BZ. The derivation of an appropriate hydrodynamic description is an interesting challenge that we leave for future work.

Conclusions and outlook.—We introduced the notion of *spatially modulated symmetries* that generalize both multipole and subsystem symmetries. We provided two new classes of such symmetries: quasiperiodically modulated and exponentially localized ones. The latter are relevant for the dynamics near the boundary, playing a role similar to strong zero modes. The former lead to unusual behavior in bulk correlations: subdiffusive decay and long-lived short-wavelength oscillations, providing a hydrodynamic analog of the phenomenon of UV-IR mixing, making long-time dynamics sensitive to lattice-scale features. While here we discussed thermal correlations, these models also appear to host interesting examples of fragmentation which we plan to explore in a future publication.

Although we focused on classical cellular automata, each of our models can be easily related to a corresponding quantum Hamiltonian, by mapping a gate $G_{\mathbf{r}}$, characterized by integers $\{n_{\mathbf{a}}\}$, to a local Hamiltonian term $\bigotimes_{\vec{a}}(\hat{S}^{\mathrm{sgn}(n_{\vec{a}})}_{\vec{r}+\vec{a}})^{|n_{\vec{a}}|}+\mathrm{h.c.}$ These quantum Hamiltonians possess the same set of symmetries as their classical counterparts. These also include additional $\mathbb{Z}_2$ symmetries, which anticommute with the modulated ones, leading to strong zero modes [22] for systems with exponentially localized symmetries. Understanding their low-energy physics and the relationship to previous studies of UV-IR mixing is an exciting challenge. Moreover, 1D systems with such quasiperiodic symmetries might be realized as effective descriptions in the strong detuning limit of experimental realizations of the Aubry-André model [48,49]. Finally, generalizing our analysis of long-time dynamics to 3D is another interesting open question.

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Note added.—Recently, we were informed of an independent study of two-dimensional systems with dipole and quadrupole conservation [50].

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