Editors' Suggestion

Noninvertible Global Symmetries in the Standard Model

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We identify infinitely many noninvertible generalized global symmetries in QED and QCD for the real world in the massless limit. In QED, while there is no conserved Noether current for the $\mathrm{U}(1)_\mathrm{A}$ axial symmetry because of the Adler-Bell-Jackiw anomaly, for every rational angle $2\pi p/N$, we construct a conserved and gauge-invariant topological symmetry operator. Intuitively, it is a composition of the axial rotation and a fractional quantum Hall state coupled to the electromagnetic U(1) gauge field. These conserved symmetry operators do not obey a group multiplication law, but a noninvertible fusion algebra. They act invertibly on all local operators as axial rotations, but noninvertibly on the 't Hooft lines. We further generalize our construction to QCD, and show that the coupling $\pi^0 F \wedge F$ in the effective pion Lagrangian is necessary to match these noninvertible symmetries in the UV. Therefore, the conventional argument for the neutral pion decay using the ABJ anomaly is now rephrased as a matching condition of a generalized global symmetry.

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Introduction.—Global symmetry is one of the few intrinsic characteristics of a quantum system that is invariantly matched across all different descriptions and dualities. The most familiar example of a global symmetry is a U(1) global symmetry with a conserved Noether current $j_\mu(x)$. Thanks to the conservation equation $\partial^\mu j_\mu = 0$, the charge $Q = \int d^3x j_0$ is conserved under time evolution, and so is the symmetry operator $U_\vartheta = \exp(i\vartheta Q)$ labeled by a U(1) group element $\vartheta \in [0, 2\pi)$. In relativistic quantum field theory (QFT), time is on footing equal to any other direction in spacetime. We can therefore define the symmetry operator on a general closed three-manifold M in the $3+1$-dimensional spacetime

$$U_\vartheta(M) = \exp\left(i\vartheta \oint_M \star j\right), \tag{1}$$

where $\star$ is the Hodge dual of a differential form. In this relativistic setting, the conservation under time evolution is upgraded to the statement that $U_\vartheta(M)$ is a *topological* operator that depends on the choice of the three-manifold M only topologically [1]. In the case of a U(1) symmetry, the topological nature simply follows from the divergence theorem.

Given a quantum system with a global symmetry, one can attempt to gauge the symmetry to obtain a different system. The obstruction to gauging the said global symmetry is sometimes referred to as the 't Hooft anomaly of a global symmetry. In contrast, the Adler-Bell-Jackiw (ABJ) anomaly [2,3] (see Ref. [4] for a review) is the statement that a classical global symmetry fails to persist at the quantum level. The ABJ anomaly has many important phenomenological consequences, including the determination of the neutral pion decay coupling $\pi^0 F \wedge F$ in the effective pion Lagrangian. However, if the punchline of the ABJ anomaly is the absence of a global symmetry, why does it imply anything nontrivial in the IR effective Lagrangian? More generally, is it meaningful to discuss the ABJ anomaly in a QFT without a Lagrangian description in terms of a fermion path integral? In this Letter, for ABJ anomalies where all the participating symmetries are U(1), we will reinterpret them in terms of certain generalized global symmetries.

We start with the ABJ anomaly of the axial $\mathrm{U}(1)_\mathrm{A}$ symmetry in the $3+1$D massless QED. While there is no gauge-invariant Noether current, for every rational angle $\alpha = 2\pi p/N$, we can dress the naive $\mathrm{U}(1)_\mathrm{A}$ operator with a fractional quantum Hall state to construct a *conserved* and *gauge-invariant* topological symmetry operator, denoted by $\mathcal{D}_{p/N}(M)$, which can be supported on any closed, oriented three-manifold M. Interestingly, the topological operator $\mathcal{D}_{p/N}$ does not obey a group multiplication law and does *not* have an inverse operator. In particular, it is not a unitary operator. Can we still think of it as a global symmetry?

This question echoes with the recent developments of a novel kind of generalized global symmetries. (See Refs. [5,6] for reviews.) While every ordinary global symmetry is associated with a topological symmetry operator [such as (1)], the converse is not true. Building on the earlier work of Refs. [7–12] in two spacetime dimensions, it has been advocated in Refs. [13–15] that these more general topological operators should be viewed as generalized global symmetries [16–24]. Since they do not have an inverse, they are commonly referred to as *noninvertible symmetries*. In the past year, noninvertible symmetries have also been constructed in many familiar continuum and lattice gauge theories in higher spacetime dimensions [25–32].

From this modern viewpoint, the new topological operators $\mathcal{D}_{p/N}$ in QED are some of the first examples of noninvertible symmetries realized in nature. They give an invariant characterization of the ABJ anomaly in terms of the *existence* of a generalized global symmetry, rather than the *absence* thereof.

We further extend our analysis to QCD of the first generation in the massless limit. Below the electroweak scale, QCD has a U(1) symmetry suffering from the ABJ anomaly with the electromagnetic gauge symmetry, which we now interpret as an infinite, discrete, noninvertible symmetry. We demonstrate that the coupling $\pi^0 F\wedge F$ in the IR pion Lagrangian is necessary to match the noninvertible symmetry in the QCD Lagrangian. Therefore, we have reinterpreted the conventional argument for the neutral pion decay using the ABJ anomaly as a matching condition for a noninvertible global symmetry.

This Letter is accompanied by Supplemental Material [33], in which we provide more detailed derivations as well as alternative constructions, including Refs. [34–51].

ABJ anomaly and the fractional quantum Hall state.— Consider QED of a unit charge, massless Dirac fermion Ψ. The Euclidean Lagrangian is

$$\mathcal{L}_{\mathrm{QED}}[\Psi,\bar\Psi,A]=\frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu}+i\bar\Psi(\partial_\mu-iA_\mu)\gamma^\mu\Psi, \quad (2)$$

where A_μ is the dynamical (compact) U(1) one-form gauge field. We normalize the gauge field such that the flux $\oint_\Sigma F\in 2\pi\mathbb{Z}$ is properly quantized for any closed two-manifold Σ.

Classically, there is a $\mathrm{U}(1)_\mathrm{A}$ axial global symmetry that acts on the fermion as

$$\Psi\to e^{i\alpha\gamma_5/2}\Psi. \quad (3)$$

The normalization in the exponent is chosen in such a way that the periodicity of the axial rotation angle α is 2π. This is because the $\alpha=2\pi$ axial rotation acts on the fermion as $\Psi\to e^{i\pi\gamma_5}\Psi=(-1)\Psi$, which is part of the U(1) gauge symmetry and is therefore a trivial transformation.

Quantum mechanically, the $\mathrm{U}(1)_\mathrm{A}$ axial symmetry is broken by the ABJ anomaly [2,3]. Let the axial current be

$$j^{\mathrm{A}}_\mu=\bar\Psi\gamma_5\gamma_\mu\Psi. \quad (4)$$

Its conservation equation is violated by the dynamical gauge field, i.e., $\partial^\mu j^{\mathrm{A}}_\mu=(1/16\pi^2)\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}$. In terms of differential forms, we have

$$d\star j^{\mathrm{A}}=\frac{1}{4\pi^2}F\wedge F. \quad (5)$$

One can still attempt to define a $\mathrm{U}(1)_\mathrm{A}$ operator

$$U_\alpha(M)=\exp\left(\frac{i\alpha}{2}\oint_M\star j^{\mathrm{A}}\right), \quad (6)$$

where M is a closed, oriented three-dimensional submanifold in spacetime on which the operator is supported. When M is the whole space at a fixed time, the ABJ anomaly (5) implies that this naive $\mathrm{U}(1)_\mathrm{A}$ symmetry operator is not conserved under time evolution. In a relativistic QFT such as QED, (5) further implies that $U_\alpha(M)$ is generally not topological.

Consider instead the combination

$$\star\hat{j}^{\mathrm{A}}\equiv\star j^{\mathrm{A}}-\frac{1}{4\pi^2}A\wedge dA \quad (7)$$

as a new current, which now formally satisfies the conservation equation $d\star\hat{j}^{\mathrm{A}}=d\star j^{\mathrm{A}}-(1/4\pi^2)F\wedge F=0$. In components, we have $\hat{j}^{\mathrm{A}}_\mu\equiv j^{\mathrm{A}}_\mu-(1/4\pi^2)\epsilon_{\mu\nu\rho\sigma}A^\nu\partial^\rho A^\sigma$. However, this new current is not gauge invariant. It appears that there is no way to restore the $\mathrm{U}(1)_\mathrm{A}$ symmetry.

Nonetheless, let us naively proceed and define a gauge noninvariant symmetry operator as

$$\hat{U}_\alpha(M)=\exp\left[\frac{i\alpha}{2}\oint_M\left(\star j^{\mathrm{A}}-\frac{1}{4\pi^2}A\wedge dA\right)\right]. \quad (8)$$

Since the Chern-Simons level of $A\wedge dA$ in (8) is not quantized, the exponent is not invariant under large gauge transformations when M is a general compact three-manifold in spacetime. The operator $\hat{U}_\alpha(M)$ would have been topological, but it is generally not well defined since it is not gauge invariant.

Interestingly, there is a simple modification of (8) for rational angles $\alpha\in 2\pi\mathbb{Q}$. Let us start with the simplest case where $\alpha=2\pi/N$ for some integer N. In this case, the gauge noninvariant term in $\hat{U}_{2\pi/N}(M)$ is

$$-\frac{i}{4\pi N}\oint_M A\wedge dA. \quad (9)$$

TABLE I. The noninvertible symmetry operator $\mathcal{D}_{p/N}$ is both conserved (topological) and gauge invariant, but it does not obey a group multiplication law under parallel fusion. In contrast, the operator $\hat{U}_\alpha$ is not gauge invariant, and U_α is not conserved due to the ABJ anomaly. (Since U_α is not topological, its fusion is subject to short-distance singularities, and it is not meaningful to discuss its invertibility.).

	$U_\alpha(M)$	$\hat{U}_\alpha(M)$	$\mathcal{D}_{p/N}(M)$
Conserved (Topological)	No	Yes	Yes
Gauge invariant	Yes	No	Yes
Invertible	N/A	Yes	No

Roughly speaking, this is the action for the fractional quantum Hall state in $2+1$d at filling fraction $\nu = 1/N$. [In that context, A is regarded as a classical U(1) background gauge field, whereas in the current context A is a dynamical gauge field in the bulk.] However, this action on M is not well defined due to the fractional Chern-Simons level. Fortunately, there is a well-known solution to this inaccuracy in the condensed matter physics literature. (See, for example, Ref. [52] for a review.) Instead of (9), the precise gauge-invariant action for the fractional quantum Hall state is

$$i\oint_M \left(\frac{N}{4\pi} a\wedge da + \frac{1}{2\pi} a\wedge dA\right), \tag{10}$$

where a is a dynamical U(1) gauge field on M. It is a $\mathrm{U}(1)_N$ Chern-Simons theory of a coupled to A. Integrating out a naively gives us $a = -A/N$, which upon substitution returns (9). However, this is not a rigorous equation since $-A/N$ is not a properly quantized U(1) gauge field. It is therefore more precise to take (10) as the action for the fractional quantum Hall state.

Motivated by this discussion of the fractional quantum Hall state, we define a new operator $\mathcal{D}_{1/N}$ in QED by replacing (9) in $\hat{U}_{2\pi/N}(M)$ with (10):

$$\mathcal{D}_{\frac{1}{N}}(M) = \exp\left[i\oint_M \left(\frac{2\pi}{2N}\star j^{\mathrm{A}} + \frac{N}{4\pi} a\wedge da + \frac{1}{2\pi} a\wedge dA\right)\right], \tag{11}$$

where a is a dynamical one-form gauge field that only lives on the three-manifold M. [Here and throughout we omit the path integral over a in the expression for $\mathcal{D}_{1/N}(M)$.] This new operator can be viewed as dressing the naive axial symmetry operator $U_{2\pi/N}(M)$ by a fractional quantum Hall state on M coupled to the bulk dynamical gauge field A. We emphasize that since a only lives on the support of the operator $\mathcal{D}_{1/N}$, it can be viewed as an auxiliary field which does not change the physics of the bulk QED; in particular, there is no additional asymptotic state introduced by a.

The operator $\mathcal{D}_{1/N}$ is distinguished from the previous trials in that it satisfies all the following properties: (i) it acts as an axial rotation on fermions with $\alpha = 2\pi/N$ in (3), (ii) it is gauge invariant since the Chern-Simons levels are properly quantized, and (iii) it is topological, and in particular conserved under time evolution.

We will give a rigorous proof on the topological nature of (11) in the Supplemental Material [33]. For now, we can understand it heuristically from the relation between (9) and (10), and the anomalous conservation equation (5).

Since $\mathcal{D}_{1/N}$ is a topological operator, it should be viewed as a generalized global symmetry in the spirit of Refs. [1,13,15]. Interestingly, it is not a usual grouplike symmetry. That is, this operator does not follow the group multiplication law under parallel fusions. Indeed, we have

$$\mathcal{D}_{\frac{1}{N}}(M)\times\mathcal{D}^\dagger_{\frac{1}{N}}(M) = \exp\left[i\oint_M\left(\frac{N}{4\pi}a\wedge da - \frac{N}{4\pi}\bar{a}\wedge d\bar{a} + \frac{1}{2\pi}(a-\bar{a})\wedge dA\right)\right], \tag{12}$$

where $\bar{a}$ only lives on the support of $\mathcal{D}^\dagger_{1/N}(M)$. In particular, $\mathcal{D}_{1/N}$ is not a unitary operator, and it does not have an inverse operator $(\mathcal{D}_{1/N})^{-1}$ such that $\mathcal{D}_{1/N}\times(\mathcal{D}_{1/N})^{-1} = (\mathcal{D}_{1/N})^{-1}\times\mathcal{D}_{1/N} = 1$. For this reason, $\mathcal{D}_{1/N}$ is a *noninvertible symmetry.*

How do we generalize this construction to any rational angle $\alpha = 2\pi p/N$ where p and N are two coprime integers? A natural generalization of the $\mathrm{U}(1)_N$ Chern-Simons theory is the minimal $\mathbb{Z}_N$ topological quantum field theory (TQFT) $\mathcal{A}^{N,p}$ [53] (see also Refs. [54–56]). The defining feature of $\mathcal{A}^{N,p}$ is that it is the minimal TQFT with a $\mathbb{Z}_N^{(1)}$ one-form global symmetry with its 't Hooft anomaly labeled by p. See the Supplemental Material [33] for a review of the minimal $\mathbb{Z}_N$ TQFT. When $p=1$, we have $\mathcal{A}^{N,1} = U(1)_N$.

Let $\mathcal{A}^{N,p}[B]$ denote the Lagrangian of the $\mathbb{Z}_N$ minimal TQFT coupled to a $\mathbb{Z}_N$ background two-form gauge field B for the $\mathbb{Z}_N^{(1)}$ one-form global symmetry. The natural generalization of the Lagrangian of (10) is $\mathcal{A}^{N,p}[dA/N]$, where we activate the two-form background gauge field by the electromagnetic one-form gauge field A, properly normalized. With all these preparations, the new topological operator $\mathcal{D}_{p/N}$ associated with the axial rotation $2\pi p/N$ is defined as

$$\mathcal{D}_{p/N}(M) = \exp\left[\oint_M\left(\frac{2\pi i p}{2N}\star j^{\mathrm{A}} + \mathcal{A}^{N,p}[dA/N]\right)\right]. \tag{13}$$

Since $\mathcal{A}^{N,p+N} = \mathcal{A}^{N,p}$, we have $\mathcal{D}_{p+N/N} = \mathcal{D}_{p/N}$, and therefore the noninvertible symmetry is labeled by an element $p/N \in \mathbb{Q}/\mathbb{Z}$.

We can replace $\mathcal{A}^{N,p}[B]$ in (13) by any $2+1$d TQFT $\mathcal{T}[B]$ [e.g., p copies of $\mathrm{U}(1)_N$] with a $\mathbb{Z}_N^{(1)}$ one-form symmetry and anomaly p. This defines another topological operator $\mathcal{D}_{\mathcal{T}}$. It was shown in Ref. [53] that any such TQFT $\mathcal{T}[B]$ is factorized as $\mathcal{T}[B] = \mathcal{A}^{N,p}[B] \otimes \mathcal{T}'$, where $\mathcal{T}'$ is a decoupled TQFT. It follows that $\mathcal{D}_{\mathcal{T}}$ is a composite operator of $\mathcal{D}_{p/N}$ and a decoupled $2+1$d TQFT $\mathcal{T}'$, i.e., $\mathcal{D}_{\mathcal{T}} = \mathcal{D}_{p/N} \times \mathcal{T}'$. In this sense, $\mathcal{D}_{p/N}$ defined in (13) is the minimal topological operator.

To summarize, in massless QED, for every rational angle $\alpha = 2\pi p/N$, there is a gauge-invariant and conserved topological symmetry operator $\mathcal{D}_{p/N}$ that acts on the fermions as axial rotations. However, there is no gauge-invariant Noether current or charge. Indeed, the exponents of (11) and (13), which would have been the conserved charges, are not gauge invariant because of the Chern-Simons terms. Rather, their exponentiations are gauge-invariant symmetry operators. Therefore, the noninvertible symmetries from $\mathcal{D}_{p/N}$ are discrete, rather than continuous. We conclude that the continuous, invertible axial $\mathrm{U}(1)_{\mathrm{A}}$ symmetry is broken by the ABJ anomaly to a discrete, noninvertible symmetry. See Table I for the summary.

Let us discuss the action of the noninvertible symmetry. From the first term in (13), we see that $\mathcal{D}_{p/N}$ acts invertibly on the fermions as axial rotations. It leads to selection rules on scattering amplitudes, which explain the familiar helicity conservation of electrons and positrons from a global symmetry principle. In contrast, in the Supplemental Material [33], we show that it acts noninvertibly on the 't Hooft lines via the Witten effect [57].

Finally, we comment on the noninvertible symmetry $\mathcal{D}_{p/N}$ and $\hat{U}_\alpha$ in (8) on noncompact space such as $\mathbb{R}^3$. (See, for example, Ref. [58] for recent discussions.) In this case, the operator $\hat{U}_\alpha(\mathbb{R}^3)$ is actually gauge invariant because there is no nontrivial gauge transformation on $\mathbb{R}^3$ or S^3 since $\pi_3[U(1)] = 0$. In fact, on $\mathbb{R}^3$, we can integrate out a in (13) and equate $\mathcal{D}_{p/N}(\mathbb{R}^3) = \hat{U}_{2\pi p/N}(\mathbb{R}^3)$. However, $\hat{U}_\alpha(M)$ is not gauge invariant on a more general compact three-manifold M. In contrast, our noninvertible symmetry $\mathcal{D}_{p/N}(M)$ (13) is gauge invariant and conserved (topological) for any compact three-manifold M, but it is only defined for rational angles $\alpha = 2\pi p/N$.

QCD and the pion decay.—Let us take the UV theory to be the QCD Lagrangian of the massless up and down quarks at an energy scale far above the pion scale, but below the electroweak scale so the $\mathrm{SU}(2)\times \mathrm{U}(1)$ gauge symmetry has been Higgsed to the electromagnetic $\mathrm{U}(1)_{\mathrm{EM}}$ gauge symmetry. Let u, d be the Dirac fermions for the up and down quarks, respectively. The $\mathrm{U}(1)_{\mathrm{EM}}$ charges of the u and d are $+2/3$ and $-1/3$, respectively. We will suppress the SU(3) color indices. Classically, the QCD Lagrangian has a global symmetry. [The subscript 3 is to distinguish this symmetry from the other axial symmetry that acts as $\binom{u}{d} \to \exp\left[i(\alpha/2)\gamma_5\right]\binom{u}{d}$, which suffers from an ABJ anomaly with the SU(3) gauge symmetry.]

$$\mathrm{U}(1)_{\mathrm{A3}}\colon \begin{pmatrix} u \\ d \end{pmatrix} \to \exp\left(i\beta\gamma_5\sigma_3\right)\begin{pmatrix} u \\ d \end{pmatrix} = \begin{pmatrix} \exp\left(i\beta\gamma_5\right)u \\ \exp\left(-i\beta\gamma_5\right)d \end{pmatrix}, \tag{14}$$

where $\beta \sim \beta + 2\pi$. The axial current is conventionally normalized as

$$j^{\mathrm{A3}}_\mu = \frac{1}{2}\bar{u}\gamma_5\gamma_\mu u - \frac{1}{2}\bar{d}\gamma_5\gamma_\mu d. \tag{15}$$

It suffers from the ABJ anomaly with the electromagnetic $\mathrm{U}(1)_{\mathrm{EM}}$ gauge symmetry:

$$d\star j^{\mathrm{A3}} = \frac{1}{8\pi^2} F \wedge F. \tag{16}$$

The naive, gauge noninvariant symmetry operator is $\exp\{2i\beta\oint_M [\star j^{\mathrm{A3}} - (1/8\pi^2) A\wedge dA]\}$. Note that for $\beta = \pi$, the $-(i/4\pi)A \wedge dA$ term is actually properly quantized, and it generates an invertible $\mathbb{Z}_2$ symmetry.

For a more generic rational angle β, say, $\beta = \pi/N$, we can apply the same construction in QED to define a gauge-invariant and conserved topological operator:

$$\mathcal{D}_{\frac{1}{N}}(M) = \exp\left[i\oint_M \left(\frac{2\pi}{N}\star j^{\mathrm{A3}} + \frac{N}{4\pi} a\wedge da + \frac{1}{2\pi} a \wedge dA\right)\right]. \tag{17}$$

How are these infinitely many noninvertible symmetries in QCD captured by the low-energy pion Lagrangian? In the pion Lagrangian, the axial current becomes

$$j^{\mathrm{A3}}_\mu = -f_\pi \partial_\mu \pi^0 + \cdots, \tag{18}$$

which shifts the neutral pion by $\pi^0 \to \pi^0 - 2\beta f_\pi$. Here $f_\pi \sim 92.4$ MeV is the pion decay constant. Since the neutral pion field is compact with periodicity $\pi^0 \sim \pi^0 + 2\pi f_\pi$, the $\beta = \pi$ transformation, which generates a $\mathbb{Z}_2$ global symmetry in QCD, now acts trivially in the IR pion Lagrangian.

The relevant terms in the pion Lagrangian in Euclidean signature are

$$\mathcal{L}_{\mathrm{IR}} = \frac{1}{2} d\pi^0 \wedge \star d\pi^0 + \frac{1}{2e^2} F\wedge \star F + i g \pi^0 F \wedge F + \cdots. \tag{19}$$

Here g is a coefficient we will fix by the noninvertible symmetry.

To proceed, we insert $\mathcal{D}_{1/N}$ as a defect at $x = 0$ in the IR pion effective theory. In other words, M is chosen to be the three-manifold defined as $x = 0$ in Euclidean spacetime. For this to be a consistent defect, we need to investigate the equations of motion. Because of the first term in (17), the pion field is discontinuous across the defect:

$$\pi^0|_{x=0^+} = \pi^0|_{x=0^-} - \frac{2\pi}{N} f_\pi. \tag{20}$$

The equation of motion for the gauge field a on M gives

$$N da + F = 0. \tag{21}$$

On the other hand, the equation of motion for the bulk gauge field A includes a boundary contribution on the defect $x = 0$:

$$2ig(\pi^0|_{x=0^+} - \pi^0|_{x=0^-})F = \frac{i}{2\pi} da. \tag{22}$$

Combining with (20) and (21), we find

$$g = \frac{1}{8\pi^2 f_\pi}. \tag{23}$$

Hence, the $\pi^0 F \wedge F$ term in the pion Lagrangian (19) is necessary to match the noninvertible symmetry in the UV QCD Lagrangian.

Let us compare our reasoning with the usual derivation in the literature. The $\pi^0 F \wedge F$ term in the effective pion Lagrangian is conventionally argued using the ABJ anomaly. Since the fine structure constant is small, one can effectively treat the $\mathrm{U}(1)_{\mathrm{EM}}$ gauge field as a background gauge field, and interpret the ABJ anomaly as an 't Hooft anomaly between $\mathrm{U}(1)_{\mathrm{A3}}$ and $\mathrm{U}(1)_{\mathrm{EM}}$. The 't Hooft anomaly matching condition then determines the $\pi^0 F \wedge F$ coupling. This term can also be derived from the Wess-Zumino term in the chiral Lagrangian when coupled to the electromagnetic gauge field [59]. In this Letter, we provide an alternative derivation of the neutral pion decay from a matching condition for the noninvertible global symmetries for any finite and nonzero fine structure constant.

Conclusion and outlook.—In the past few years, there have been a lot of exciting developments on generalized global symmetries in high energy physics and condensed matter physics. In this Letter, we identify some of the first examples of noninvertible global symmetries in nature.

In QED, the continuous, invertible classical $\mathrm{U}(1)_{\mathrm{A}}$ symmetry turns into a discrete, noninvertible global symmetry generated by the topological operators $\mathcal{D}_{p/N}$, each labeled by a rational number $p/N \in \mathbb{Q}/\mathbb{Z}$. The noninvertible symmetry operator $\mathcal{D}_{p/N}$ is a composition of the naive axial rotation with a rational angle $\alpha = 2\pi p/N$, together with a $\nu = p/N$ fractional quantum Hall state.

We similarly construct these noninvertible symmetries in QCD of the first generation in the massless limit and below the electroweak scale. The coupling $\pi^0 F \wedge F$ in the IR pion Lagrangian is necessary to match these noninvertible symmetries in the QCD Lagrangian. Therefore, the neutral pion decay $\pi^0 \to \gamma\gamma$ is a direct consequence of the noninvertible global symmetries.

Our noninvertible global symmetries are only exact when the fermions are massless. Said differently, electrons and quarks are naturally massless in QED and QCD, respectively, because of the noninvertible global symmetry.

There are several future directions: (i) What is the full noninvertible fusion algebra of $\mathcal{D}_{p/N}$ and the condensation operator? (ii) How do we understand the spontaneous symmetry breaking of noninvertible symmetries? Even though the noninvertible symmetry is discrete, it has infinitely many elements each labeled by a rational number, which is dense in U(1). Is it possible to interpret π^0 as a Goldstone boson for this discrete but infinite noninvertible global symmetry? It is intriguing to speculate that this might be the reason why π^0 can be so light as a Goldstone boson, but also admits the nonderivative coupling $\pi^0 F \wedge F$. (iii) It would be interesting to extend our construction to nonabelian gauge groups. However, for simply connected nonabelian gauge groups, there is no magnetic one-form symmetry, and our construction does not generalize to these cases straightforwardly. (iv) If we interpret the pion field π^0 of (19) as an axion, we immediately conclude that there are infinitely many noninvertible symmetries $\mathcal{D}_{p/N}$ in the axion-Maxwell theory. (The higher group structure of axion gauge theory has been explored in Refs. [60–64].) See Ref. [65] for applications of noninvertible symmetries on axion physics. (v) In Ref. [28], it was shown that the higher gauging of a higher-form symmetry leads to a noninvertible symmetry, generated by the condensation operators. Using this construction, in addition to the symmetries discussed here, there are many other noninvertible symmetries from the higher gauging of higher-form symmetries (e.g., $\mathbb{Z}_6^{(1)}$) in the standard model. We leave these condensation operators for future investigations. (vi) The axial $\mathrm{U}(1)_A$ global symmetry of a free massless Dirac fermion has a mixed gravitational anomaly. It would be interesting to understand if there is a similar gravitational anomaly for our noninvertible symmetry. See Ref. [30] for a discussion on the mixed gravitational anomaly for noninvertible global symmetries in other models. Relatedly, one is free to dress a properly quantized $2+1$d gravitational Chern-Simons term on $\mathcal{D}_{p/N}$. Such a topological counterterm has interesting consequences for lower dimensional topological defects (see, for example, Refs. [15,28,66]). We leave the investigations of these effects for the future. (We thank the referee for this point.)

Strictly speaking, our noninvertible symmetries only exist in QCD below the electroweak scale, but not in the full standard model. It would be exciting to explore other possible noninvertible global symmetries in the standard model and their dynamical applications.

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[1] D. Gaiotto, A. Kapustin, N. Seiberg, and B. Willett, J. High Energy Phys. 02 (2015) 172.
[2] S. L. Adler, Phys. Rev. **177**, 2426 (1969).
[3] J. S. Bell and R. Jackiw, Nuovo Cimento A **60**, 47 (1969).
[4] G. 't Hooft, Phys. Rep. **142**, 357 (1986).
[5] J. McGreevy, arXiv:2204.03045.
[6] C. Cordova, T. T. Dumitrescu, K. Intriligator, and S.-H. Shao, in 2022 Snowmass Summer Study (2022), arXiv:2205.09545.
[7] E. P. Verlinde, Nucl. Phys. **B300**, 360 (1988).
[8] V. B. Petkova and J. B. Zuber, Phys. Lett. B **504**, 157 (2001).
[9] J. Fuchs, I. Runkel, and C. Schweigert, Nucl. Phys. **B646**, 353 (2002).
[10] J. Frohlich, J. Fuchs, I. Runkel, and C. Schweigert, Phys. Rev. Lett. **93**, 070601 (2004).
[11] J. Frohlich, J. Fuchs, I. Runkel, and C. Schweigert, Nucl. Phys. **B763**, 354 (2007).
[12] J. Frohlich, J. Fuchs, I. Runkel, and C. Schweigert, *XVIth International Congress on Mathematical Physics* (World Scientific, Singapore, 2010).
[13] L. Bhardwaj and Y. Tachikawa, J. High Energy Phys. 03 (2018) 189.
[14] Y. Tachikawa, SciPost Phys. **8**, 015 (2020).
[15] C.-M. Chang, Y.-H. Lin, S.-H. Shao, Y. Wang, and X. Yin, J. High Energy Phys. 01 (**2019**) 026.
[16] D. Aasen, R. S. K. Mong, and P. Fendley, J. Phys. A **49**, 354001 (2016).
[17] R. Thorngren and Y. Wang, arXiv:1912.02817.
[18] W. Ji, S.-H. Shao, and X.-G. Wen, Phys. Rev. Res. **2**, 033317 (2020).
[19] Y.-H. Lin and S.-H. Shao, J. Phys. A **54**, 065201 (2021).
[20] Z. Komargodski, K. Ohmori, K. Roumpedakis, and S. Seifnashri, J. High Energy Phys. 03 (2021) 103.
[21] R. Thorngren and Y. Wang, arXiv:2106.12577.
[22] D. Aasen, P. Fendley, and R. S. K. Mong, arXiv:2008.08598.
[23] T.-C. Huang, Y.-H. Lin, and S. Seifnashri, J. High Energy Phys. 12 (2021) 028.
[24] I. M. Burbano, J. Kulp, and J. Neuser, arXiv:2112.14323.
[25] M. Koide, Y. Nagoya, and S. Yamaguchi, Prog. Theor. Exp. Phys. **2022**, 013B03 (2022).
[26] Y. Choi, C. Cordova, P.-S. Hsin, H. T. Lam, and S.-H. Shao, Phys. Rev. D **105**, 125016 (2022).
[27] J. Kaidi, K. Ohmori, and Y. Zheng, Phys. Rev. Lett. **128**, 111601 (2022).
[28] K. Roumpedakis, S. Seifnashri, and S.-H. Shao, arXiv:2204.02407.
[29] L. Bhardwaj, L. Bottini, S. Schafer-Nameki, and A. Tiwari, arXiv:2204.06564.
[30] Y. Hayashi and Y. Tanizaki, J. High Energy Phys. 08 (2022) 036.
[31] Y. Choi, C. Cordova, P.-S. Hsin, H. T. Lam, and S.-H. Shao, arXiv:2204.09025.
[32] J. Kaidi, G. Zafrir, and Y. Zheng, J. High Energy Phys. 08 (2022) 053.
[33] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.129.161601, for detail.
[34] J. Gomis, Z. Komargodski, and N. Seiberg, SciPost Phys. **5**, 007 (2018).
[35] A. Kapustin and R. Thorngren, arXiv:1309.4721.
[36] R. Thorngren and C. von Keyserlingk, arXiv:1511.02929.
[37] L. Kong and X.-G. Wen, arXiv:1405.5858.
[38] D. V. Else and C. Nayak, Phys. Rev. B **96**, 045136 (2017).
[39] L. Kong, T. Lan, X.-G. Wen, Z.-H. Zhang, and H. Zheng, Phys. Rev. Res. **2**, 043086 (2020).
[40] T. Johnson-Freyd, arXiv:2011.11165.
[41] L. Bhardwaj, Y. Lee, and Y. Tachikawa, J. High Energy Phys. 11 (2020) 141.
[42] C. Cordova, P.-S. Hsin, and N. Seiberg, SciPost Phys. **4**, 021 (2018).
[43] L. Fidkowski and A. Kitaev, Phys. Rev. B **81**, 134509 (2010).
[44] J. M. Maldacena, G. W. Moore, and N. Seiberg, J. High Energy Phys. 10 (2001) 005.
[45] T. Banks and N. Seiberg, Phys. Rev. D **83**, 084019 (2011).
[46] A. Kapustin and N. Seiberg, J. High Energy Phys. 04 (2014) 001.
[47] R. Dijkgraaf and E. Witten, Commun. Math. Phys. **129**, 393 (1990).
[48] J. Kaidi, Z. Komargodski, K. Ohmori, S. Seifnashri, and S.-H. Shao, arXiv:2107.13091.
[49] K. Fujikawa, Phys. Rev. D **29**, 285 (1984).
[50] G. 't Hooft, NATO Sci. Ser. B **59**, 135 (1980).
[51] G. Mahlon, Phys. Rev. D **49**, 2197 (1994).
[52] D. Tong, arXiv:1606.06687.
[53] P.-S. Hsin, H. T. Lam, and N. Seiberg, SciPost Phys. **6**, 039 (2019).
[54] G. W. Moore and N. Seiberg, Commun. Math. Phys. **123**, 177 (1989).
[55] P. Bonderson, K. Shtengel, and J. K. Slingerland, Ann. Phys. (Amsterdam) **323**, 2709 (2008).
[56] M. Barkeshli, P. Bonderson, M. Cheng, and Z. Wang, Phys. Rev. B **100**, 115147 (2019).
[57] E. Witten, Phys. Lett. **86B**, 283 (1979).
[58] D. Harlow and H. Ooguri, Commun. Math. Phys. **383**, 1669 (2021).
[59] E. Witten, Nucl. Phys. **B223**, 422 (1983).
[60] Y. Hidaka, M. Nitta, and R. Yokokura, J. High Energy Phys. 01 (2021) 173.
[61] T. D. Brennan and C. Cordova, J. High Energy Phys. 02 (2022) 145.
[62] Y. Hidaka, M. Nitta, and R. Yokokura, Phys. Lett. B **823**, 136762 (2021).
[63] Y. Hidaka, M. Nitta, and R. Yokokura, Prog. Theor. Exp. Phys. **2022**, 04A109 (2022).
[64] S. Kaya and T. Rudelius, arXiv:2202.04655.
[65] C. Cordova and K. Ohmori, arXiv:2205.06243.
[66] C. Córdova, K. Ohmori, S.-H. Shao, and F. Yan, Phys. Rev. D **102**, 045019 (2020).