Four-Loop Rapidity Anomalous Dimension and Event Shapes to Fourth Logarithmic Order

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We obtain the quark and gluon rapidity anomalous dimension to fourth order in QCD. We calculate the N^{3}LO rapidity anomalous dimensions to higher order in the dimensional regulator and make use of the soft and rapidity anomalous dimension correspondence in conjunction with the recent determination of the N^{4}LO threshold anomalous dimensions to achieve our result. We show that the results for the quark and gluon rapidity anomalous dimensions at four loops are related by generalized Casimir scaling. Using the N^{4}LO rapidity anomalous dimension, we perform the resummation of the energy-energy correlation in the back-to-back limit at N^{4}LL, achieving for the first time the resummation of an event shape at this logarithmic order. We present numerical results and observe a reduction of perturbative uncertainties on the resummed cross section to below 1%.

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Introduction.—Over the last decade, we have entered a new era in QCD phenomenology where we can perform high-precision computations for key observables, ranging from nonperturbative determinations of form factors from lattice QCD to high-order perturbative computations for high-energy collider processes. In many cases, perturbative computations develop large logarithms order by order in perturbation theory that may spoil the convergence of the perturbative series in the coupling constant. In such a scenario, these logarithms need to be resummed to all orders by a renormalization group equation (RGE) governed by certain anomalous dimensions. The most prominent examples of QCD anomalous dimensions are the QCD beta function and the cusp anomalous dimension, which control the running of the strong coupling and the structure of infrared singularities. The knowledge of QCD anomalous dimensions is therefore not only important to obtain precise phenomenological predictions, but the dimensions are also a unique window into the all-order perturbative structure of the strong interactions. In the remainder of this Letter we compute the four-loop corrections of such an anomalous dimension, and we apply them to resum for the first time large logarithmic corrections to an event shape and a transverse momentum dependent (TMD) observable to fourth logarithmic order.

Examples of the class of collider observables characterized by the presence of large rapidity logarithms are transverse momentum distributions at hadron colliders. Their all-order resummation is dictated by the so-called rapidity anomalous dimension [1], which is closely related to the Collins-Soper kernel [2–4]. Let us consider for concreteness the case of Drell-Yan-like processes. The leading power factorization theorem for transverse momentum distributions, which encodes the all-order behavior in the coupling in the $q_T \to 0$ limit, can be written as [1–18]

$$\frac{d\sigma}{dQ^2 dY d^2\vec{q}_T} = \sigma_0 \sum_{a,b} H_{ab}(Q^2,\mu) \int \frac{d^2\vec{b}_T}{(2\pi)^2} e^{i\vec{q}_T\cdot\vec{b}_T} \times \tilde{B}_a(x_1^B, b_T,\mu,\nu)\tilde{B}_b(x_2^B, b_T,\mu,\nu) S_q(b_T,\mu,\nu). \quad (1)$$

The logarithmic dependence on the transverse momentum can be resummed by deriving RGEs for the objects appearing in the factorization theorem. For example, the soft function in Eq. (1) obeys the following RGEs:

$$\begin{aligned} \mu\frac{d}{d\mu}\ln S_i(\vec{b}_T,\mu,\nu) &= 4\Gamma^i_{\mathrm{cusp}}[\alpha_s(\mu)]\ln\mu/\nu + \gamma^i_{\mathrm{th}}[\alpha_s] \\ \nu\frac{d}{d\nu}\ln S_i(\vec{b}_T,\mu,\nu) &= -4\int_{b_0/b_T}^{\mu}\frac{d\mu'}{\mu'}\Gamma^i_{\mathrm{cusp}}[\alpha_s(\mu')] + \gamma^i_r[\alpha_s], \end{aligned} \quad (2)$$

where $i \in \{q,g\}$ labels the parton species and Γ_{cusp} is the cusp anomalous dimension [19–22], γ^i_{th} is called the threshold anomalous dimension, and γ^i_r is the rapidity anomalous dimension [1]. The rapidity anomalous dimension respects itself an RGE,

$$\mu \frac{d}{d\mu} \gamma_r^i(b_T, \mu) = -4\Gamma_{\mathrm{cusp}}^i[\alpha_s(\mu)], \tag{3}$$

with the solution given by

$$\gamma_r^i(b_T, \mu) = -4 \int_{\mu_0}^{\mu} \frac{d\mu'}{\mu'} \Gamma_{\mathrm{cusp}}^i[\alpha_s(\mu')] + \gamma_r^i(\mu_0, b_T), \tag{4}$$

where μ_0 is an arbitrary scale that marks the starting point of the RGE in Eq. (3). Choosing $\mu_0 = b_0/b_T$ sets the logarithms in $\gamma_r^q(\mu_0, b_T)$ to zero and allows one to express the boundary of the RGE as

$$\gamma_r^i(\mu_0 = b_0/b_T, b_T) \equiv \gamma_r^i[\alpha_s(b_0/b_T)]. \tag{5}$$

The rapidity anomalous dimension boundary $\gamma_r^i[\alpha_s(b_0/b_T)]$ has been calculated to three loops in Refs. [15,23], and its calculation to four loops is one of the main results of this Letter.

Rapidity anomalous dimension at N^{4}LO.—In Refs. [24,25] an identity relating the threshold and rapidity anomalous dimensions using a conformal mapping of matrix elements of Wilson lines was proposed. In $d = 4 - 2\epsilon$ dimensions the QCD beta function reads

$$\beta[\alpha_s, \epsilon] = -2\alpha_s \left[\epsilon + \frac{\alpha_s}{4\pi}\beta_0 + \left(\frac{\alpha_s}{4\pi}\right)^2 \beta_1 + \ldots\right]. \tag{6}$$

Massless QCD is conformal up to the running of the strong coupling. Consequently, there exists a critical point ϵ^* such that $\beta[\alpha_s, \epsilon^*] = 0$ and QCD can be rendered conformal at this point. Following Refs. [24,25], at this critical point the sum of the rapidity and threshold anomalous dimension vanishes:

$$\gamma_r^i[\alpha_s, \epsilon^*] + \gamma_{\mathrm{th}}^i[\alpha_s, \epsilon^*] = 0. \tag{7}$$

Since $\epsilon^* \sim \mathcal{O}(\alpha_s)$, Eq. (7) allows one to obtain the standard rapidity anomalous dimension $\gamma_r^i[\alpha_s, 0]$ in $d = 4$ at $\mathcal{O}(\alpha_s^n)$ from the $d = 4$ threshold anomalous dimension at $\mathcal{O}(\alpha_s^n)$ and the d-dimensional rapidity anomalous dimension $\gamma_r^i[\alpha_s, \epsilon]$ at $\mathcal{O}(\alpha_s^{n-1})$. In this Letter we have calculated the d-dimensional rapidity anomalous dimension to three loops, extending the computation of the transverse momentum dependent soft function [1] of Refs. [17,23,26] to higher orders in the dimensional regulator. Together with the four-loop threshold anomalous dimension [27,28], we use it to extract the rapidity anomalous dimension for a parton species i in representation R to N^{4}LO. Identifying the coefficients of the perturbative expansion of the rapidity anomalous dimension as

$$\gamma_r^i[\alpha_s(\mu)] = \sum_n \left(\frac{\alpha_s(\mu)}{4\pi}\right)^n \gamma_{r,n}^i, \tag{8}$$

the four-loop coefficient reads

$$\begin{aligned}
\gamma_{r,4}^i = {} & C_A^3 C_R \Bigg(-\frac{21164}{9}\zeta_3^2 - \frac{26104}{9}\zeta_2\zeta_3 + \frac{4228}{3}\zeta_4\zeta_3 + \frac{2752}{3}\zeta_2\zeta_5 + \frac{1201744\zeta_3}{81} + \frac{778166\zeta_2}{243} + \frac{8288\zeta_4}{9} - \frac{181924\zeta_5}{27} \\
& - \frac{63580\zeta_6}{27} + \frac{11071\zeta_7}{3} - \frac{28290079}{2187} - \frac{b^4_{q,C^4_{AF}}}{6} \Bigg) \\
& + C_A C_R n_f^2 \left(\frac{224}{9}\zeta_3\zeta_2 + \frac{6752\zeta_2}{243} - \frac{22256\zeta_3}{81} + \frac{160\zeta_4}{9} + \frac{1472\zeta_5}{9} - \frac{898033}{2916} \right) + C_R n_f^3 \left(\frac{160\zeta_3}{9} - \frac{16\zeta_4}{9} + \frac{10432}{2187} \right) \\
& + C_R C_A^2 n_f \Bigg(-\frac{8584}{9}\zeta_3^2 + \frac{2080}{3}\zeta_2\zeta_3 - \frac{247652\zeta_3}{81} - \frac{182134\zeta_2}{243} + \frac{43624\zeta_4}{27} - \frac{17936\zeta_5}{27} + \frac{1582\zeta_6}{27} + \frac{10761379}{2916} \\
& - \frac{b^4_{q,C^4_{FF}}}{12} - 2b^4_{q,n_f C_F^2 C_A} - b^4_{q,n_f C_F^3} \Bigg) + C_R C_F n_f^2 \left(\frac{6928\zeta_3}{27} + \frac{160\zeta_4}{3} + 32\zeta_5 - \frac{110059}{243} \right) + \frac{C^4_{AR}}{d_R} \Bigg(\frac{6688\zeta_3^2}{3} + 3584\zeta_2\zeta_3 + 736\zeta_4\zeta_3 \\
& + \frac{15616\zeta_3}{9} - \frac{224\zeta_4}{3} + \frac{4352\zeta_2}{3} - 2048\zeta_2\zeta_5 + \frac{3680\zeta_5}{9} - \frac{6952\zeta_6}{9} - 6968\zeta_7 - 384 + 4b_{4,d4AF} \Bigg) \\
& + \frac{C^4_{FR}}{d_R} n_f \left(-\frac{2432}{3}\zeta_3^2 - 256\zeta_2\zeta_3 + \frac{10624\zeta_3}{9} - \frac{9088\zeta_2}{3} + \frac{1600\zeta_4}{3} + \frac{43520\zeta_5}{9} - \frac{2368\zeta_6}{9} + 768 + 4b^4_{q,C^4_{FF}} \right) \\
& + C_A C_F C_R n_f \left(4b_{4,n_f C_F^2 C_A} + \frac{6800\zeta_3^2}{3} - \frac{8864}{9}\zeta_2\zeta_3 - \frac{1892\zeta_3}{9} + \frac{5122\zeta_2}{27} - \frac{122216\zeta_4}{27} + \frac{21904\zeta_5}{9} - 1436\zeta_6 + \frac{2149049}{486} \right) \\
& + C_F^2 C_R n_f \left(4b^4_{q,n_f C_F^3} - 736\zeta_3^2 + \frac{1024}{3}\zeta_2\zeta_3 + \frac{2240\zeta_3}{9} - 648\zeta_2 + 668\zeta_4 - \frac{7744\zeta_5}{3} + \frac{29336\zeta_6}{9} - \frac{27949}{54} \right).
\end{aligned} \tag{9}$$

We include the quark and gluon rapidity anomalous dimension in analytic form as electronically readable files together with the journal submission of this Letter [29]. Note that the rapidity anomalous dimension is expressed in terms of four coefficients, which currently have only been determined numerically in Table 1 of Ref. [28]. We print their values here for convenience:

$$\begin{aligned} b^4_{q,n_f C_F^2 C_A} &= -455.247 \pm 0.005, \\ b^4_{q,C^4_{AF}} &= -998.0 \pm 0.2, \\ b^4_{q,C^4_{FF}} &= -143.6 \pm 0.2, \\ b^4_{q,n_f C_F^3} &= 80.780 \pm 0.005. \end{aligned} \tag{10}$$

The high numerical precision allows us to obtain a very precise determination of the rapidity anomalous dimension for adjoint and fundamental Wilson lines:

$$\begin{aligned} \gamma_r^q(n_f=3) &= 0.399\,12\alpha_s^2 + 0.525\,16\alpha_s^3 \\ &\quad + (0.601\,005 \pm 5\times 10^{-5})\alpha_s^4, \\ \gamma_r^q(n_f=4) &= 0.469\,24\alpha_s^2 + 0.616\,48\alpha_s^3 \\ &\quad + (0.613\,623 \pm 5\times 10^{-5})\alpha_s^4, \\ \gamma_r^q(n_f=5) &= 0.539\,29\alpha_s^2 + 0.689\,47\alpha_s^3 \\ &\quad + (0.535\,95 \pm 5\times 10^{-5})\alpha_s^4, \\ \gamma_r^g(n_f=3) &= 0.898\,19\alpha_s^2 + 1.181\,62\alpha_s^3 \\ &\quad + (1.553\,15 \pm 5\times 10^{-4})\alpha_s^4, \\ \gamma_r^g(n_f=4) &= 1.055\,80\alpha_s^2 + 1.387\,08\alpha_s^3 \\ &\quad + (1.588\,44 \pm 5\times 10^{-4})\alpha_s^4, \\ \gamma_r^g(n_f=5) &= 1.213\,41\alpha_s^2 + 1.551\,30\alpha_s^3 \\ &\quad + (1.420\,59 \pm 5\times 10^{-4})\alpha_s^4, \end{aligned} \tag{11}$$

where the uncertainty on the N^{4}LO coefficients is estimated by propagating the uncertainties in Eq. (10). Since they affect the fourth significant digit of the N^{4}LO correction, we will treat them as negligible for the rest of this Letter. Equations (9) and (11) are some of the main results of this Letter.

Let us emphasize that our result for $\gamma^i_{r,4}$ is essentially identical, analytically, for quarks and gluons, and only depends on the color representation through the quadratic and quartic Casimir operators

$$\begin{aligned} C_R &= \frac{1}{d_R}\mathrm{tr}(T_R^a T_R^a), \\ C^4_{R'R} &= \frac{1}{(4!)^2}\mathrm{tr}(T_{R'}^{\{a_1}\cdots T_{R'}^{a_4\}})\mathrm{tr}(T_R^{\{a_1}\cdots T_R^{a_4\}}), \end{aligned} \tag{12}$$

where $R' \in \{F, A\}$, T_R^a are the generators of the representation R and d_R is the dimension of the color representation. This property is referred to as "generalized Casimir scaling," which has also been observed to hold for the four-loop cusp anomalous dimension [21,22,30]. We stress that we have computed $\gamma^i_{r,4}$ independently for $i \in \{q, g\}$, so that generalized Casimir scaling was not used as an input to our computation.

Energy-energy correlation at N^{4}LL.—In this section we use our new result for γ_r^q to obtain the first resummation for an event shape at N^{4}LL. In particular, we consider the energy-energy correlation [31] (EEC) in electron-positron annihilation,

$$\mathrm{EEC}(\chi) = \sum_{a,b}\int d\sigma_{e^+e^-\to a+b+X}\frac{E_a E_b}{Q^2}\delta(\cos\chi_{ab} - \cos\chi), \tag{13}$$

which was one of the first infrared and collinear safe observables proposed for an e^+e^- collider. The EEC measures the angle χ_{ab} between two final state particles weighted by the energies of the particles relative to the total center-of-mass energy of the colliding e^+e^- pair. Furthermore, the EEC is symmetrized over all possible final state particle pairs, as implemented by the sum in Eq. (13). It is convenient to introduce a change of variables and to express the EEC in terms of $z \equiv \frac{1}{2}(1 - \cos\chi)$, $z \in [0, 1]$. The small angle limit ($\chi \to 0$) is reproduced by the $z \to 0$ limit, and the $z \to 1$ limit describes the dijet or back-to-back ($\chi \to \pi$) configuration. In these limits, the observable becomes strongly sensitive to collinear configurations of the QCD radiation generating large logarithms whose presence spoils the convergence of the perturbative expansion in the strong coupling constant. An all-order understanding in the coupling, which allows for the resummation of these logarithms, can be achieved using factorization theorems [2,3,32–41].

Throughout its history the EEC has provided the playground for exploring a variety of crucial aspects of QCD and non-Abelian quantum field theories in general, such as maximally supersymmetric Yang-Mills theory ($\mathcal{N}=4$ sYM). As a matter of fact, not only has the EEC been measured in multiple experiments [42–51], but it has been at the intersection of a variety of different theoretical fields. The EEC has been studied at strong coupling using the AdS/CFT correspondence [52], perturbatively in $\mathcal{N} = 4$ sYM [53–59], and in QCD [33,35–40,60–64], and it constitutes one of the simplest examples of energy correlators that have spurred renewed interest in exploring the connections between QCD and $\mathcal{N} = 4$; see, for example, [65–70]. Moreover, the EEC can be used for the extraction of the strong coupling constant (see, for example, [49,51,71]), and its generalizations to ep and hadron colliders as high precision probes for TMD physics at present and future colliders [72–77].

EEC in the back-to-back limit: The back-to-back asymptotics of the EEC can be described using soft and collinear effective theory (SCET) [78–81] via the following factorization theorem [41]:

$$\frac{d\sigma}{dz} = \frac{\hat{\sigma}_0}{8} H_{q\bar{q}}(Q,\mu) \int_0^\infty d(b_T Q)^2 J_0(b_T Q\sqrt{1-z}) \times \mathcal{J}_q\left(b_T,\mu,\frac{Qb_T}{\upsilon}\right) \mathcal{J}_{\bar{q}}\left(b_T,\mu,Qb_T\upsilon\right)[1+\mathcal{O}(1-z)]. \tag{14}$$

In Eq. (14), J_0 is the Bessel function arising from the Fourier transform due to the azimuthal symmetry of the EEC measurement, $H_{q\bar{q}}$ is the quark color singlet SCET hard function, which is related to the IR finite part of the quark form factors [82–91] and can be extracted up to four loops from the recent result of Ref. [91], and $\mathcal{J}_q$ is the quark EEC jet function, which is known up to N^{3}LO [40,41].

The EEC in the back-to-back limit is a SCET$_{\rm II}$ observable, and therefore requires the handling of rapidity divergences [1,2,12,15,92–98]. Equation (14) is derived in pure rapidity renormalization [41,98], with υ being the pure rapidity renormalization scale. In this renormalization scheme, the soft function is dropped since it is 1 to all orders, while the collinear and anticollinear jet functions are identical up to $\upsilon \to 1/\upsilon$, and the rapidity scale dependence cancels exactly at each order in perturbation theory in the product of the jet functions.

The hard and jet functions in Eq. (14) obey the following renormalization group equations [41]:

$$\mu\frac{d}{d\mu}\ln H_{q\bar{q}}(Q,\mu) = \gamma_H^q(Q,\mu),$$
$$\mu\frac{d}{d\mu}\ln \mathcal{J}_q\left(b_T,\mu,\frac{Qb_T}{\upsilon}\right) = \gamma_{\mathcal{J}_q}(\mu,\upsilon\mu/Q), \tag{15}$$

with the anomalous dimensions

$$\gamma_H^q(Q,\mu) = 4\Gamma_{\rm cusp}^q[\alpha_s(\mu)]\ln\frac{Q}{\mu} + 4\gamma_H^q[\alpha_s(\mu)],$$
$$\gamma_{\mathcal{J}_q}(\mu,\upsilon\mu/Q) = 2\Gamma_{\rm cusp}^q[\alpha_s(\mu)]\ln\frac{\upsilon\mu}{Q} - 2\gamma_H^q[\alpha_s(\mu)], \tag{16}$$

where $\Gamma_{\rm cusp}^q$ is the cusp anomalous dimension in the fundamental representation [19–21], the quark anomalous dimension $\gamma_H^q[\alpha_s(\mu)]$ is related to the quark collinear anomalous dimension [99], and we differentiated the anomalous dimensions from their noncusp part by the number of arguments as commonly done in SCET literature. The EEC jet function also obeys a rapidity RGE, governed by the rapidity anomalous dimension

TABLE I. Resummation accuracy in terms of the perturbative order of boundary terms, anomalous dimensions, and beta function.

Accuracy	H, $\mathcal{J}$	$\Gamma_{\rm cusp}(\alpha_s)$	$\gamma_H^q(\alpha_s)$	$\gamma_r^q(\alpha_s)$	$\beta(\alpha_s)$
LL	Tree level	1-loop			1-loop
NLL	Tree level	2-loop	1-loop	1-loop	2-loop
NLL′	1-loop	2-loop	1-loop	1-loop	2-loop
NNLL	1-loop	3-loop	2-loop	2-loop	3-loop
NNLL′	2-loop	3-loop	2-loop	2-loop	3-loop
N^{3}LL	2-loop	4-loop	3-loop	3-loop	4-loop
N^{3}LL′	3-loop	4-loop	3-loop	3-loop	4-loop
N^{4}LL	3-loop	5-loop	4-loop	4-loop	5-loop
N^{4}LL′	4-loop	5-loop	4-loop	4-loop	5-loop

$$\upsilon\frac{d}{d\upsilon}\ln \mathcal{J}_q\left(b_T,\mu,\frac{Qb_T}{\upsilon}\right) = -\frac{1}{2}\gamma_r^q(b_T,\mu). \tag{17}$$

We solve these RGEs to obtain the resummed cross section for the EEC explicitly in terms of the anomalous dimensions and boundary functions:

$$\frac{d\sigma}{dz} = \frac{\hat{\sigma}_0}{8}\int_0^\infty d(b_TQ)^2 J_0(b_TQ\sqrt{1-z})H_{q\bar{q}}(Q,\mu_H) \times \mathcal{J}_q\left(b_T,\mu_J,\frac{Qb_T}{\upsilon_n}\right)\mathcal{J}_{\bar{q}}\left(b_T,\mu_J,Qb_T\upsilon_{\bar{n}}\right)\left(\frac{\upsilon_n}{\upsilon_{\bar{n}}}\right)^{\frac{1}{2}\gamma_r^q(b_T,\mu_J)} \times \exp\left\{4\int_{\mu_J}^{\mu_H}\frac{d\mu'}{\mu'}\Gamma_{\rm cusp}^q[\alpha_s(\mu')]\ln\frac{\mu'}{Q} - \gamma_H^q[\alpha_s(\mu')]\right\}. \tag{18}$$

The logarithmic accuracy of the resummed cross section is defined in terms of the perturbative order at which the ingredients entering Eq. (18) are computed as shown in Table I. Explicitly, N^{4}LL resummation requires the cusp anomalous dimension and the QCD beta function to five loops [100,101], the collinear dimension at four loops [99], the jet function boundaries at three loops [40], the hard function at three loops [89,90], and the four-loop rapidity anomalous dimension, which we obtained in this Letter. In combination with an approximation of the five-loop cusp anomalous dimension [100], we have now all anomalous dimensions for N^{4}LL resummation at our disposal and can apply them toward realistic observables.

Numerical results:—We have implemented the resummed cross section of Eq. (18) in a private PYTHON code and performed the resummation of this observable up to N^{4}LL. Note that this constitutes the first ever resummation for an event shape at this level of accuracy. On top of all the necessary ingredients for N^{4}LL resummation, we also include the four-loop hard function, which we have extracted from the four-loop form factor calculation of Ref. [91]. Figure 1 shows the rapidity anomalous dimension of Eq. (8) in the fundamental representation. Figure 2 shows our results as a function of the scattering angle χ through different logarithmic orders. We observe that increasing

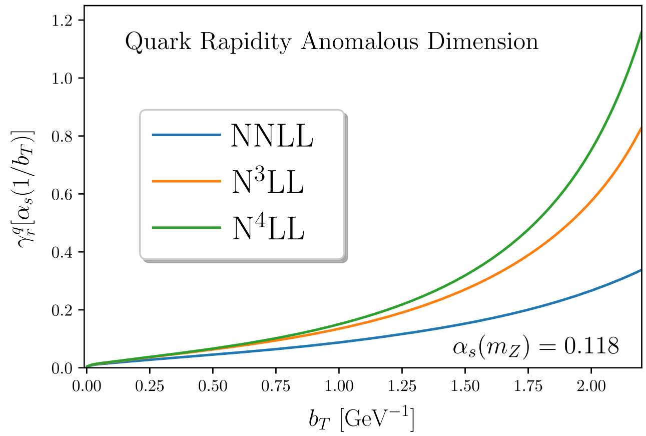

FIG. 1. Boundary term of the rapidity quark anomalous dimension as a function of b_T through four loops. The b_T dependence enters only through the coupling constant. The diverging behavior at large b_T is due to approaching the Landau pole. For recent work on extracting the anomalous dimension nonperturbatively at large b_T, see Refs. [102–112].

the logarithmic order leads to an improved description of the EEC. We indicate uncertainty estimates due to the truncation of the logarithmic accuracy by colored bands and observe that successively higher order bands are contained within the estimates based on previous orders. We conclude that the our computation of the EEC in the limit of $z \to 1$ at N^{4}LL yields a highly precise determination of the perturbative contribution to the scattering observable in this limit.

Our uncertainty estimates are based on the variation of renormalization scales. As expected, the explicit dependence on the renormalization scales μ and υ exactly cancels in the resummed cross section in Eq. (18). The result depends on the boundary scales $\{\mu_H, \mu_J, \upsilon_{n,\bar{n}}\}$ marking the starting points of the renormalization group evolution.

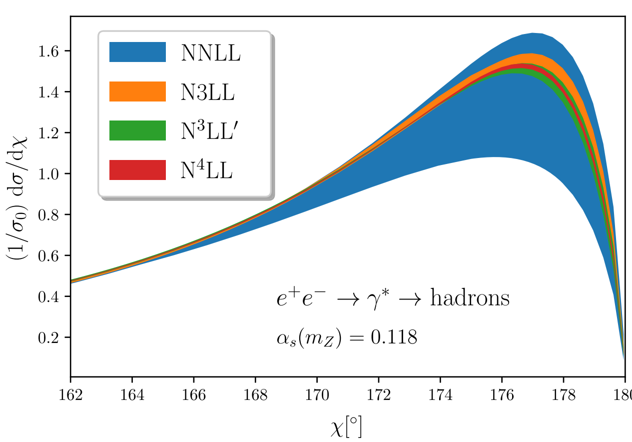

FIG. 2. Resummed result for the EEC in the back-to-back region up to N^{4}LL accuracy. Uncertainty bands reflect the residual perturbative uncertainty and are obtained with a 15-point scale variation of the resummation scales. See text for details.

The choice of these boundary scales is in principle arbitrary and, at any given logarithmic accuracy, the resummed cross sections obtained with different choices of boundary scales would give results that differ by terms that are beyond this logarithmic accuracy. We select the following scales:

$$\{\mu_H^* = Q, \mu_J^* = b_0/b_T, \upsilon_n^* = Qb_T/b_0 = 1/\upsilon_{\bar{n}}^*\}. \quad (19)$$

When choosing these values for the boundary scales, all explicit logarithms in the boundary functions vanish identically. Equation (18) evaluated with this canonical choice constitutes our central value of the resummed prediction. We estimate perturbative uncertainties on the resummed cross section by evaluating Eq. (18) with different boundary scales. Here, we vary the scales individually by a factor of $\frac{1}{2}$ or 2 around their canonical value and remove the configurations with simultaneous variations of factors greater than 2 or smaller than $\frac{1}{2}$. Next, we take the envelope of the results as our estimate of the perturbative uncertainty. This results in a 15-point scale variation procedure very analogous to the usual 7-point scale variation employed to estimate perturbative uncertainties in fixed order calculations. To treat the large b_T behavior in the Fourier transform we use the b^* prescription [2,3] employed in Ref. [40].

Note that the cusp anomalous dimension is known at five loops only in approximate form [100] with an 80% relative uncertainty, $\Gamma^{(5)}_{\rm cusp} = 0.21 \pm 0.17$, but it is in general expected that its numerical impact will be very small. In Fig. 3 we show the effect of varying the five-loop cusp anomalous dimension coefficient around the values of the uncertainty, $\{\Gamma^{(5)}_{\rm Cusp,+} = 0.38, \Gamma^{(5)}_{\rm Cusp} = 0.21, \Gamma^{(5)}_{\rm Cusp,-} = 0.04\}$. We see that it generates a sub-per-mille variation, confirming that it is indeed the case that its numerical impact is small and that the approximation of Ref. [100] is more than enough for current phenomenological studies.

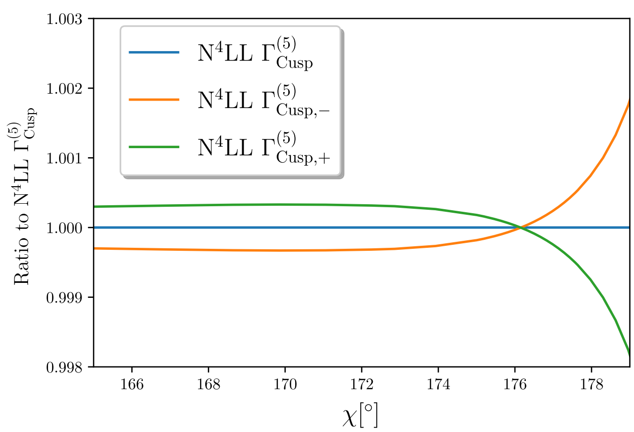

FIG. 3. Comparison of the central value for the EEC distribution between the resummed result computed with different values of the five-loop cusp anomalous dimension.

We leave a full phenomenological study of the EEC—including fixed order predictions [37,60,61], state of the art resummation in the $z \to 0$ limit [34,39], and estimation of parametric and nonperturbative uncertainties—to future work.

Conclusion.—Throughout this Letter, we have discussed the computation of the four-loop corrections to the quark and gluon rapidity anomalous dimensions, which control the all-order structure of large logarithms for several quantities of phenomenological interest, including transverse momentum distributions at proton colliders and event shape observables at e^+e^- colliders. Our computation is built on our recent determination of the four-loop soft anomalous dimension and the conjectured duality between the soft and rapidity anomalous dimensions. Our result is fully analytic, up to four constants that are only known numerically. Remarkably, our results exhibit generalized Casimir scaling, a property that was observed to hold also for the cusp anomalous dimension through four loops. We also applied our results for the rapidity anomalous dimension to obtain for the first time phenomenological results for the EEC in the back-to-back region at N^{4}LL, providing the most precise resummed calculation for this observable to date and the first example of the resummation of a TMD observable to fourth logarithmic order. This shows that our result will play an important role in the future in precisely determining several quantities of phenomenological interest.

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