Fracton Hydrodynamics without Time-Reversal Symmetry

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We present an effective field theory for the nonlinear fluctuating hydrodynamics of a single conserved charge with or without time-reversal symmetry, based on the Martin-Siggia-Rose formalism. Applying this formalism to fluids with only charge and multipole conservation, and with broken time-reversal symmetry, we predict infinitely many new dynamical universality classes, including some with arbitrarily large upper critical dimensions. Using large scale simulations of classical Markov chains, we find numerical evidence for a breakdown of hydrodynamics in quadrupole-conserving models with broken time-reversal symmetry in one spatial dimension. Our framework can be applied to the hydrodynamics around stationary states of open systems, broadening the applicability of previously developed ideas and methods to a wide range of systems in driven and active matter.

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Introduction.—In the past few years, infinitely many universality classes of hydrodynamics have been discovered [1–14], with exotic conservation laws such as the conservation of multipole charges or charges along subdimensional manifolds. Dubbed "fracton fluids," as such universality classes describe the thermalization of generic models of interacting fractons (particles with mobility constraints) [15–25], a careful study of these new hydrodynamic universality classes is likely to give valuable insight into the foundational underpinnings of hydrodynamics as an effective field theory (EFT) [26–28], especially in nonthermal systems with unusual symmetries.

In this Letter, we find new universality classes of fracton hydrodynamics with broken time-reversal symmetry. To understand why this construction is subtle, let us consider the simplest fracton fluid: a 1D system with charge and dipole symmetry [1–4], which can be experimentally realized in tilted optical lattices [29]. Letting ρ denote the density of conserved charge, one finds that dipole conservation $\partial_t \int dx\, x\rho = 0$ mandates

$$\partial_t \rho + \partial_x^2 J_{xx} = 0. \tag{1}$$

With time-reversal symmetry,

$$J_{xx} = D\partial_x^2 \rho + \cdots \tag{2}$$

is necessary, where the dots denote subleading terms in derivative expansion. Thus far, this result is justified using effective field theory methods based on coupling this fluid to background (mixed-rank) gauge fields [1]; a more straightforward argument is that J_{xx} is time-reversal odd: since there is no T-odd parameter in the model that could relate J_{xx} to ρ within ideal hydrodynamics, only derivatives of ρ can appear in J_{xx}. When time-reversal symmetry is broken, is it possible to write $J_{xx} = -D'\rho + \cdots$?

Our purpose in this Letter is to give a systematic and highly generalizable framework capable of answering this question (negatively). We will develop a systematic effective field theory framework for studying hydrodynamics of nonthermal systems, with or without time-reversal symmetry. Studying many different examples of fracton fluids without time-reversal symmetry, we will discover an infinite new family of dynamical universality classes, which generalize the KPZ (Kardar-Parisi-Zhang) fixed point [30–33] and multipolar extensions thereof [9].

Effective field theory.—We first develop a user-friendly EFT for a nonthermal fluid (one in which energy is not conserved, and temperature is not well defined). We focus on systems with a single conserved charge with density ρ, which is a scalar under rotations, inversions, and time reversal; generalizations will appear elsewhere. We assume that the dynamics is local in space, ergodic, and that there exists a steady state probability distribution on the classical state space (or quantum density matrix) invariant under the microscopic dynamics. In contrast, we will *not* assume that the dynamics is invariant under time-reversal or any spatial symmetry.

For pedagogical purposes, consider nonlinear fluctuating hydrodynamics from a traditional perspective via classical stochastic differential equations [34]. (Note that our eventual EFT will also describe the hydrodynamics of microscopically quantum systems.) It is useful (for now) to think of ρ_x as the discretization of a continuum function $\rho(x)$ onto some d-dimensional lattice. We write

$$\frac{d\rho_x}{dt} = F_x(\boldsymbol{\rho}) + \zeta_x(t), \tag{3}$$

where F_x is some nonlinear function of ρs on nearby lattice sites, consistent with all necessary symmetries, and $\zeta_x(t)$ corresponds to stochastic fluctuations. Equation (3) is in the Ito interpretation. Eventually, we'll want a rulebook for how to calculate F_x and the statistics of ζ_x. For now, assume that the noise is white, with zero mean and

$$\langle \zeta_x(t)\zeta_{x'}(t')\rangle = \epsilon Q_{xx'}(\boldsymbol{\rho})\delta(t-t'), \tag{4}$$

with ϵ a perturbatively small "bookkeeping" parameter, and Q_{ij} symmetric and positive semidefinite. It will be useful to replace (3) by the equivalent Fokker-Planck equation for $P(\boldsymbol{\rho}, t)$,

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial \rho_x}\left[-F_x(\boldsymbol{\rho})P + \frac{\epsilon}{2}\frac{\partial}{\partial \rho_{x'}}(Q_{xx'}(\boldsymbol{\rho})P)\right], \tag{5}$$

where summation over repeated indices is understood.

Now we bring in our first key assumption: the existence of a steady state distribution,

$$P_{\mathrm{eq}}(\boldsymbol{\rho}) \propto \exp[-\Phi(\boldsymbol{\rho})/\epsilon]. \tag{6}$$

If $\epsilon \to 0$, this distribution becomes tightly peaked near minima of Φ at small ϵ. This limit is both technically convenient and physically sensible: on very long scales, a fluid should be approximately described by noise-free partial differential equations (e.g., Fick's Law). Combining (5) and (6) we conclude that [35]

$$\begin{aligned} 0 + O(\epsilon^0) &\equiv \frac{1}{\epsilon}\left(-F_x\mu_x + \frac{1}{2}Q_{xx'}\mu_x\mu_{x'}\right) \\ &= -iH(-i\boldsymbol{\mu}/\epsilon, \boldsymbol{\rho}), \end{aligned} \tag{7}$$

where we have defined

$$\mu_x \equiv -\frac{\partial \Phi}{\partial \rho_x}. \tag{8}$$

Already, we can see sharp connections to thermodynamics and statistical mechanics: Φ plays the role of entropy S, the thermodynamic potential in the microcanonical ensemble, while μ is the chemical potential conjugate to ρ_x. This emergent thermodynamics does not require finite temperature, energy conservation, or time-reversal symmetry. Moreover, the noise variance $Q_{xx'}$ is not arbitrary: (7) mandates a fluctuation-dissipation theorem [36,37] relating $Q_{xx'}$ to F_x; the consequences of this will be especially clear in the EFT language.

In (7), we also defined a function $H(-i\boldsymbol{\mu}/\epsilon, \boldsymbol{\rho})$. We will now show that it can be interpreted as a "Hamiltonian." The path integral of the system described by (3) is given by the Martin-Siggia-Rose method [38]:

$$Z = \int D\boldsymbol{\rho} D\boldsymbol{\zeta}\, \delta[\partial_t \boldsymbol{\rho} - \boldsymbol{F}(\boldsymbol{\rho}) + \boldsymbol{\zeta}] e^{-\int dt \frac{1}{2\epsilon}\boldsymbol{\zeta}\boldsymbol{Q}^{-1}\boldsymbol{\zeta}}, \tag{9}$$

which is equivalent to

$$\begin{aligned} Z &= \int D\boldsymbol{\rho} D\boldsymbol{\pi} D\boldsymbol{\zeta}\, e^{i\int dt(\boldsymbol{\pi}\partial_t\boldsymbol{\rho} - \boldsymbol{F}(\boldsymbol{\rho})\cdot\boldsymbol{\pi} + \frac{1}{2\epsilon}\boldsymbol{\zeta}\boldsymbol{Q}^{-1}\boldsymbol{\zeta} + \boldsymbol{\zeta}\boldsymbol{\pi})} \\ &= \int D\boldsymbol{\rho} D\boldsymbol{\pi}\, e^{i\int dt \mathcal{L}}. \end{aligned} \tag{10}$$

In the last equation we get the effective Lagrangian

$$\mathcal{L} = \boldsymbol{\pi}\partial_t\boldsymbol{\rho} - \boldsymbol{F}(\boldsymbol{\rho})\boldsymbol{\pi} + \frac{i\epsilon}{2}\boldsymbol{\pi}\boldsymbol{Q}\boldsymbol{\pi} = \boldsymbol{\pi}\partial_t\boldsymbol{\rho} - H(\boldsymbol{\pi}, \boldsymbol{\rho}). \tag{11}$$

Note that $H(\boldsymbol{\pi} = -i\boldsymbol{\mu}/\epsilon, \boldsymbol{\rho})$ is simply (7) up to $O(\epsilon)$.

From now on, we replace ρ_x with its continuum limit $\rho(x)$. F_x, $Q_{xx'}$, and $P(\boldsymbol{\rho}, t)$ then become functionals of $\rho(x)$. The Hamiltonian in the continuum limit is

$$H = \int dx\left(F(x,\rho)\pi(x) - \frac{i\epsilon}{2}\pi(x)Q(x,\rho)\pi(x)\right), \tag{12}$$

where F and Q can include spatial derivatives acting on ρ and/or π.

There are three important transformations and/or symmetries we wish to impose within EFT:

Charge and/or multipole conservation: For any integrable function $f(x)$, we define a "multipolar" charge as

$$Q_f \coloneqq \int d^d x\, f(x)\rho(x). \tag{13}$$

Q_{f_i} is conserved if the system is invariant under

$$\pi(x) \to \pi(x) + f(x)c(t), \tag{14}$$

where $c_i(t)$ is an arbitrary function of time. Under this transformation, the action transforms as

$$S \to S + \int dt\, d^d x\, f(x)c(t)\partial_t\rho(x). \tag{15}$$

The invariance of the action gives

$$\frac{\delta S}{\delta c(t)} = \frac{d}{dt}\int d^d x f(x)\rho(x) = \frac{d}{dt}Q_f = 0. \tag{16}$$

Parity: Under parity, $x \to -x$ and $\rho(x) \to \rho(-x)$. We further demand the canonical momentum $\pi(x) \to \pi(-x)$.

Time reversal: Under time reversal, $t \to -t$ and $\rho(x,t) \to \rho(x,-t)$. Supposing for the moment that we have time-reversal symmetry and satisfy (17), in order for the Lagrangian to be invariant under time reversal, the term $\pi\partial_t\rho$'s contribution to the action should remain the same (up to a total derivative). Under time reversal, $\partial_t\rho \to -\partial_t\rho$. If $\pi \to -\pi$ under time reversal, from the invariance of the Hamiltonian, $H(\pi,\rho) = H(-\pi,\rho)$, we would find that the leading order of π in the time-derivative free terms of the Hamiltonian is $H \sim \pi^2$, which means the dynamics of the system is fully stochastic.

If we want a system whose dynamics is not fully stochastic, we have to change the behavior of π under time reversal, namely, $\pi(x,t) \to -\pi(x,-t) + ig(x)$, so now $H(\pi,\rho) = H(-\pi + ig, \rho)$. From the above analysis, we know that only when $g\partial_t\rho$ is a total derivative can the equations of motion be invariant. According to (7), the Hamiltonian satisfies

$$H(0,\rho) = H(-i\mu/\epsilon,\rho) + O(\epsilon^0) = 0. \tag{17}$$

A natural choice is therefore

$$\pi(x,t) \to -\pi(x,-t) - i\mu(x)/\epsilon. \tag{18}$$

Note that (18) is a $\mathbb{Z}_2$ transformation reminiscent of the Kubo-Martin-Schwinger (KMS) symmetry used to implement time-reversal symmetry in dissipative thermal systems at temperature T. It is consistent with the condition that two applications of the time reversal should return dynamical fields to their original values. Equation (18) is the unique kind of $\mathbb{Z}_2$ transformation on functions (also called an involution) not requiring an infinite order series in π. Since μ is a total derivative, assuming that H is invariant under (18), the change in the action is a total derivative:

$$S \to S + i\Delta\Phi/\epsilon, \tag{19}$$

where $\Delta\Phi$ denotes the difference in the thermodynamic potential Φ in the initial and final state.

Remarkably, our EFT-based guess for how to implement time reversal can also be justified *microscopically*. Assuming statistical time-translation invariance for simplicity, time-reversal symmetry is microscopically implemented via detailed balance: if at time t the microstate of the system is $\boldsymbol{\rho}'$, and at time $t = 0$ the microstate is $\boldsymbol{\rho}_0$, then

$$P(\boldsymbol{\rho}', t|\boldsymbol{\rho}_0, 0)P_{\rm eq}(\boldsymbol{\rho}_0) = P(\boldsymbol{\rho}_0, t|\boldsymbol{\rho}', 0)P_{\rm eq}(\boldsymbol{\rho}') \tag{20}$$

Here $P(\cdots)$ denotes the transition probability, which can be calculated via path integral [38]

$$P(\boldsymbol{\rho}', t|\boldsymbol{\rho}_0, 0) = \int_{\boldsymbol{\rho}(0)=\boldsymbol{\rho}_0, \boldsymbol{\rho}(t)=\boldsymbol{\rho}'} D\boldsymbol{\rho} D\boldsymbol{\pi}\, e^{i\int dt\mathcal{L}}. \tag{21}$$

Observe that the transformation (18) is accompanied with $t \to -t$, which flips the two boundary conditions in the path integral. Combining (19) with (6) we obtain (20).

So far, our discussion has focused on theories with Gaussian noise, which are described by a quadratic Hamiltonian $H(\pi,\rho)$. However, it is straightforward to consider higher order Hamiltonians from the EFT perspective. What is highly nontrivial is to convert the action $S[\pi,\rho]$ back to the Fokker-Planck equation, once we consider nonlinearities in π; this task will be done in a future Letter. In the Supplemental Material (SM) [39], we give the generalization of (7) to nonperturbatively large noise without time reversal.

Equation (18) is also the correct definition of time reversal in situations where detailed balance is broken. In this case, it may not be a symmetry (S need not be invariant). Still, within linear response, we can cleanly separate out the time-reversal even and odd contributions to S: $F_x = F_x^{(e)} + F_x^{(o)}$, where $F_x^{(o)}$ satisfies $\int_x F_x^{(o)}\mu_x = 0$. It is easy to verify that (18) still holds if, instead of (19), we have

$$S \to S^* + i\Delta\Phi/\epsilon, \tag{22}$$

where S^* is the original action with $F_x^{(o)} \to -F_x^{(o)}$, and where $F_x^{(e)}$ obeys (7). $F_x^{(o)}$ is unrelated to the noise $Q_{xx'}$, and correspond to time-reversal breaking terms [hence the sign flip in (22)] that are not dissipative. In hydrodynamics such terms can arise from quantum anomalies [40,41], Hall transport [42], and more general situations when boost invariance is broken [43]. We conclude that any hydrodynamic theory for ρ, with a stationary homogeneous distribution, transforms in a "covariant" way under the symmetry (22) at leading orders in the derivative expansion (see SM). This allows us to provide quite strong constraints on fracton hydrodynamics without time-reversal symmetry.

Fracton fluids.—We now begin to classify the new universality classes of fracton hydrodynamics with or without P or T symmetry. Here we will systematically discuss systems with only three kinds of multipole charge conservation: monopole, dipole, and quadrupole conservation, but our framework can be easily generalized to other systems. At least for multipole conserving theories, it appears that all of the peculiar possible phenomena can be found already within one of these three theories.

We start by writing down all possible leading-order terms in Hamiltonian; namely, we will consider at most quadratic terms in π, and keep as few derivatives and nonlinearities in ρ or μ as possible.

Charge conserving: the action is invariant under the transformation $\pi \to \pi + c(t)$, so the Hamiltonian should be a function of $\partial_x\pi$ or higher order derivative terms:

$$H = A(\rho)\partial_x\pi + \sigma(\rho)\partial_x\mu\partial_x\pi - i\epsilon Q(\rho)(\partial_x\pi)^2 + \cdots. \tag{23}$$

Dipole conserving: H contains $\partial_x^2\pi$ or higher order terms, since it is invariant under $\pi \to \pi + xc(t)$:

$$H = A(\rho)\partial_x^2\pi + \partial_x B(\rho)\partial_x^2\pi + \sigma(\rho)\partial_x^2\mu\partial_x^2\pi - i\epsilon Q(\rho)(\partial_x^2\pi)^2 + \cdots. \tag{24}$$

Quadrupole conserving: Now H should consist only of $\partial_x^3\pi$ or higher order terms:

$$H = A(\rho)\partial_x^3\pi + \partial_x B(\rho)\partial_x^3\pi + \partial_x^2 C(\rho)\partial_x^3\pi + \sigma(\rho)\partial_x^3\mu\partial_x^3\pi - i\epsilon Q(\rho)(\partial_x^3\pi)^2 + \cdots. \tag{25}$$

TABLE I. Leading order terms in H for a fracton fluid. $f(\mu)$ represents an arbitrary function of μ. The slash marks denote terms which are subleading in the derivative expansion and are not considered.

Conservation	Symmetry	$A(\rho)$	$B(\rho)$	$C(\rho)$	$\sigma(\rho)$
Monopole	T or P	0			Q
	PT	$f(\mu)$	/	/	Q
	None	$f(\mu)$			Q
Dipole	T or P	0	0		Q
	PT	0	μ	/	Q
	None	0	μ		Q
Quadrupole	T or P	0	0	0	Q
	PT	μ	0	μ	Q
	None	μ	0	μ	Q

In the above equations, $A(\rho)$, $B(\rho)$, $C(\rho)$, $\sigma(\rho)$, and $Q(\rho)$ are (as of yet) undetermined functions of ρ, which do not include any derivatives. Combining all the other constraints we imposed to the system, (7), (17), and (18), we list all possible forms of the undetermined functions in Table I. From the table, we see that with or without P or T, the leading order dissipative terms $\sigma(\rho)$ are always the same and are fixed by the conditions (7) and (17). This is the fluctuation-dissipation theorem [36]. Second, when the systems have PT symmetry or neither, there always exists a nonzero leading order term, which is dissipationless. The monopole charge-conserving case allows for a generic function $f(\mu)$: indeed, for F antiderivative of f, the corresponding term in the Hamiltonian transforms as $f\partial_x\pi \to f\partial_x\pi + if\partial_x\mu = f\partial_x\pi + \partial_x F(\mu)$, where the second term is a total derivative, and similar steps lead to the other nonzero entries of Table I. These terms can lead to instabilities. In the charge-conserving case, the endpoint of this instability is the KPZ fixed point [30–33]; in higher dimensions, we have found a new generalization of the KPZ fixed point.

To estimate the critical dimensions for these fixed points, we assume that the charge susceptibility $\int d^d x\langle\rho^2(x)\rangle$ is finite, which implies the scaling $\rho \sim L^{-(d/2)}$, where L is the system size. In the charge conserving case, the leading nonlinearity in the current is $J_x = A(\rho) \sim \rho^2 \sim L^{-d}$, while the leading dissipative term is Fick's law $J_x = -\sigma\partial_x\mu \sim \partial_x\rho \sim L^{-1-(d/2)}$. We see that, as $L \to \infty$, the nonlinearity dominates over the dissipative term below $d = 2$. Taking $B(\rho)$ to be the leading nonlinearity in the dipole conserving case (see Table I), a similar reasoning gives $d = 2$ as the critical dimension; while in the quadrupole conserving case, with the leading nonlinearity being $A(\rho)$, the upper critical dimension is $d = 6$. For n-pole conserving systems in general, we find upper critical dimension $d = 2(1 + n)$ if n is even, and $d = 2n$ if n is odd. Hence, for sufficiently large n, the upper critical dimension for hydrodynamics can be arbitrarily large.

We can also answer the question we posed at the beginning of the Letter, under (2). We cannot write $J_{xx} = -D'\rho + \cdots$, because the dissipationless part of the dispersion relation can change if we break T or P symmetry, but the leading order dissipative terms in the systems (within linear response) do not change. This follows from the requirement of stationarity, (7).

Numerical simulations.—We now present large-scale simulations of classical Markov chains in one-dimensional lattice models with quadrupole conservation, and with or without time-reversal symmetry. The time-reversal symmetric chain is constructed generalizing [3,6]: we allow charges of value $q_x = 0, \pm 1, \ldots, \pm 4$ to exist on each of the L sites of a 1D lattice, with periodic boundary conditions; at each time step, we act with "gates" on each q-tuple of adjacent sites, and replace the configuration of charges present with another one with identical charge, dipole, and quadrupole moment. We have taken $q = 6$ in our simulations to ensure the dynamics does not get frozen [44,45] and that the late-time physics is captured by hydrodynamics.

We analyze the correlator

$$C(x, t) = \langle q_{x+y}(t + s)q_y(s)\rangle_y, \tag{26}$$

with the average taken over position y, and random realizations of the gates and initial conditions; the correlator is insensitive to the value of $s \lesssim 10^5$. With time-reversal symmetry, by dimensional analysis we know that $C(0, t) \sim t^{-1/z}$ with $z = 6$, which can be obtained by comparing the scaling of charge $q \sim L^{-\frac{1}{2}}$ with the dynamical scaling exponent $\partial_t \sim \partial_x^z$. As in [3,6] we can confirm this scaling readily in numerics: see Fig. 1.

Now let us sketch how we break time-reversal symmetry: details are found in the SM. If we only had charge conservation, then we could break time-reversal symmetry by simply hopping a unit of charge to the right neighbor with some finite probability at the end of each round of random gates. Importantly, this rule does not modify the fact that the uniform distribution (taken over all many-body configurations in each fixed charge sector) is the stationary distribution of the classical Markov chain: thus, we can readily numerically evaluate $C(x, t)$ by sampling uniformly random initial conditions. To generalize this idea to n-pole conserving models, first observe that the charge conserving chain can be understood as operating by always trying to increase the local dipole moment. We modify this picture by finding gates which try to increase the $(n + 1)$-pole moment in each charge sector, yet do so disturbing the uniform distribution as little as possible. While we could not find a Markov chain which provably has a uniform many-body stationary distribution once $n > 0$, the chains which we did find exhibit behaviors which are consistent with our qualitative expectations: namely, breaking P and T leads to a dissipationless "drift" term. The effect of this term on correlators can be estimated by balancing the time

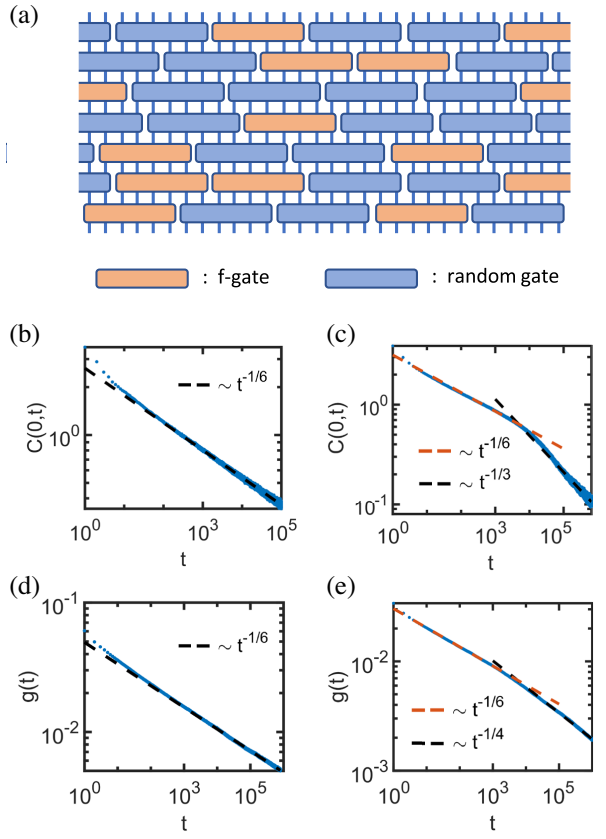

FIG. 1. (a) Sketch of the Markov chains we simulate. At each time step, we act with f gates (that break time-reversal symmetry) with probability $p = 0.1$, and random gates with $1 - p = 0.9$, on blocks of size 6. (b) $C(0, t)$ with time-reversal symmetry. (c) $C(0, t)$ without time-reversal symmetry. (d) $g(t)$ with time-reversal symmetry. (e) $g(t)$ without time-reversal symmetry.

derivative with the linear part of the drift term: $\partial_t \rho \sim \partial_x^{n+1} \rho$ for n even and $\partial_t \rho \sim \partial_x^{n+2} \rho$ for n odd. These lead to a power-law decay:

$$C(0, t) \sim \begin{cases} t^{-1/(n+1)} & n \text{ even} \\ t^{-1/(n+2)} & n \text{ odd} \end{cases}. \tag{27}$$

To estimate the dynamical critical exponent, we must discard this added drift, so we calculate

$$g(t) \equiv \int dx\, C(x, t)^2 \sim t^{-1/z}, \tag{28}$$

as our estimate for the dynamical exponent z. Intuitively this correlator will capture the "width" of an initial charge distribution at time t. Figure 1 shows that after an initial transient period of $z = 6$ scaling, at sufficiently late times the chain exhibits anomalous scaling with $z \approx 4$. A crude estimate of z can be attempted by balancing the time derivative in the conservation equation with the leading nonlinearity induced by the drift $\partial_t \rho + \partial_x^3 \rho^2$ which, accounting for $\rho \sim L^{-1/2}$, leads to $z = 3.5$, which is reasonably close to the measured value, and very far from the value $z = 6$ predicted by linear response. Therefore, our simulations are consistent with the existence of a new dynamical universality class, whose upper critical dimension will be $d = 6$.

We emphasize that the simulations of classical models are sufficient to study hydrodynamics, even as "fracton fluids" were originally inspired by quantum phases of matter. First, any possible quantum corrections to hydrodynamics become important only at frequency scales $\hbar\omega \sim T$, which are generally beyond the hydrodynamic regime of validity anyway. For the Markov chains we simulated, one would take $T \to \infty$ and one sees hydrodynamics break down at $\omega \sim 1$. Second, previous studies [3,6] on quantum automaton circuits find the same hydrodynamics as the T-invariant theories of this Letter.

Outlook.—In this Letter we have described the systematic construction of nonlinear fluctuating hydrodynamics without time-reversal symmetry. Our construction is valid with or without a well-defined temperature, generalizing recent field theories of hydrodynamics [26–28] to a broad range of theories which cannot be coupled to a spacetime metric. A nontrivial example of this is to multipole-conserving theories, where we have shown that hydrodynamics can break down in PT-symmetric models; the late time physics is described by exotic dynamical universality classes. We hope to report on additional applications of our formalism in the near future.

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