Unitarity and Low Energy Expansion of the Coon Amplitude

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The Coon amplitude is a deformation of the Veneziano amplitude with logarithmic Regge trajectories and an accumulation point in the spectrum, which interpolates between string theory and field theory. With string theory, it is the only other solution to duality constraints explicitly known and it constitutes an important data point in the modern S-matrix bootstrap. Yet, its basics properties are essentially unknown. In this Letter, we fill this gap and derive the conditions of positivity and the low energy expansion of the amplitude. On the positivity side, we discover that the amplitude switches from a regime where it is positive in all dimensions to a regime with critical dimensions, which connects to the known $d = 26$, 10 when the deformation is removed. Incidentally, we find that the Veneziano amplitude can be extended to massive scalars of masses up to $m^2 = 1/3$, where it has critical dimension 6.3. On the low-energy side, we compute the first few couplings of the theory in terms of q-deformed analogs of the standard Riemann zeta values of the string expansion. We locate their location in the EFT-hedron, and find agreement with a recent conjecture that theories with accumulation points populate this space. We also discuss their relation to low spin dominance. Finally, we comment on the length of the Coon parton.

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The Coon amplitude [1–3] is, together with the Veneziano amplitude, the only explicitly known four-point tree-level amplitude that describes an infinite exchange of higher-spin resonances that solve the duality constraints. It was discovered as a deformation of the Veneziano amplitude to non-linear Regge trajectories. The deformation is given in terms of a parameter q ($0 \leq q \leq 1$), which characterizes a family of amplitudes defined by (in units $\alpha' = 1$)

$$A_q(s,t) = (q-1)q^{\frac{\log(\sigma)\log(\tau)}{\log(q)\log(q)}} \prod_{n=0}^{\infty} \frac{(\sigma\tau - q^n)(1-q^{n+1})}{(\sigma - q^n)(\tau - q^n)} \quad (1)$$

with

$$\sigma = 1 + (s - m^2)(q-1), \qquad \tau = 1 + (t - m^2)(q-1), \quad (2)$$

where s, t are the Mandelstam variables (cf. the Supplemental Material [4]). This amplitude describes the scattering of four identical scalars of mass m^2. At $q = 0$, it reduces to a scalar theory, and at $q = 1$ it gives back the Veneziano model:

$$\lim_{q\to 0} A_q(s,t) = \frac{1}{s - m^2} + \frac{1}{t - m^2} - 1, \quad (3)$$

$$\lim_{q\to 1} A_q(s,t) = A^V(s,t) = -\frac{\Gamma(-s+m^2)\Gamma(-t+m^2)}{\Gamma(-s-t+2m^2)}. \quad (4)$$

Unlike for the Veneziano model, no world sheet theory was found for the Coon amplitude, and to this day, its physical origin remains mysterious. In addition, and what concerns us in this Letter, its basic properties—unitarity conditions and low-energy expansion—are essentially unknown.

In more recent times, the Coon amplitude was brought forward as an exception to the universality of linear Regge trajectories in [9], where the authors traced to the presence of an accumulation point in its spectrum.

Related bootstrap constraints applied to the Wilson coefficients of effective field theories (EFTs) coming from unitarity, crossing, and analyticity are known to impose bounds [10] that carve theory islands [11–15], and it appears that they are bigger than what is required to describe the known physical theories [16–19]. Even more interestingly, Ref. [19] recently conjectured that the space of gravitational EFTs is actually populated generically by theories with accumulation points. Since the Coon amplitude has an accumulation point and connects, continuously, string theory and field theory, it provides an interesting testing ground to investigate these questions.

First, we map the positivity region of the amplitude, see Fig. 1: for each point (q, m^2) we determine the maximal dimension in which no ghosts are exchanged as intermediate states. This generalizes the known $d = (10)26$ critical dimensions of (super)string theory for $m^2 = (0)$, -1.

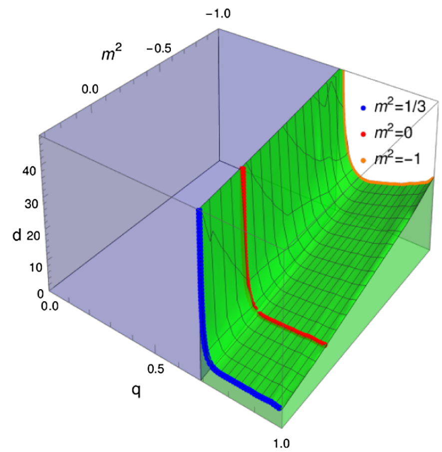

FIG. 1. Green surface: Numerical boundary of the unitary region. Blue shaded region: Unitary region determined by $q \leq q_c(m^2)$.

Our results show the existence of two regimes. In the first one, we prove that the amplitude is ghost-free in all dimensions. While this goes against intuition from standard string theory, null strings are known to possess similar features [20–25]. In the other regime, we determine numerically the positivity surface, which interpolates from infinite critical dimensions to the standard critical dimensions of string theory. Along the way, we realized that the Coon and thus Veneziano amplitudes can be extended to positive $m^2 \to 1/3$, with critical dimension $d \simeq 6.3$ for the latter [26].

Second, we compute some low energy couplings of the Coon amplitude in terms of q-polylogarithm values, which generalize the known zeta values of the string low-energy expansion and suggest the possible existence of a q-deformed world sheet theory where we relate q to the Coon parton in Eq. (11). We compute explicitly the first coefficients $g_2(q), g_3(q), g_4(q)$ and map their location in the space of couplings, comparing to [11]. We also comment on the connection to the notion of low-spin dominance (LSD) of [16].

The Coon amplitude.—We start by reviewing some facts about the Coon amplitude. As defined in (1), it describes color-ordered scalar scattering amplitude, being stripped of a color factor in analogy to [27]. Later, to compare its IR regime to [11], we consider an (s,t,u) symmetrized amplitude, corresponding to ordinary external scalars.

The spectrum of the amplitude can be read off Eq. (1). It has single poles located at $\sigma = q^n$ which correspond to [28]

$$s = m_n^2 := m^2 + \frac{q^n - 1}{q - 1}. \tag{5}$$

These poles build up an accumulation point as $n \to \infty$,

$$s_* = m^2 + \frac{1}{1 - q}. \tag{6}$$

At the accumulation point starts a cut, coming from the nonmeromorphic prefactor in Eq. (1). As we explain below, this factor is crucial to ensure polynomial residues [3]. Analogies with atomic physics of such an analytic structure were drawn in [29] but never made precise [30]. This makes this amplitude depart from the strict tree-level case studied in [9]. It would be interesting to understand if such non-analyticities can help to model the important phenomenon of bending of hadron Regge trajectories, as in [31].

On a given pole $s = m_n^2$, the nonmeromorphic q factor reduces to τ^n and cancels an inverse factor τ^{-n} coming from the denominator [32] and the residue of the amplitude reduces to a polynomial of degree n in t, corresponding to the exchange of particles of spins 0 to n, which reads

$$\mathrm{Res}(A_q(s,t))|_{s=m_n^2} = \prod_{j=0}^{n-1} \left(\frac{\tau - q^{j-n}}{1 - q^{-j-1}} \right). \tag{7}$$

This factor was missed in the original papers and in the more recent Refs. [33,34]. This aspect of the Coon amplitude indicates that accumulation points are a clear sign of tension between weak coupling (meromorphy) and locality, and could be useful to understand the species bound [35,36]. This tension also suggests a physical rationale as to why amplitudes with accumulation points evade the theorem of Ref. [9] (Supplemental Material [4]) on the linearity of Regge trajectories.

The Regge and fixed-angle regimes are readily extracted [1,37]. In the Regge regime, where $s \gg -t$ and t fixed, $1/\sigma$ vanishes and the amplitude behaves as

$$A_q(s,t) \sim f(t) s^{j(t)}, \qquad j(t) = \frac{\log[(t - m^2)(q-1) + 1]}{\log(q)}, \tag{8}$$

where $f(t) \sim \prod(1 - q^n/\tau)$. Since $q < 1$, the amplitude is *suppressed* at physical negative t for $m^2 \geq 0$. In particular, all bounds on couplings derived assuming standard twice-subtracted dispersion relations are expected to hold. We plot $j(t)$ in the Supplemental Material [4] for a specific value of the mass.

In the fixed angle regime, $(s/-t)$ fixed, both $1/\sigma$ and $1/\tau$ vanish, therefore we get

$$A_q(s,t) \sim e^{\log(s)\log(-t)/\log(q)} \sim s^{\log[s\cos(\theta)]/\log(q)}. \tag{9}$$

Finally, we also worked out the impact parameter transform of the amplitude at large b (and $q < 1$), which reads

$$\hat{A}_q(s,b) \sim \left(\frac{b}{\sqrt{t_*}} \right)^{-\frac{\log s}{\log q}} e^{-b\sqrt{t_*}}, \qquad t_* = m^2 + \frac{1}{1-q}. \tag{10}$$

It is similar to the Yukawa potential of a single particle of mass $m = \sqrt{t_*}$, dressed by power corrections. When $q \to 1$, these resum to the Gaussian behavior of string theory amplitudes [38]. In the Supplemental Material [4] we provide numerical arguments that essentially relate q to the length of the Coon parton, as

$$\ell_C \simeq \ell_0 + q\sqrt{\log(s)}. \qquad (11)$$

Positivity of the amplitude.—We present now our results on the unitarity of the Coon amplitude, i.e., the conditions under which no negative-norm states, or ghosts, decouple. Ghosts are characterized by negative residues on single poles, or more precisely by negative coefficients in the partial wave expansion of the amplitude's residues. Near a resonance exchange in the s channel, the amplitude takes the form

$$A(s,t) \sim_{s\to m_n^2} \frac{\mathrm{Res}_n(t)}{s-m_n^2}. \qquad (12)$$

As is standard, [9,13] Lorentz invariance implies that $\mathrm{Res}_n(t)$ is a polynomial whose degree corresponds to the highest spin among the modes of mass m_n being exchanged, and that it can be further decomposed into Gegenbauer polynomials $\mathcal{P}_J^{(d)}$ which are angular eigenfunctions in dimension d:

$$\mathrm{Res}_n(t) = \sum_{J=0}^{n} c_{n,J}\mathcal{P}_J^{(d)}(\cos(\theta)). \qquad (13)$$

Here $\cos(\theta) = 1 + [2t/(m_n^2 - 4m^2)]$ is the cosine of the scattering angle at $s = m_n^2$, and each term in the sum corresponds to a different particle of mass m_n and spin J being exchanged.

The coefficients $c_{n,J}$ can be obtained using the orthogonality of the Gegenbauer polynomials, and unitarity implies that these should be positive.

In the case of string theory, the no-ghost theorem [39,40] guarantees that such states decouple from all scattering amplitudes in $d \le 26$ or 10. At the level the Veneziano amplitude, a recent paper showed that residues should all be positively expandable on Gegenbauer polynomials [41] in $d \le 6$ [42]. It would still be desirable to be able to bridge the gap to $d = 10$ or 26 and maybe the Coon amplitude in its $q \to 1$ limit could be useful for this.

Regarding Coon, the authors of [43] investigated the presence of ghosts in the amplitude in $d = 4$, and observed numerically that some regions in the q, M^2 parameter space are ghost-free. While their (numerical) method finds ghosts in $d = 4$, we do not, for any values of q. This is because our set up is different: for them, the mass of the external particles M^2 is distinct from m^2 (the first pole of the amplitude), while for us, $M^2 = m^2$. The subsequent work [33] also studied the problem (while being unaware of [43]), but was inconclusive. Overall, before the present work, nothing was known on the critical dimensions of the Coon amplitude.

Our results are summarized in Fig. 1. They show the existence of two regimes for q, distinguished critical value $q \lessgtr q_\infty(m^2)$, where

$$q_\infty(m^2) = \frac{m^2 - 3 + \sqrt{9 + 2m^2 + m^4}}{2m^2}. \qquad (14)$$

Using elementary properties of the Gegenbauer polynomials and the residues given by Eq. (7), it is easy to show that for $q < q_\infty(m^2)$ the amplitude is ghost-free in all dimensions. Then, for $q > q_\infty(m^2)$, critical dimensions exist and we numerically determined them, backed by an estimate of the envelope of the critical dimensions near $q_\infty(m^2)$. The details of these proofs are given in the Supplemental Material [4].

Requiring that the residues are positive also puts an upper bound on the mass of the external particle: if $m^2 > \frac{1}{3}$, the Coon amplitude exhibits ghosts at all values of q and d. There is no such strict lower bound on the mass, but for $m^2 < -1$ the unitary region of the Coon amplitude becomes disconnected from the Veneziano limit, and thus we restricted our analysis to the region $-1 \le m^2 \le \frac{1}{3}$. More details on this can also be found in the Supplemental Material [4].

On the numerics side, for $q > q_\infty(m^2)$ we performed an extensive numerical study of the signs of the residues. We computed the Gegenbauer coefficients for the first 50 resonances and all spins $J \le 50$ for a grid of values of $m^2 \in [-1, \frac{1}{3}]$ and $q \in [0; 1]$ and determined with finite 4% accuracy the critical dimension $d(m^2, q)$ for which all the coefficients become positive. This draws the green surface of Fig. 1. When $q \to 1$, the critical dimensions of the $m^2 = -1$ and $m^2 = 0$ models match the values for the Veneziano and Neveu-Schwarz models. Moreover, our results suggest a possible extremal unitary amplitude with $m^2 = \frac{1}{3}$ and critical dimension $d \simeq 6.3$. Those three curves are plotted specifically in Fig. 2.

One curious observation from that study is that the scalar ghost sector seems to always define the unitarity surface. In the Supplemental Material [4] we give a proof of this fact in the large d limit and an estimate of the critical dimensions for q near the critical line which matches the numerics; see Fig. 2. While our arguments fail as $q \to 1$, we observed that this continues to hold. Proving this fact, maybe using the methods of [41], would allow to prove analytically the unitarity of the Coon and of Veneziano amplitude as a function of m^2.

Low energy expansion and EFT hedron.—Recent times have witnessed a renewal of activity revolving around implications of dispersion relations, crossing symmetry, and unitarity. Following the ideas of [10], various studies explored how the Wilson coefficients of weakly coupled

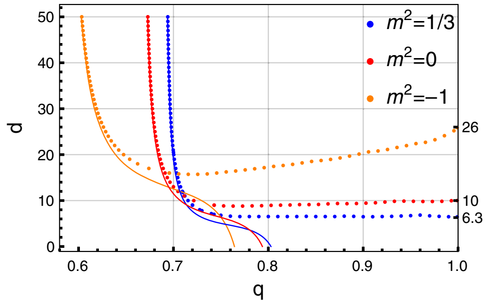

FIG. 2. Dots: Sections of the surface mapped in Fig. 1. Solid lines: Large d envelope computed analytically (see Supplemental Material [4]).

EFTs are constrained [11–15,44] and live in some regions of some positive region dubbed "EFThedron."

In this section, we compute the first few low energy couplings of the Coon amplitude. Because the Coon amplitude is well behaved at infinity, and respects analyticity and crossing, it admits dispersion relations and must fall in those positivity regions. We will see that the couplings indeed draw one-dimensional varieties within those regions, parametrized by the value of q.

The first amplitude we consider is an (s, t, u) symmetric version of Coon, for external *massless* scalars ($m^2 = 0$) with no color indices:

$$M_q(s,t,u) = A_q(s,t) + A_q(t,u) + A_q(u,s). \tag{15}$$

The (s, t, u) symmetry and momentum conservation $s + t + u = 0$ allow expansion at small s, t, u of this function in terms of $\sigma_2 = s^2 + t^2 + u^2$ and $\sigma_3 = stu$, so that

$$M_q(s,t,u) = \frac{1}{s} + \frac{1}{t} + \frac{1}{u} + g_0(q) + g_2(q)\sigma_2 + g_3(q)\sigma_3 + g_4(q)(\sigma_2)^2 + \dots, \tag{16}$$

where the coefficients of this expansion are classically interpreted as low energy Wilson coefficients.

A lengthy but straightforward explicit calculation gave us the first few coefficients, up to $g_4(q)$. Trivially, $g_0(q) = 1 - q$. The next ones are given by functions related to q-zeta values, for instance, $g_2(q)$ reads

$$g_2(q) = \frac{1}{2}(q-1)^3(3h_1(q) + 5h_2(q) + 2h_3(q)) - \frac{(q-1)^3}{\log(q)}, \tag{17}$$

where

$$h_m(q) = \sum_{n=1}^{\infty} \frac{q^{nm}}{(1-q^n)^m} \coloneqq \mathrm{Li}_m(q^m; q) \tag{18}$$

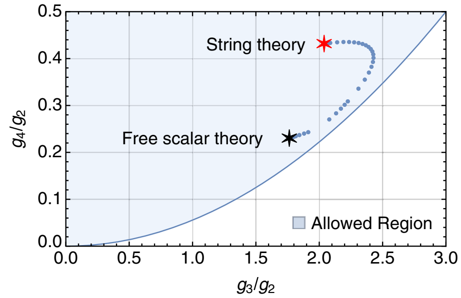

FIG. 3. Plot of Wilson coefficients with cutoff $M^2 = 1$.

can be written in terms of q-deformed polylogarithms as defined, for instance, in [45], and whose q-zeta values are classically defined as q values of those functions. Note that, compared to string theory, different orders of q transcendentality appear to be mixed. The other couplings $g_3(q)$ and $g_4(q)$ are given in the Supplemental Material [4]. We also verified that when $q \to 1$, they descend to the values given by the symmetrized sum $A^V(s,t) + A^V(t,u) + A^V(u,s)$:

$$g_2(1) = -\zeta_3, \qquad g_3(1) = 9/4\zeta_4, \qquad g_4(1) = -\zeta_5/2. \tag{19}$$

For generic EFTs, the allowed range of coefficients g_2, g_3, g_4 was determined in [11], Fig. 8, in terms of dimensionless ratios $\tilde{g}_3 = g_3 M^2/g_2$ and $\tilde{g}_4 = g_4 M^4/g_2$ with M^2 given by the scale of the first massive mode, which in our conventions is $M^2 = 1$. We show in Fig. 3 the value of those ratios. They fall neatly in the domain determined in [11], albeit approaching tangentially the boundary at intermediate values of q.

One can also couple a massless scalar to a *massive* Coon amplitude, since $0 \le m^2 \le 1/3$ are allowed. These amplitudes reduce to the extreme case of [11], coupling the massless scalar to a single massive scalar of mass $M^2 = m^2$ and therefore accumulate to the upper right corner of their Fig. 8. It is not surprising, and the same happens when coupling a massless scalar to an amplitude made of massive Veneziano blocks.

Low spin dominance.—The Coon amplitude $A_q(s,t)$, together with its Veneziano limit, exhibit a form of LSD, albeit weaker than that of [16]. It is a LSD where not only the scalar state dominates the partial waves, but also the spin 1 state [46]. Let us explain how this comes about.

Following the conventions of [16], we Taylor expand the amplitude as $A_q(t,-s-t) = (1/s) + (1/t) + \sum_{p<k} a_{k,p} s^{k-p} t^p$. To compare to [16], we look at the coefficients at level $k = 2, 4, 6$, which we relate to that of an amplitude given by a sum of

$$A^{(J)}(t,u) = (-1)^J \frac{\mathcal{P}_J(1+2s/M^2)}{t-M^2} + \frac{\mathcal{P}_J(1+2t/M^2)}{u-M^2}, \tag{20}$$

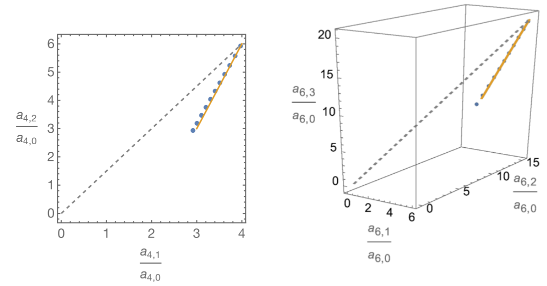

FIG. 4. Plots of a_4 and a_6. Gray dashed: LSD. Yellow: spin 0,1 model. Blue dots: actual points. Veneziano is in the middle of the graph ($q = 1$) and the scalar theory is at the intersection with LSD.

with $J = 0, 1$. Denoting by $a^{(J)}_{k,q}$ the low energy coefficients of expanding $A^{(J)}(t, -s - t)$ in powers of s, t, the model mentioned above states that $a_{k,p} \sim a^{(0)}_{k,p} + q a^{(1)}_{k,p}$. This corresponds to the straight yellow line in Fig. 4, and matches well the dots near $q = 1$, where the amplitude matches pure LSD, since only $J = 0$ contributes. Within this spin 0-1 model, it is immediate to verify that

$$\frac{a_{2,1}}{a_{2,0}} = \frac{2 - 2q}{1 + q}, \tag{21}$$

$$\frac{a_{4,1}}{a_{4,0}} = \frac{2(q + 2)}{q + 1}, \qquad \frac{a_{4,2}}{a_{4,0}} = \frac{6}{q + 1}. \tag{22}$$

For $q \in [0, 1]$, this implies in particular that $0 \le (a_{2,1}/a_{2,0}) \le 1$, $3 \le (a_{4,1}/a_{4,0}) \le 4$, and $3 \le (a_{4,2}/a_{4,0}) \le 6$. The upper bound corresponds to pure LSD, while the lower bound is pure spin-0 + spin-1 model. While for the coefficients $a_{2,1}/a_{2,0}$, the bounds are exactly satisfied, at $k = 4$ it can be seen that string theory lies a bit away from that, at $[(a_{4,1}/a_{4,0}), (a_{4,2}/a_{4,0})] \simeq (2.9, 2.9)$. The relative accuracy of the model is explained by the fact that spin J exchanges come with $q^{J(J+1)/2}$, for which the linear approximation (spin 0 and 1) is a good approximation away from $q = 1$.

Perspectives.—This study opens many perspectives, some mentioned in the text. First, we clarified the definition of the amplitude and our results on unitarity and the q-deformed low energy expansion very similar to that of string theory suggest strongly that there might exist a unitary Coon theory, with maybe a q-deformed world sheet theory. Very few results exist in the literature on world sheet theory and N-point functions (see Refs. [47,48] and [2,49–51], respectively) and it would be really interesting to understand this better. This amplitude could also be used to revisit the analysis of [41] and study the unitarity of the Veneziano amplitude thanks to the q dependence. It would be nice to understand and relate to the q deformation the feature of the disappearance of critical dimensions when $q \to 0$.

Furthermore, our study resonates neatly with a conjecture of [19] that amplitudes with accumulation points populate the EFT-hedron of gravitational theories away from the small portion, where usual theories seem to live. It would be very important to study this question in more detail, in relation with the results presented here. In view of the species bound mentioned above, there is clearly a paradox here, whose resolution might provide us with new physical principles to understand better the space of EFTs and Coon is a nice and controlled toy model for this problem.

It would also be interesting to study the Coon version of the Lovelace-Shapiro amplitude [52,53]. This amplitude, only recently understood from string theory [54], constitutes an interesting example where the N-point function shows defaults of unitarity.

Finally, and maybe in relation to the species bound mentioned above for gravitational theories, we have not been able to write down a unitary Coon amplitude via the Kawai-Lewellen-Tye formula [55]. It would be very interesting to see if one can be found and study the question of the species bound in this context.

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[4] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.129.121602 we define our kinematic conventions and provide some properties of the Gegenbauer polynomials that are relevant for this work. Furthermore we give a proof of the existence of a critical value of q below which the amplitude has no critical dimension and the explicit form of the curve separating the unitary and non-unitary regimes in the large dimension limit. We also show further details regarding the numerical techniques used in this work, explicit expressions for the first low energy coefficients in the EFT expansion of the amplitude and review the relation between the Coon and Veneziano amplitudes. Finally we present some results regarding the impact parameter representation of the Coon amplitude supporting the idea that it interpolates between string and field theory, and that the deformation parameter q can be interpreted as the size of the Coon particle, which includes Refs. [5–8].
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