Intrinsic Pathology of Self-Interacting Vector Fields

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We show that self-interacting vector field theories exhibit unphysical behavior even when they are not coupled to any external field. This means any theory featuring such vectors is in danger of being unphysical, an alarming prospect for many proposals in cosmology, gravity, high energy physics, and beyond. The problem arises when vector fields with healthy configurations naturally reach a point where time evolution is mathematically ill defined. We develop tools to easily identify this issue, and provide a simple and unifying framework to investigate it.

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Introduction.—Classical electromagnetic waves simply pass through each other when they meet since they obey a linear equation. The picture changes in quantum electrodynamics where two photons can scatter off of each other in principle, they are *self-interacting* in this picture [1]. The use of self-interacting vector fields goes beyond this example. They are prevalent in fundamental theories of gravity and cosmology [2–12], and in effective field theories encountered in a wide range of research from astrophysics to plasma physics [13–16], including the photon-photon scattering we mentioned [17]. These theories have interesting mathematical structure in their own right [12,18–20], and there are systematic efforts to classify all possible self-interacting generalizations of the photon, building on the massive vector theory of Proca [21–25]. In short, self-interacting vector fields can be encountered in all corners of physics. We will, however, show that some of the simplest and most widely encountered forms of vector self-interaction cannot be included in physical theories, hence, many of the ideas we counted above are in need of reevaluation.

The unphysical aspects of self-interacting vector fields arise because their time evolution is not possible beyond a finite duration. Specifically, we show that the field equations that provide the dynamics become unusable, as they no longer define a time evolution. We demonstrate this for vectors that are not coupled to any external fields, which means our results are independent of the context in which the vector is considered, hence they apply to all conceivable cases. These results build on, and widely generalize, a series of studies which first showed that specific self-interacting theories break down near certain astrophysical objects [2,26–29], and more recently generalized this breakdown to simpler couplings and dynamical cases [29,30].

A central idea to understand the problem is that the dynamics of the vector field can sometimes be formulated as if governed by a so-called *effective metric* that depends on the field itself, even when the gravitational coupling is turned off [2,27–29]. That is, the vector can behave as if in curved spacetime, even when it is not, and this metric can become singular in finite time, at finite vector field values for regular spacetime metrics.

We show for the first time that the effective metric can be constructed *exactly* if spacetime is $1+1$ dimensional, and most likely not in any other case, but surprisingly it still controls the breakdown of time evolution in any dimension. Our approach improves upon earlier approximate methods [27–29], and we also dispel some of the confusion in the literature. We demonstrate that without proper analysis, unphysical coordinate effects can be misidentified as problems in time evolution, or a true breakdown can be overlooked in numerical computations, hence the framework we provide is an essential tool for any future work on the topic.

These results are highly surprising since they demonstrate that the vector field theories that can exist in nature are tightly constrained, providing a novel appreciation of the Maxwell and Proca theories. We show that heuristic reasoning in field theories, which is commonly based on scalars, can mislead us and mask problems in general, even in the next simplest example of vectors. Furthermore, we show that the analysis of the dynamics of self-interacting vector fields can reveal anomalies that are not apparent in static solutions or a basic counting of the propagating degrees of freedom, hence, it can be a powerful tool to test a wide variety of theoretical ideas.

Our metric signature is $(-,+,\cdots,+)$.

Explicitly hyperbolic formulation of the nonlinear Proca theory.—A simple generalization of the Proca theory, which we dub the "nonlinear Proca theory" (NPT), is given by the Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \overbrace{\left(\frac{\mu^2}{2}X^2 + \frac{\lambda\mu^2}{4}(X^2)^2\right)}^{V(X^2)}, \tag{1}$$

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where $F_{\mu\nu} = \nabla_\mu X_\nu - \nabla_\nu X_\mu$ and $X^2 = X_\mu X^\mu$ for the real vector field X_μ. The corresponding field equation is

$$\nabla_\mu F^{\mu\nu} = 2V' X^\nu = \mu^2 (1 + \lambda X^2) X^\nu, \qquad (2)$$

with $V' = dV/d(X^2)$. We can scale the coordinates and the fields, and without loss of generality set $\mu^2 = \pm 1$, $\lambda = \pm 1$ henceforth.

Note that the potential is unbounded from below in some cases. A major point of this study is that the notion of boundedness from below that is central to scalar field theories is insufficient for vectors, as we shall explain. Nevertheless, we still consider $V(X^2)$ to be supplemented by the term $\epsilon (X^2)^4$ for the sake of argument, for some sufficiently small ϵ. One can also physically motivate different parameter signs. For example, $\mu^2 = 1$, $\lambda = 1$ has a convex self-interaction potential without any intrinsic instabilities, and hence is an analog of the nonlinear Klein-Gordon equation. $\mu^2 = 1$, $\lambda = -1$ is an effective field theory for the Abelian Higgs mechanism (e.g., [14,15,31]). The expansion breaks down at $z = 0$ but the problems we discuss occur before this point. $\mu^2 = -1$, $\lambda = -1$ is an analog of the famous Mexican hat potential.

It is not trivial to judge the well posedness of NPT from Eq. (2) since it is not manifestly hyperbolic, i.e., not in the form of a generalized wave equation. To obtain this form, we first observe that X_μ obeys the (generalized) Lorenz condition [27,29]

$$\nabla_\nu \nabla_\mu F^{\mu\nu} = 0 \Rightarrow \nabla_\mu (z X^\mu) = 0 \qquad (3)$$

due to the antisymmetry of $F^{\mu\nu}$, where $z = 2V'/\mu^2 = 1 + \lambda X^2$.

Using a calculation detailed in Supplemental Material, Sec. A [32], we show that in $1 + 1$ dimensions the principal part of Eq. (2) can be rewritten as

$$\bar{g}_{\alpha\beta} \nabla^\alpha \nabla^\beta X^\mu + \ldots = \mathcal{M}^\mu_\alpha X^\alpha, \qquad (4)$$

where the ellipses represent single derivative terms, and the effective metric and the mass square tensor are, respectively,

$$\bar{g}_{\mu\nu} = z g_{\mu\nu} + 2z' X_\mu X_\nu \qquad (5)$$

$$\mathcal{M}^\mu_\nu = z^2 \mu^2 \delta^\mu_\nu + \text{curvature terms}. \qquad (6)$$

The overall factor z is optional in the definition of $\bar{g}_{\mu\nu}$, i.e., our results also hold for $\bar{g}_{\mu\nu} = g_{\mu\nu} + 2z^{-1} z' X_\mu X_\nu$. We demonstrate in Supplemental Material, Sec. A [32] that, despite some recent approximate computations in $3 + 1$ dimensions, the above result most likely cannot be generalized beyond $1 + 1$ dimensions, and we also discuss the exact form of $\mathcal{M}$. However, the effective metric still determines when the loss of hyperbolicity occurs in any spacetime dimension as detailed in Supplemental Material, Sec. B [32].

The breakdown of time evolution in NPT.—Once it is established that the effective metric governs the dynamics, we immediately see that the time evolution cannot continue to the future of a point where $\bar{g}_{\mu\nu}$ becomes singular. Hence, our main task is identifying if and when this occurs.

Our main result is that, starting from problem-free initial data, NPT can naturally evolve to a configuration where the effective metric becomes singular in finite time. Mathematically, this happens when the determinant vanishes

$$\bar{g} = g (1 + \lambda X^2)^d (1 + 3\lambda X^2) = g z^d z_3 = 0, \qquad (7)$$

where $g = \det(g_{\mu\nu})$, and we used the determinant lemma $\det(A + uv^T) = (1 + v^T A^{-1} u) \det A$ in $d + 1$ dimensions [33]. Hence, $\bar{g}$ vanishes when $z_3 = 0$, which is encountered earlier than $z = 0$ starting from small field amplitudes. Note that the problem is encountered even when $g_{\mu\nu}$ is regular everywhere, and X^2 can have either sign, hence, the breakdown is possible for any $\lambda \neq 0$. We emphasize that a point with $z_3 = 0$ signifies a physical effect, not a coordinate one. Even though the determinant might vanish due to divergent coordinate transformations in some cases, the physical importance of $z_3 = 0$ can also be seen in the Ricci scalar of $\bar{g}_{\mu\nu}$ which can only diverge at a physical singularity. $\bar{R} = F(g_{\mu\nu}, X_\mu, \nabla_\mu X_\nu, \nabla_\mu \nabla_\nu X_\rho)/(z z_3)$ indeed diverges, since F, whose exact form is given in Supplemental Material, Sec. A [32], is generically nonvanishing at points with $z_3 = 0$, demonstrating our point. Since X_μ behaves as if it lives in the spacetime with metric $\bar{g}_{\mu\nu}$, its time evolution cannot be continued beyond $z_3 = 0$, the same way any time evolution cannot be continued beyond a spacetime singularity. Last, our analysis in Supplemental Material, Sec. B [32] also identifies $z_3 = 0$ as the point where hyperbolicity is lost in any dimension, even when the field equations cannot be posed in a manifestly hyperbolic form as in Eq. (4), this is based on the methods described in Refs. [34,35].

We should highlight that the above results only employ the covariant field equation (2) and its necessary implication Eq. (3), hence the loss of well posedness is not a coordinate effect. The appearance of a curvature singularity additionally signals that there is no formulation of NPT which can evolve beyond this point, see Supplemental Material, Sec. B [32].

$z_3 = 0$ requires the growth of λX^2, which can have various causes, e.g., energy transfer to the vector field from an outside source [29]. Since we investigate *intrinsic* pathologies, we do not consider such factors. Rather, we will see that tachyonic instabilities for $\mu^2 < 0$, or simply the initial “momentum” of the fields in terms of nonzero time derivatives suffice. Last, note that the growth of the components of X_μ is not sufficient by itself, since λX^2

can stay small or strictly positive, both of which imply $z_3 = 0$ is not achieved.

$\bar{g} = 0$ is the only form of breakdown in NPT to the best of our knowledge, however, there has been another criterion discussed in the recent literature [29,30], which is based on the $d + 1$ *decomposition* [36,37]. In this approach we first represent the spacetime as a collection of spatial hypersurfaces in a process called *foliation*, and decompose all tensors into space and time components

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt),$$
$$X_\mu = n_\mu \phi + A_\mu, \quad \phi = -n_\mu X^\mu, \quad A_i = (\delta^\mu_i + n^\mu n_i) X_\mu. \quad (8)$$

The details of this process, some of which can be found in Supplemental Material, Sec. C [32] following the logic of Ref. [38], is not central to our discussion, aside from the fact that $n^\mu = \alpha^{-1}(1, -\beta^i)$ is a normalized vector field that is orthogonal to the set of spatial hypersurfaces forming our foliation, and defines the slicing of spacetime. $n_\mu \phi$ is orthogonal to the spatial surfaces, and A_μ lies on them. Introducing the "electric field" $E_i = (\delta^\mu_i + n^\mu n_i) n^\nu F_{\mu\nu}$, Eqs. (2) and (3) imply [29]

$$\partial_t \phi = \beta^i D_i \phi - A^i D_i \alpha - \frac{\alpha}{\bar{g}_{nn}} z (K\phi - D_i A^i) + \frac{2\lambda\alpha}{\bar{g}_{nn}} [A^i A^j D_i A_j - \phi(E_i A^i - K_{ij} A^i A^j + 2A^i D_i \phi)],$$
$$0 = D_i E^i + \mu^2 z \phi = \mathcal{C}, \quad (9)$$

where

$$\bar{g}_{nn} = n^\mu n^\nu \bar{g}_{\mu\nu} = -z + 2\lambda\phi^2 = -z_3 + 2\lambda A_i A^i. \quad (10)$$

D_i is the covariant derivative compatible with the induced metric on spatial slices (γ_{ij}), and K_{ij} and K are the extrinsic curvature and its trace, respectively. $\mathcal{C} = 0$, called the "constraint equation," is a result of the $\nu = t$ component of Eq. (2), and does not provide time evolution. However, it has to be satisfied at all times, i.e., on all spatial hypersurfaces.

Recent studies noted that Eq. (9) cannot be solved beyond a point where $\bar{g}_{nn} = 0$, which was interpreted as a breakdown of time evolution [29,30,39]. The significance of $\bar{g}_{nn} = 0$ is that the constraint, $\mathcal{C} = 0$, is a polynomial equation in ϕ and the number of roots changes at $\bar{g}_{nn} = 0$, as $\bar{g}_{nn} = \partial\mathcal{C}/\partial\phi$. This means that ϕ will generically be discontinuous at $\bar{g}_{nn} = 0$, and leads to the more apparent issue that $\partial_t \phi$ diverges.

Before detailing our argument, note that $\bar{g}_{nn} = -z_3 + 2\lambda A_i A^i$ implies that for $\lambda > 0$, $\bar{g}_{nn} = 0$ is generically encountered before $\bar{g} = 0$, and the order is reversed for $\lambda < 0$. Thus, for $\lambda < 0$ we never encounter $\bar{g}_{nn} = 0$ during hyperbolic evolution. Thus, we will consider the $\lambda > 0$ case in the following discussion.

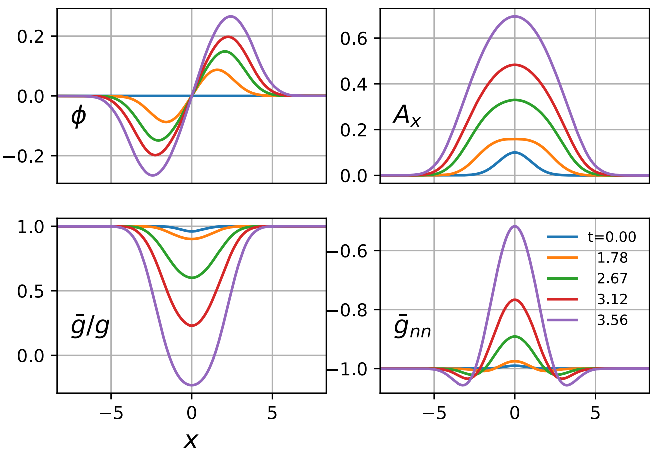

FIG. 1. Snapshots of X_μ and $\bar{g}_{\mu\nu}$ for $\mu^2 = -1$, $\lambda = -1$. The initial growth of the vector is due to a tachyonic instability, which eventually carries the solution to breakdown at $\bar{g} = 0$. The physical meaning of the solution is lost in the region $\bar{g} > 0$, where numerical computation artificially continues due to limited resolution.

We believe the issue at $\bar{g}_{nn} = 0$ to be a "coordinate singularity" which does not imply a physical problem in the time evolution. Namely, $\bar{g}_{nn} = 0$ arises when one uses a foliation which is not suitable for $\bar{g}_{\mu\nu}$, possibly because it is adapted to $g_{\mu\nu}$. $\bar{g}_{\mu\nu}$ controls the dynamics of X_μ, hence the time evolution appears to be problematic for an ill-constructed foliation, similar to coordinate singularities in general relativity [37,40]. That $\bar{g}_{nn} = 0$ implies the inability of the solution to satisfy the constraint does not change this fact, since the form of the constraint equation, hence its root structure, is also foliation dependent.

Our point can be seen directly in the dependence of $\bar{g}_{nn}$ on $\phi = n_\mu X^\mu$, which changes with foliation, unlike X^2. Consider a point where $\bar{g}_{nn} = 0$, $X_\mu = n_\mu \phi + A_\mu$, and $X^2 = A_i A^i - \phi^2$ for a foliation defined by the normal vector n^μ. We are free to change our foliation, i.e., choose a new normal vector $\tilde{n}^\mu$, without changing the physics. This provides a new decomposition $X_\mu = \tilde{n}_\mu \tilde{\phi} + \tilde{A}_\mu$. Then, if $X^2 > 0$, we can choose $\tilde{n}^\mu$ to be orthogonal to X^μ so that $\tilde{\phi} = \tilde{n}_\mu X^\mu = 0 \Rightarrow X^2 = \tilde{A}_i \tilde{A}^i$. Whereas if $X^2 < 0$ we can choose $\tilde{n}^\mu$ to be parallel to X^μ so that $\tilde{\phi} = \mathrm{sign}(\phi)\sqrt{\phi^2 - A_i A^i} \Rightarrow X^2 = -\tilde{\phi}^2$. In $1 + 1$ dimensions this can be done globally with some modifications around $X_\mu X^\mu = 0$, but more generally it can at least be performed at the point where $\bar{g}_{nn} = 0$. In other words, we can always find a new foliation where $\tilde{A}_i \tilde{A}^i \le A_i A^i$ (equivalently $\tilde{\phi}^2 \le \phi^2$), hence $\bar{g}_{\tilde{n}\tilde{n}} \le \bar{g}_{nn}$, the equality only being possible if A_i vanishes. Thus, in the generic case, the time evolution can be continued in the tilde foliation without issue, thanks to $\bar{g}_{\tilde{n}\tilde{n}} < 0$, proving our point that $\bar{g}_{nn} = 0$ is a result of an ill-suited foliation. This is not relevant for earlier studies with diagonal effective metrics [26–28], for which $\bar{g}_{nn} = 0$

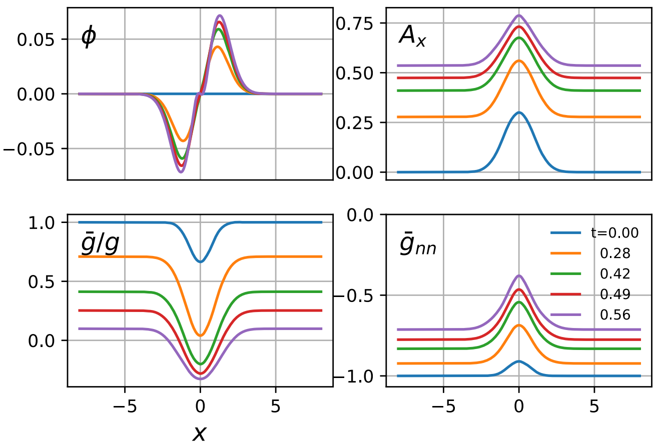

FIG. 2. Snapshots of X_μ and $\bar{g}_{\mu\nu}$ for $\mu^2 = 1$, $\lambda = -1$. The initial value of E_x drives A_x, and in turn X^2, so that hyperbolicity is lost, $\bar{g} > 0$.

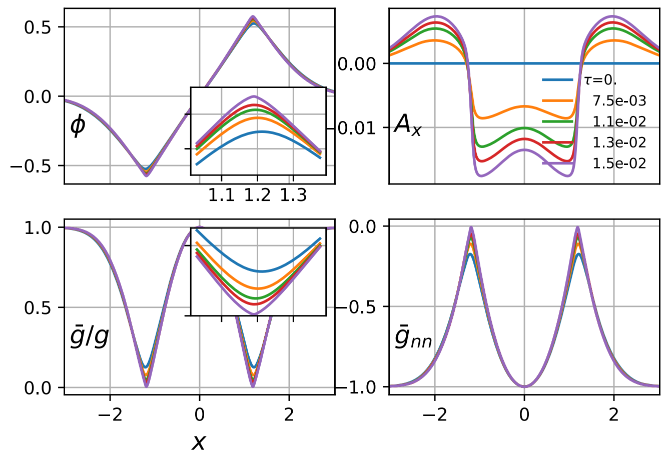

FIG. 3. Snapshots of X_μ and $\bar{g}_{\mu\nu}$ for $\mu^2 = 1$, $\lambda = 1$. We start with initial data close to breakdown and $A_i = 0$. This way, we encounter the coordinate singularity $\bar{g}_{nn} = 0$ shortly before the true singularity $\bar{g} = 0$, and infer that the solution is indeed evolving toward breakdown.

implies $\bar{g} = 0$. The exception, $A_i = 0$, leads to $\bar{g}_{nn} = \bar{g}_{\tilde{n}\tilde{n}} = -z_3 = 0$. However, this also implies $\bar{g} = 0$, hence, the time evolution indeed breaks down in this case, not due to $\bar{g}_{nn} = 0$, but rather due to $\bar{g}_{\mu\nu}$ becoming singular.

Numerical results.—We evolved the vector fields of the Lagrangian (1) on a $1 + 1$ dimensional flat spacetime background, $g_{\mu\nu} = \eta_{\mu\nu}$, using a first order formulation as in Eq. (9). Overall, we confirm that there exist initial data configurations for any value of (μ^2, λ) for which hyperbolicity is lost. Technical details are in Supplemental Material, Sec. C [32].

Sample evolutions for $\lambda = -1$ can be seen in Fig. 1 ($\mu^2 = -1$) and Fig. 2 ($\mu^2 = 1$), where we encounter $\bar{g} = 0$ without any foliation issues, as expected. The main difference between the cases is that $\mu^2 = -1$ breaks down even for arbitrarily low-amplitude initial data due to its tachyonic instability, whereas $\mu^2 = 1$ requires relatively high initial amplitudes and/or nonzero momentum in the form of E_x. Note that the evolution continues beyond $\bar{g} = 0$ as an artifact of the numerics which cannot resolve the problematic fast-growing modes. Hence, these parts of the solutions are not physical (see Supplemental Material, Sec. C [32]).

From a physical perspective, $\mu^2 = -1, \lambda = -1$ is a vector analog of the Higgs potential, where the classical "false vacuum," $X_\mu = 0$, is unstable, but is not dynamically connected to any true vacuum. The effective metric becomes singular well before X_μ reaches the minimum of $V(X^2)$ at $X^2 = 1$, at which $\bar{g} = 0$.

The $\lambda = 1$ cases require special numerical care since $\bar{g}_{nn} = 0$ has to be encountered before $\bar{g} = 0$. Even though physical time evolution is not affected by $\bar{g}_{nn} = 0$, numerical computation fails to continue beyond such a point, hence we cannot investigate the physical breakdown using generic foliations. However, we also saw that, $\bar{g}_{nn} = 0$ and $\bar{g} = 0$ can be coincident if $A^i = 0$ at this point. Therefore, to get as close as possible to $\bar{g} = 0$, we used initial data that satisfies $|A_i| \ll 1$, and $\phi = 1/\sqrt{3\lambda} + \delta\phi$ chosen so that we are already somewhat close to the loss of hyperbolicity. The question is whether the time evolution proceeds toward breakdown starting from this configuration, or away from it.

Analytically, the leading behavior of Eq. (9), $\partial_t \delta\phi = -(\alpha K / 9\lambda)(\delta\phi)^{-1} + \cdots$, already implies that $\delta\phi$ evolves toward 0 if $K > 0$, which is the case for an appropriate choice of foliation. Thus, we expect the time evolution to break down for $\mu^2 = \pm$, $\lambda = 1$. See a sample numerical evolution in Fig. 3 which uses the coordinates of Ref. [41].

Last, our results can be generalized to any dimensions, e.g., by using our specific initial configurations along one spatial direction and translation symmetry along the rest. Whether more generic initial data can still lead to loss of hyperbolicity in higher dimensions remains to be seen.

Discussion.—The key part of our Letter was a careful construction of the effective metric, and identifying its singularity as the appropriate criterion for the loss of hyperbolicity. We also revealed the foliation-dependent nature of the commonly used breakdown criterion $\bar{g}_{nn} = 0$, which can be easily misidentified as a physical breakdown in numerical studies. In essence, the effective metric is generically curved even when the spacetime metric is not. Thus, even in Minkowski spacetime, the usual setting of high energy theories, tools from general relativity are likely required. We explained some of the basic principals for choosing a well-suited foliation for NPT, but future studies will likely require novel approaches.

The problems we revealed can be traced back to the constrained nature of the time evolution and the Lorenz condition, and these do not rely on the specific form of

$V(X^2)$, only that it is not linear in X^2. Derivative self-interactions also generally lead to generalized Lorenz conditions and constrained evolutions, hence, we expect most, if not all, self-interacting vector field theories to suffer from the same issues.

We demonstrate that the intuition gained by studying scalar fields cannot be directly applied to vectors. For example, the ϕ^4 scalar field theory can be evolved indefinitely for all (μ^2, λ), even if the field amplitude grows without bound. Contrast this with our results, showing that for all (μ^2, λ) the evolution breaks down at finite field values. This is despite the fact that NPT is a member (perhaps the simplest) of the generalized Proca theories, which are explicitly constructed to be ghost free [22]. Therefore we suggest that simply counting the degrees of freedom is not sufficient, and our results are essential in investigating the viability of such theories.

All our conclusions about the pathology of NPT considered the theory at face value, i.e., not as an effective approximation to a yet more fundamental theory. Nevertheless, self-interacting vectors, for $\lambda \leq 0$ [42,43], can appear as such effective fields in some contexts, hence the problems may be resolved in a complete theory. Thus, NPT can still be useful as long as such limitations are taken into account. Exploration of these topics is a lengthy endeavor by itself and an important part of ongoing [44] and future research.

This study identified the problematic nature of one of the simplest classical field theories, that of a self-interacting vector. We hope our results lead to further research on mathematical constraints on field theories, and the physical implications of such results.

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[32] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.129.151103 for a discussion of the causal structure and well-posedness of the theory, and numerical details.
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