Splitting Probabilities of Symmetric Jump Processes

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We derive a universal, exact asymptotic form of the splitting probability for symmetric continuous jump processes, which quantifies the probability $\pi_{0,\underline{x}}(x_0)$ that the process crosses x before 0 starting from a given position $x_0 \in [0, x]$ in the regime $x_0 \ll x$. This analysis provides in particular a fully explicit determination of the transmission probability ($x_0 = 0$), in striking contrast with the trivial prediction $\pi_{0,\underline{x}}(0) = 0$ obtained by taking the continuous limit of the process, which reveals the importance of the microscopic properties of the dynamics. These results are illustrated with paradigmatic models of jump processes with applications to light scattering in heterogeneous media in realistic 3D slab geometries. In this context, our explicit predictions of the transmission probability, which can be directly measured experimentally, provide a quantitative characterization of the effective random process describing light scattering in the medium.

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The splitting probability quantifies the likelihood of a specific outcome out of several alternative possibilities for a random process [1–4]. While these quantities can be defined for general d-dimensional stochastic processes and any number of possible outcomes [5,6], most examples of applications concern one-dimensional processes with two outcomes; one then defines $\pi_{0,\underline{x}}(x_0)$ as the probability that the process crosses x [underlined subscript in $\pi_{0,\underline{x}}(x_0)$] before 0 starting from x_0. A celebrated example is given by the Gambler's ruin problem [1], schematically quantified by the splitting probability that a one-dimensional random walker (figuring the gambler's fortune) reaches 0 (complete ruin) before a fixed given threshold; other examples are given by the fixation probability of a mutant in the context of population dynamics [7], or the melting probability of a heteropolymer [8], which can be reexpressed in terms of splitting probabilities. A key example, to which we will refer through this Letter, is given by the transmission probability of particles (e.g., photons or neutrons) through a slab of a scattering medium, which has important applications in various fields [9–13]; in this case the transmission probability is nothing but the splitting probability for the particle to reach the exit side rather than being backscattered (see Fig. 1).

There is to date no explicit determination of the splitting probability for general jump processes [2,14,15]. Jump processes are defined as follows for $d = 1$: at each discrete time step n, the walker performs a jump of extension $\ell \in \mathbb{R}$ drawn according to a distribution $f(\ell)$ whose Fourier transform will be denoted $\tilde{f}(k) = \int_{-\infty}^{\infty} e^{ik\ell} f(\ell) d\ell$. For jump processes, the splitting probability [16] is known to satisfy the following integral backward equation [2]:

$$\pi_{0,\underline{x}}(x_0) = \int_{x-x_0}^{\infty} dx' f(x') + \int_{-x_0}^{x-x_0} dx' \pi_{0,\underline{x}}(x_0 + x') f(x'), \tag{1}$$

which results from a partition over the first jump. Note that this equation for the splitting probability, because it does not involve time, also holds for continuous time extensions of jump processes, for which each jump takes an arbitrary time $t(\ell)$; in particular the splitting probability of jump processes also applies to processes with constant speed v [$t(\ell) = \ell/v$]. Even if Eq. (1) is linear, there is to date no available solution with the exception of the exponential distribution $f(\ell) = e^{-|\ell|/\gamma}/(2\gamma)$; the main

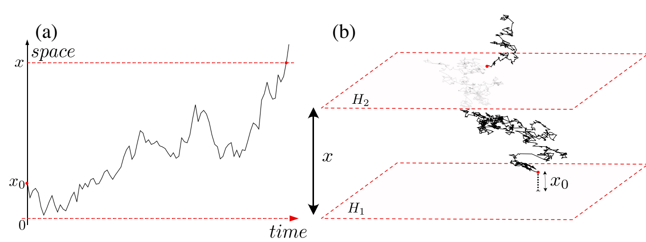

FIG. 1. (a) The one-dimensional jump process evolves in the bounded interval $[0, x]$. One is interested in the probability $\pi_{0,\underline{x}}(x_0)$ of crossing x before 0 starting from x_0, as shown in the diagram. (b) The jump process is now evolving in a three-dimensional space and is bound to stay between two hyperplanes H_1 and H_2 distant of x. We now want to evaluate the probability of crossing H_2 before H_1, starting from x_0.

difficulty lies in the finite integration range, which prevents the use of classical integral transforms [17].

An important simplification of the problem is achieved by taking a continuous limit (see definition below). For symmetric jump processes, considered in what follows, the small k expansion of $\tilde{f}(k)$ reads

$$\tilde{f}(k) \underset{k\to 0}{=} 1-(a_\mu|k|)^\mu + o(k^\mu), \tag{2}$$

where a_μ defines the microscopic characteristic length scale of the process. Two limit behaviors emerge [18,19]. For $\mu = 2$ the variance of the jump distribution is finite and the process is known to converge at large times to Brownian motion; for $0<\mu<2$, the process converges instead to an α-stable Levy process of parameter μ. Hence, there are three independent length scales in the problem: x_0, x, and a_μ, which can lead to two distinct asymptotic regimes. Taking $a_\mu \ll x_0 < x$ defines the continuous limit of the problem (1) (we recall that time is irrelevant to determine the splitting probability), whose solution can be obtained and reads [20–22]:

$$\pi_{0,\underline{x}}(x_0) = \frac{\Gamma(\mu)}{\Gamma^2(\frac{\mu}{2})}\int_0^{x_0/x}[u(1-u)]^{\mu/2-1}du. \tag{3}$$

The regime $x_0 \ll x$ is of particular interest and has received marked attention [22]. One obtains from (3) that this regime is given by [23]:

$$\pi_{0,\underline{x}}(x_0) \underset{a_\mu\ll x_0 \ll x}{\sim} \frac{2\Gamma(\mu)}{\mu\Gamma^2(\frac{\mu}{2})}\left(\frac{x_0}{x}\right)^{\mu/2}. \tag{4}$$

As explained above, a key application of splitting probabilities is the determination of the transmission probability of particles through a slab, that can be defined as $\pi_{0,\underline{x}}(x_0=0)$. The blunt use of the continuous limit (4) yields $\pi_{0,\underline{x}}(x_0=0)=0$, in clear contradiction with the expected result for a jump process with finite microscopic length scale a_μ, for which $\pi_{0,\underline{x}}(x_0{=}0) > \int_x^\infty dx' f(x') > 0$. The determination of the transmission probability thus requires one to consider the second, distinct regime $x_0 \ll a_\mu$ and to go beyond the continuous limit (4); this is the main purpose of this Letter.

Jump processes with finite microscopic length scale a_μ have proved to be relevant in various contexts [24]. They provide emblematic models of transport of photons or neutrons in scattering media [9]. More recently, they have gained renewed interest in the context of self-propelled particles, be they artificial or living, such as active colloids, cells, or larger scale animals [14,25–28]. In what follows, we derive a universal form for the splitting probability for continuous jump processes of finite length scale a_μ in the regime $x_0 \ll a_\mu \ll x$, which provides in particular an explicit determination of the transmission probability ($x_0=0$), and reveals the importance of the microscopic properties of the process. These results are illustrated with paradigmatic models of jump processes with applications to light scattering in heterogeneous media.

General results.—We first derive an asymptotic expression of the splitting probability $\pi_{0,\underline{x}}(x_0)$ for general 1D continuous [29] symmetric jump processes of characteristic microscopic length scale a_μ as defined above in the limit $x\to\infty$. Denoting $F_{\underline{0},x}(n|x_0)$ the probability that the process starting from $x_0 \in [0,x]$ crosses 0 before $x>0$ for the first time after exactly n steps, and making a partition over the crossing time yields

$$1-\pi_{0,\underline{x}}(x_0) \equiv \pi_{\underline{0},x}(x_0) = \sum_{n=1}^{\infty} F_{\underline{0},x}(n|x_0). \tag{5}$$

This exact equation expresses the splitting probability in terms of two targets first-passage time distributions $F_{\underline{0},x}(n|x_0)$, for which no explicit solutions are available for general jump processes. Adapting the approach introduced for scale invariant processes in 1D [22] and then extended to d-dimensional compact cases [30], we next show that in the asymptotic limit $x\to\infty$, the splitting probability of jump processes can in fact be re-expressed in terms of one target first-passage time distributions. We first note that in (5) the right hand side involves trajectories that cross 0 before x; most of these events thus occur within the typical number of steps $n_{\rm typ}$ needed to cross x. In the regime $x \gg a_\mu, x_0$, we argue that $n_{\rm typ}$ is simply the timescale to cover a distance x [18] and thus satisfies $n_{\rm typ}\sim \alpha x^\mu$ where α is a process dependent constant (independent of x_0). We next remark that for timescales $n<n_{\rm typ}$, the target at x is irrelevant so that $F_{\underline{0},x}(n|x_0) \simeq F_{\underline{0},\infty}(n|x_0)$, which leads to

$$\pi_{\underline{0},x}(x_0) \sim \sum_{n=1}^{n_{\rm typ}} F_{\underline{0},\infty}(n|x_0) \equiv 1-q(x_0,n_{\rm typ}), \tag{6}$$

where $q(x_0,n)=\sum_{k=n+1}^{\infty}F_{\underline{0},\infty}(k|x_0)$ is the survival probability, i.e., the probability that the process never crosses 0 during its n first steps, and $F_{\underline{0},\infty}(k|x_0)$ is the probability of crossing 0 on the kth step exactly. We next make use of the asymptotic behavior of $q(x_0,n)$ obtained in [31], which yields for $1 \ll (x_0/a_\mu)^\mu \ll n$:

$$q(x_0,n) \sim \frac{1}{\sqrt{n}}\frac{a_\mu^{-\mu/2}}{\sqrt{\pi}\Gamma(1+\frac{\mu}{2})}x_0^{\mu/2}. \tag{7}$$

Combining (4) and (7) finally yields the coefficient $\alpha \sim n_{\rm typ}/x^\mu$ defined above, and thus the following determination of $n_{\rm typ}$, valid for any $x_0 \ll x$:

$$n_{\rm typ} \sim \left[2^{\mu-1}\Gamma\left(\frac{1+\mu}{2}\right)\right]^{-2}\left(\frac{x}{a_\mu}\right)^\mu. \tag{8}$$

In order to determine the dependence on x_0 of the splitting probability, we use next the large n behavior of the survival probability given by [31]:

$$q(x_0, n) \underset{n\to\infty}{\sim} \frac{1}{\sqrt{n}}\left[\frac{1}{\sqrt{\pi}} + V(x_0)\right], \tag{9}$$

where $V(x_0)$ is defined by its Laplace transform:

$$\begin{aligned}\mathcal{L}V(\lambda) &= \int_0^\infty V(x_0)e^{-\lambda x_0}dx_0 \\ &= \frac{1}{\lambda\sqrt{\pi}}\left(\exp\left[-\frac{\lambda}{\pi}\int_0^\infty \frac{dk}{\lambda^2+k^2}\ln[1-\tilde{f}(k)]\right]-1\right),\end{aligned} \tag{10}$$

and $\tilde{f}(k)$ is the Fourier transformed jump distribution defined above.

Using Eq. (6) and the above given asymptotic behavior of $n_{\rm typ}$ (8), we finally obtain the following general explicit asymptotic determination of the splitting probability of jump processes:

$$\lim_{x\to\infty}\left[\frac{\pi_{0,\underline{x}}(x_0)}{A_\mu(x)}\right] = \frac{1}{\sqrt{\pi}} + V(x_0), \tag{11}$$

where

$$A_\mu(x) = \left(\frac{a_\mu}{x}\right)^{\mu/2} 2^{\mu-1}\Gamma\left(\frac{1+\mu}{2}\right). \tag{12}$$

This holds for *any* fixed x_0, including the regime $x_0 \lesssim a_\mu$ that we intended to determine. This result thus elucidates the dependence of the splitting probability on x (in the regime $x \gg x_0, a_\mu$), and, up to Laplace inversion, on x_0. In particular, the asymptotic behavior for $x_0 \ll a_\mu$ can be derived explicitly and yields

$$V(x_0) = \begin{cases} -\left[\pi^{-\frac{3}{2}}\int_0^\infty dk\ \log[1-\tilde{f}(k)]\right]x_0 + o(x_0) \\ \text{if } \tilde{f}(k) \underset{k\to\infty}{=} o(k^{-1}) \\ \frac{\beta}{2\sqrt{\pi}\Gamma(1+\nu)\cos(\pi\nu/2)}x_0^\nu + o(x_0^\nu) \\ \text{if } \tilde{f}(k) \underset{k\to\infty}{\sim} \beta k^{-\nu} \text{ with } \nu < 1 \\ -\frac{\beta}{\pi^{3/2}}x_0\ln(x_0) + o[x_0\ln(x_0)] \\ \text{if } \tilde{f}(k) \underset{k\to\infty}{\sim} \beta k^{-1}. \end{cases} \tag{13}$$

Of note, the linear dependence of the auxiliary function $V(x_0)$ on x_0 obtained for $\tilde{f}(k) \underset{k\to\infty}{=} o(k^{-1})$ in (13) was given in [31]. Interestingly, we find that the scaling of the splitting probability with $x_0 \ll a_\mu$ is not universal and can be sublinear depending solely on the small scale behavior of the jump distribution $f(\ell)$; in particular it is independent of the large scale behavior of $f(\ell)$, and thus of μ.

Remarkably, although $V(x_0)$ and thus $\pi_{0,\underline{x}}(x_0)$ [see Eq. (11)] generically depend on the jump process through the full jump distribution $f(\ell)$, the asymptotic transmission probability $\pi_{0,\underline{x}}(0)$ in fact depends on the jump distribution only through μ and a_μ and takes the simple, explicit form

$$\pi_{0,\underline{x}}(0) \underset{x\to\infty}{\sim} \frac{2^{\mu-1}}{\sqrt{\pi}}\Gamma\left(\frac{1+\mu}{2}\right)\left(\frac{a_\mu}{x}\right)^{\mu/2}. \tag{14}$$

Even though the above derivation involves the asymptotics (6) that we motivate physically but do not prove rigorously, we claim that our main results (11) and (14) are exact; below we confirm these results either analytically or numerically on representative examples of jump processes.

Jump processes with finite second moment.—We start by considering continuous jump processes with a finite second moment, corresponding to the case $\mu = 2$ in (2), which we illustrate by the class of gamma jump processes of order $n > -1$, whose jump distributions read

$$f(\ell) = \frac{1}{2\gamma^{n+1}\Gamma(n+1)}|\ell|^n e^{-|\ell|/\gamma}, \tag{15}$$

so that $a_2 = \gamma\sqrt{(n+1)(n+2)/2}$. For $n = 0$, this corresponds to the classical exponential jump distribution $f(\ell) = e^{-|\ell|/\gamma}/(2\gamma)$, for which, as mentioned above, the splitting probability is known exactly for all values of parameters [2], and satisfies in the regime $x_0, a_2 \ll x$:

$$\pi_{0,\underline{x}}(x_0) \underset{x\to\infty}{\sim} \frac{\gamma}{x}\left[1 + \frac{x_0}{\gamma}\right]. \tag{16}$$

Calculating $V(x_0)$ from (10), one verifies explicitly the agreement of this exact result with (11). Note that in this example $\tilde{f}(k) \underset{k\to\infty}{=} o(k^{-1})$, so that one verifies in the $x_0 \ll a_2$ regime the linear dependence on x_0 predicted by (13) [with the correct prefactor, see Supplemental Material (SM) [32]].

For $n = 1$, one obtains the so-called gamma jump process defined by the jump distribution $f(\ell) = (1/2\gamma^2)|\ell|e^{-|\ell|/\gamma}$. To the best of our knowledge the splitting probability for this jump process is not known; it can be obtained explicitly for all values of parameters as we proceed to show. Denoting D the differential operator and applying the operator $(D^2 - \gamma^{-2})^2$ to the variable x_0 in Eq. (1) yields the following ordinary differential equation (see SM [32]):

$$D^4\pi_{0,\underline{x}}(x_0) - \frac{3}{\gamma^2}D^2\pi_{0,\underline{x}}(x_0) = 0. \tag{17}$$

The splitting probability is then obtained as

$$\pi_{0,\underline{x}}(x_0) = Ae^{-(\sqrt{3}/\gamma x_0)} + Be^{+(\sqrt{3}/\gamma x_0)} + Cx_0 + E, \tag{18}$$

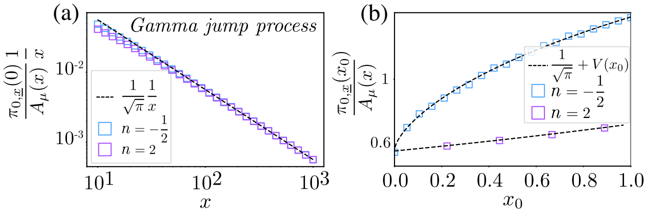

FIG. 2. (a) Transmission probability for examples of gamma jump processes. After rescaling according to (14), the transmission probabilities collapse. (b) Small x_0 behavior of the splitting probability with x fixed to 10^3, as predicted by (11) and (13): for $n=-1/2$, one has $\nu=1/2$ and a sublinear dependence on x_0, while for $n=2$, one has a linear dependence on x_0. Theoretical predictions (dashed lines) are obtained by numerical inverse Laplace transform of (10), while simulations (squares) are averaged over 10^6 trials.

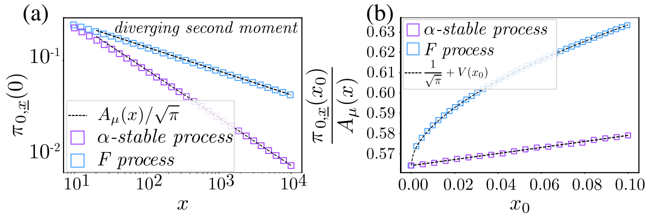

FIG. 3. (a) Transmission probability for a jump process with distribution $f(\ell)=[2\pi\sqrt{|\ell|}(1+\ell)]^{-1}$ (denoted F process), yielding $\mu=1/2$ and $\nu=1/2$ and a Levy flight with $\mu=1$ and $a_\mu=2$. The transmission probabilities (including prefactors) are accurately predicted. (b) Small x_0 behavior of the splitting probability. For the F process, x is fixed to 10^4 and the behavior is sublinear. For the Levy flight, x is fixed to 2×10^5 and one finds a linear behavior. Theoretical predictions (dashed lines) are obtained by numerical inverse Laplace transform of (10), while simulations (squares) are averaged over 10^6 trials.

where A, B, C, and E are determined by using (1). This provides finally an explicit, exact determination of the splitting probability (see SM [32] for explicit expressions) for all values of the parameters for the gamma jump process. Calculating $V(x_0)$ from (10), one verifies explicitly the agreement of this exact result for all $x_0 \ll x$ with (11) (see SM [32]). In particular, in the $x_0 \ll a_2 \ll x$ regime, the splitting probability satisfies

$$\pi_{0,\underline{x}}(x_0) \underset{x\to\infty}{\sim} \frac{1}{x}[\sqrt{3}\gamma + (2\sqrt{3}-3)x_0 + o(x_0)]. \quad (19)$$

This linear scaling with x_0 is in agreement with Eq. (13) (with the correct prefactor), as expected since $\tilde{f}(k) \underset{k\to\infty}{=} o(k^{-1})$.

Finally, these two examples for $n=0$, 1 provide analytical validations supporting the exactness of our results (11) and (14). Additionally, we show in SM [32] that the asymptotic splitting probability for higher or lower order gamma jump processes can be derived explicitly, and is confirmed by numerical simulations for $n=2$ and $n=-1/2$ in Fig. 2.

Levy flights.—For jump processes with infinite second moment, i.e., $\mu<2$ in (2)—called Levy flights [18,19,35,36], no exact results for the splitting probability are available for generic a_μ, x_0. We thus resort to numerical simulations to validate predictions (11) to (14) (see Fig. 3). First, the prediction (14) of the transmission probability is confirmed and in particular fully captures the dependence on x (including the prefactor) that is controlled by the large scale behavior of $f(\ell)$, parametrized by μ and a_μ only. In turn, (11) captures the dependence on x_0, which can lead to different scalings depending on the $\ell\to0$ behavior of the jump distribution $f(\ell)$. The linear dependence on x_0 is illustrated by the α-stable jump distribution of parameter μ defined by $\tilde{f}(k)=e^{-(a_\mu|k|)^\mu}$, which verifies $\tilde{f}(k)\underset{k\to\infty}{=}o(k^{-1})$; an example of sublinear scaling with x_0 is provided by the jump distribution $f(\ell)\propto[1/\sqrt{|\ell|}(1+|\ell|)]$, which corresponds to $\nu=1/2$ in (13) and has an infinite second moment ($\mu=1/2$). Our results are thus also validated in the case of jump processes with infinite second moment.

Application to effective 1D problems.—In this section we show how our formalism applies to higher dimensional jump processes evolving between two parallel hyperplanes H_1 and H_2; coming back to our initial example of the transmission of particles (e.g., photons or neutrons) through a slab of a scattering medium, the case $d=3$ is of particular interest. The trajectory is then naturally described as a 3D jump process, where at each step, the direction of the jump is drawn uniformly on the unit sphere and its length r is drawn according to a distribution $p(r)$; typically experiments show that exponential or Levy distributions $p(r)$ are observed, and provide as readout the transmission probability through the exit plane H_2 rather than H_1. Even if the problem is three dimensional, the determination of the transmission probability amounts to solving for the splitting probability of a one-dimensional problem, with the effective jump distribution $f(\ell)=\frac{1}{2}\int_{|\ell|}^{\infty}[p(r)/r]dr$ [28]. The above formalism is thus directly applicable and provides explicit determinations of the asymptotic splitting probability and in particular of the transmission probability (see SM [32]). In the case of an exponential jump distribution $p(r)=(1/\gamma)e^{-r/\gamma}$, relevant to classical diffusive media [9], we obtain $f(\ell)=(1/2\gamma)\Gamma[0,(|\ell|/\gamma)]$, where $\Gamma(x,y)$ stands for the incomplete gamma function, yielding $\tilde{f}(k)=[\arctan(k\gamma)/k\gamma]$ after Fourier transform. Equation (11) then provides—up to Laplace inversion—the asymptotic expression (for $x\to\infty$) of the splitting probability for any x_0. In particular (11) and (13) yield for $x_0 \ll a_2 \equiv \gamma/\sqrt{3}$:

$$\pi_{0,\underline{x}}(x_0) \underset{x\to\infty}{\sim} \frac{1}{\sqrt{3}x}\left[\gamma - \frac{x_0\ln(x_0)}{2} + o[x_0\ln(x_0)]\right]. \quad (20)$$

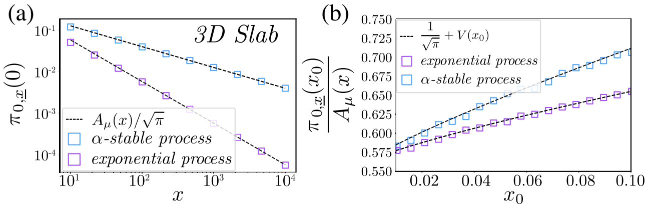

FIG. 4. (a) Transmission probability for an exponential jump process with $\gamma = 1$, and a Levy flight with $\mu = 1$ and $a_\mu = 1$. The transmission probabilities (including prefactors) are accurately predicted by (11) and (13). (b) Small x_0 behavior of the splitting probability. For both processes, x is fixed to 10^3 and the behavior is sublinear as predicted. Theoretical predictions (dashed lines) are obtained by numerical inverse Laplace transform of (10), while simulations (squares) are averaged over 10^6 trials.

In the case of α-stable jump distributions, which have been shown recently to be relevant to photon scattering in hot atomic vapors [12,13], we obtain $\tilde{f}(k) = \{[\Gamma(\mu^{-1}) - \Gamma(\mu^{-1}, (a_\mu k)^\mu)]/a_\mu \mu k\}$ (see SM [32]). As above, this provides the asymptotic expression (for $x \to \infty$) of the splitting probability for any x_0 thanks to (11), and making use of (13) one obtains for $x_0 \ll a_\mu$:

$$\pi_{0,\underline{x}}(x_0) \underset{x\to\infty}{\sim} \frac{\Gamma(\frac{1+\mu}{2})2^{\mu-1}}{\sqrt{(1+\mu)}} \left[\frac{a_\mu}{x}\right]^{\mu/2} \times \left[\frac{1}{\sqrt{\pi}} - \frac{\Gamma(\mu^{-1})}{a_\mu \mu \pi^{3/2}} x_0 \ln(x_0)[1 + o(1)]\right]. \quad (21)$$

Agreement with simulations in both cases is displayed in Fig. 4.

Conclusion.—We have derived a universal exact asymptotic form for the splitting probability for continuous symmetric jump processes characterized by a finite length scale a_μ, which have proved to be relevant in various contexts, such as transport of photons or neutrons in scattering media. This analysis covers the regime $x_0 \ll a_\mu \ll x$ and provides in particular a fully explicit determination of the transmission probability ($x_0 = 0$), in striking contrast with the trivial prediction $\pi_{0,\underline{x}}(x_0) = 0$ obtained by taking the continuous limit of the process. This reveals the importance of the microscopic properties of the dynamics. Our approach is general and can be further extended to cover examples of biased (asymmetric) jump processes, as discussed in SM [32].

These results are illustrated with paradigmatic models of jump processes with applications to light scattering in heterogeneous media in realistic 3D slab geometries. In this context, our explicit asymptotic predictions of the transmission probability (21), which can be directly measured experimentally, provides in principle a quantitative determination of not only the Levy exponent μ, as already proposed and measured in [12,13], but also of the microscopic length scale a_μ. This significantly refines the characterization of the effective random process describing light scattering in the medium.

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