Criticality and Phase Classification for Quadratic Open Quantum Many-Body Systems

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We study the steady states of translation-invariant open quantum many-body systems governed by Lindblad master equations, where the Hamiltonian is quadratic in the ladder operators, and the Lindblad operators are either linear or quadratic and Hermitian. These systems are called quasifree and quadratic, respectively. We find that steady states of one-dimensional systems with finite-range interactions necessarily have exponentially decaying Green's functions. For the quasifree case without quadratic Lindblad operators, we show that fermionic systems with finite-range interactions are noncritical for any number of spatial dimensions and provide bounds on the correlation lengths. Quasifree bosonic systems can be critical in $D > 1$ dimensions. Last, we address the question of phase transitions in quadratic systems and find that, without symmetry constraints beyond invariance under single-particle basis and particle-hole transformations, all gapped Liouvillians belong to the same phase.

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Introduction.—For closed systems, criticality and quantum phase transitions have been studied extensively [1–4]. Particularly, for one-dimensional systems, we have obtained a thorough classification of gapped states using the tensor-network ansatz [5–8].

In practice, most quantum systems are not perfectly isolated from their environment. In addition to posing challenges for quantum technology, driving and dissipation in open systems could be designed to stabilize (novel) phases of matter or particular entangled states [9–11], e.g., to facilitate measurement-based quantum computation [12,13], quantum phase estimation [14,15], and quantum simulation [16–20]. For Markovian systems, the density matrix $\hat{\rho}$ evolves according to a Lindblad master equation [21–25]:

$$\partial_t\hat{\rho} = \mathcal{L}\hat{\rho} = -i[\hat{H},\hat{\rho}] + \sum_\alpha\left(\hat{L}_\alpha\hat{\rho}\hat{L}_\alpha^\dagger - \frac{1}{2}\{\hat{L}_\alpha^\dagger\hat{L}_\alpha,\hat{\rho}\}\right).$$

In addition to the Hamiltonian part $-i[\hat{H},\hat{\rho}]$, the Liouvillian superoperator $\mathcal{L}$ captures decoherence processes with environment couplings described by the Lindblad operators $\hat{L}_\alpha$.

In this Letter, we elucidate the occurrence of criticality and phase transitions in the steady states of open quasifree and quadratic systems of fermions and bosons. Quasifree open systems are characterized by Hamiltonians that are bilinear and Lindblad operators that are linear in ladder operators. Quadratic open systems may have additional bilinear self-adjoint Lindblad operators [26,27]. A system is called "critical" if it has a unique steady state with algebraically decaying correlations. We establish that quadratic one-dimensional (1D) systems with finite-range interactions and unique steady states necessarily have exponentially decaying Green's functions. Next, we address quasifree systems with finite-range interactions. Quasifree fermionic systems are noncritical for any number of spatial dimensions [28]. Conversely, one can construct critical quasifree bosonic systems for $D \geq 2$ dimensions. Gapped quasifree systems are always noncritical. Of course, the existence of critical steady states does not necessarily imply phase transitions. In fact we show that, without symmetry constraints beyond invariance under single-particle basis and particle-hole transformations, all gapped Liouvillians of quadratic open systems belong to the same phase.

Experimentally, systems of trapped ions [30,31], Rydberg atoms [32,33], ultracold atoms in optical lattices or tweezers [34–36], and superconducting circuits [37,38] allow for the engineering of such dissipative systems [39–44]. In circuit QED systems [45–48], linear Lindblad operators arise naturally from photon loss and pump process, while the coupling of cavities can lead to bilinear Lindblad operators [49,50].

Setup and covariance matrix.—Consider a system of identical bosons or fermions with ladder operators $\hat{a}_j$ and $\hat{a}_j^\dagger$ for modes $j = 1, \ldots, N$. We employ Majorana operators $\hat{w}_{j+} := (\hat{a}_j + \hat{a}_j^\dagger)/\sqrt{2}$ and $\hat{w}_{j-} := i(\hat{a}_j - \hat{a}_j^\dagger)/\sqrt{2}$, which obey the (anti)commutation relations

$$\{\hat{w}_{i\mu},\hat{w}_{j\nu}\} = \delta_{i,j}\delta_{\mu\nu} \qquad \text{for fermions, and}$$
$$[\hat{w}_{i\mu},\hat{w}_{j\nu}] = -\mathrm{i}\mu\delta_{i,j}\delta_{\mu,-\nu} \qquad \text{for bosons.}$$

We address Markovian systems with quadratic Hamiltonians $\hat{H} = \sum_{i\mu,j\nu}\hat{w}_{i\mu}H_{i\mu,j\nu}\hat{w}_{j\nu}$. Quasifree systems only have linear Lindblad operators $\hat{L}_s = \sum_{j\nu}L_{s,j\nu}\hat{w}_{j\nu}$. Quadratic systems may feature additional bilinear self-adjoint Lindblad

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operators $\hat{M}_u = \hat{M}_u^\dagger = \sum_{i\mu,j\nu} \hat{w}_{i\mu} (M_u)_{i\mu,j\nu} \hat{w}_{j\nu}$. The $2N \times 2N$ covariance matrix

$$\Gamma_{i\mu,j\nu} := \begin{cases} \frac{\mathrm{i}}{2}\langle \hat{w}_{i\mu}\hat{w}_{j\nu} - \hat{w}_{j\nu}\hat{w}_{i\mu}\rangle & \text{for fermions,} \\ \frac{1}{2}\langle \hat{w}_{i\mu}\hat{w}_{j\nu} + \hat{w}_{j\nu}\hat{w}_{i\mu}\rangle & \text{for bosons} \end{cases} \tag{1}$$

can be shown to evolve according to the equation of motion [26,27]

$$\partial_t \Gamma = X\Gamma + \Gamma X^T + Y + \sum_u Z_u \Gamma Z_u^T, \tag{2}$$

where the real $2N \times 2N$ matrices X, Y, and Z_u depend on the coupling coefficients H, L_s, and M_u as detailed in the Supplemental Material [51]. The Z_u term vanishes for quasifree systems.

For a translation-invariant system in D dimensions, each mode i is associated with a cell location $\boldsymbol{i} \in \mathbb{Z}^D$ and a crystal-basis index $c_i = 1, \ldots, b$, where b is the number of bands. The covariance matrix elements and coupling coefficients are then functions of spatial distances such that

$$\Gamma_{i\mu,j\nu} =: \gamma_{c_i\mu,c_j\nu}(\boldsymbol{i}-\boldsymbol{j}), \qquad X_{i\mu,j\nu} =: x_{c_i\mu,c_j\nu}(\boldsymbol{i}-\boldsymbol{j})$$

etc., and the equation of motion (2) takes the form

$$\begin{aligned} \partial_t \gamma(\boldsymbol{r}) = \sum_{\boldsymbol{n}} [x(\boldsymbol{n})\gamma(\boldsymbol{r}-\boldsymbol{n}) + \gamma(\boldsymbol{r}+\boldsymbol{n})x^T(\boldsymbol{n})] + y(\boldsymbol{r}) \\ + \sum_{u,\boldsymbol{n},\boldsymbol{j},\boldsymbol{l}} z_u(\boldsymbol{r}-\boldsymbol{n},\boldsymbol{j}-\boldsymbol{n})\gamma(\boldsymbol{r}-\boldsymbol{l}) z_u^T(-\boldsymbol{n},\boldsymbol{l}-\boldsymbol{n}), \end{aligned} \tag{3}$$

where γ, x, y, and z_u are $2b \times 2b$ matrices depending on lattice translation vectors $\boldsymbol{r} \in \mathbb{Z}^D$.

Correlations in quadratic 1D systems.—As a first result, let us establish the following:

Proposition 1: If a quadratic 1D system with translation-invariant finite-range couplings has a unique steady state, then its single-particle Green's function $\gamma(r)$ cannot follow a power-law decay with respect to the distance $|r|$.

For the steady-state covariance matrix $\gamma(r)$, the right-hand side of Eq. (3) needs to be zero. For distances r large enough such that the local z_u and y terms vanish, $\gamma(r)$ obeys a matrix difference equation of the form

$$C_0 \boldsymbol{\gamma}(r) + C_1 \boldsymbol{\gamma}(r+1) + \cdots + C_R \boldsymbol{\gamma}(r+R) = \boldsymbol{0}. \tag{4}$$

Here, $\boldsymbol{\gamma}(r)$ is the vectorization of $\gamma(r)$, the $4b^2 \times 4b^2$ matrices C_m are determined by the coupling matrices $x(n)$, and R denotes the interaction range [51].

In the simplest scenario, C_R is invertible such that we can solve Eq. (4) for $\boldsymbol{\gamma}(r+R)$ and

$$\boldsymbol{g}_{r+1} = \begin{bmatrix} A_{R-1} & A_{R-2} & \cdots & A_1 & A_0 \\ \mathbb{1} & 0 & \cdots & 0 & 0 \\ 0 & \mathbb{1} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \mathbb{1} & 0 \end{bmatrix} \boldsymbol{g}_r \tag{5}$$

with $\boldsymbol{g}_r^T := [\boldsymbol{\gamma}^T(r+R-1), \ldots, \boldsymbol{\gamma}^T(r)]$ and $A_m := -C_R^{-1} C_m$. The spectrum of the $4b^2 R \times 4b^2 R$ transfer matrix in Eq. (5) characterizes the spatial decay of $\gamma(r)$. As the spectrum is discrete, all elements of $\gamma(r)$ must decay exponentially, converge to a constant, or oscillate with constant amplitude. An algebraic decay that characterizes critical systems is not possible. The transfer matrix may have eigenvalues β with $|\beta| > 1$. These are, however, irrelevant as physical systems cannot feature indefinitely growing $\gamma(r)$. For fermions, this is also prohibited by the constraint that all covariance matrix elements lie in the interval $[-1/2, 1/2]$ [26]. The Supplemental Material [51] gives a more general proof based on generating functions, which does not require invertibility of C_R.

Criticality in quasifree systems.—Stronger results hold for the systems that have no quadratic Lindblad operators and, hence, no Z_u term in Eq. (2). Let us first consider "gapped" systems, where the Liouvillian $\mathcal{L}$ has a single zero eigenvalue and the other eigenvalues λ have a nonzero "dissipative gap" $\Delta := -\max_{\lambda \neq 0} \mathrm{Re}\lambda > 0$.

Proposition 2: Gapped quasifree systems with translation-invariant finite-range couplings are never critical.

Note that, using quasilocality [59], this proposition can be generalized to interacting systems. But quasifree systems allow for a more direct proof that provides bounds on correlation lengths to be reused for Proposition 3:

Because of translation invariance, we can transform to a momentum-space representation with quasimomenta $k_a = (2\pi/L), (4\pi/L), \ldots, 2\pi$ for $a = 1, \ldots, D$. With

$$\tilde{\gamma}(\boldsymbol{k}) := \sum_{\boldsymbol{r}} e^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} \gamma(\boldsymbol{r}), \qquad \tilde{x}(\boldsymbol{k}) := \sum_{\boldsymbol{r}} e^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} x(\boldsymbol{r}) \tag{6}$$

and an analogous definition of $\tilde{y}$, according to Eq. (3), the steady state obeys the continuous Lyapunov equation

$$\tilde{x}(\boldsymbol{k})\tilde{\gamma}(\boldsymbol{k}) + \tilde{\gamma}(\boldsymbol{k})\tilde{x}^T(-\boldsymbol{k}) = -\tilde{y}(\boldsymbol{k}). \tag{7}$$

For a quasifree system to be gapped, all eigenvalues of X in Eq. (2) or, equivalently, all eigenvalues of $\tilde{x}(\boldsymbol{k})\ \forall\ \boldsymbol{k}$ in Eq. (7) need to have negative real parts [26]. But this means that we can solve Eq. (7) for $\tilde{\gamma}(\boldsymbol{k})$ by inverting the matrix $\tilde{x}(\boldsymbol{k}) \otimes \mathbb{1} + \mathbb{1} \otimes \tilde{x}(-\boldsymbol{k})$. Because of the finite interaction range, $\tilde{x}(\boldsymbol{k})$ and $\tilde{y}(\boldsymbol{k})$ are polynomials in variables $z_a := e^{\mathrm{i}k_a} \in \mathbb{C}$ and $1/z_a$. Hence, $\tilde{\gamma}(\boldsymbol{k})$ is a rational function of the z_a which, according to the invertibility of $\tilde{x}(\boldsymbol{k})$, has no poles on the manifold $|z_a| = 1$ which corresponds to real momenta $k_a \in (0, 2\pi]$ in the Brillouin zone. For concreteness, let us discuss $D=2$ dimensions; the generalization to $D\neq 2$ is trivial. The established property of $\tilde{\gamma}(\boldsymbol{k}) =: \tilde{\gamma}(z_1,z_2)$ allows us to determine $\gamma(r_1,r_2)$ using Cauchy's residue theorem from complex analysis.

In the thermodynamic limit, the inverse of Eq. (6) is

$$\gamma(r_1,r_2) = -\oint\!\!\oint_{|z_1|=|z_2|=1} \frac{d^2z}{(2\pi)^2} z_1^{r_1-1} z_2^{r_2-1} \tilde{\gamma}(z_1,z_2). \quad (8)$$

For fixed z_2, let $\varrho(z_2) := \mathrm{i}\sum_m \mathrm{Res}(\tilde{\gamma}(\zeta_m(z_2),z_2))$ denote the sum over the residues of $\tilde{\gamma}$ at pole locations $z_1=\zeta_m(z_2)$ inside the unit circle $|z_1|=1$ [60]. With $|\zeta| := \max_{m,|z_2|=1}|\zeta_m(z_2)| < 1$, it follows that

$$|\gamma(r_1,r_2)| \leq |\zeta|^{r_1-1} \oint_{|z_2|=1} \frac{dz_2}{2\pi} |z_2^{r_2-1}\varrho(z_2)|. \quad (9)$$

As the contour integral is independent of r_1, this bound establishes an exponential decay of $\gamma(r_1,r_2)$ with correlation length

$$\xi_1 \leq -1/\ln|\zeta| = -1/\max_{m,|z_2|=1} \ln|\zeta_m(z_2)| \quad (10)$$

in the positive r_1 direction. An exponential bound for negative r_1 is obtained by using $z_1 := e^{-\mathrm{i}k_1}$ instead of $e^{\mathrm{i}k_1}$, and the same arguments apply to r_2 or further dimensions.

The steady states of quasifree systems are Gaussian [26]. Hence, according to Wick's theorem [61,62], the steady state is fully characterized by $\gamma(\boldsymbol{r})$, and the exponential decay of $\gamma(\boldsymbol{r})$ implies the exponential decay of all connected real-space correlation functions. This concludes the proof of Proposition 2. Let us now drop the constraint of a nonzero dissipative gap.

Proposition 3: Quasifree fermionic systems with translation-invariant finite-range couplings are never critical.

For a unique steady state, the momentum-space covariance matrix $\tilde{\gamma}(\boldsymbol{k})$ solving Eq. (7) is again a rational function. Furthermore, it cannot have poles at real $\boldsymbol{k}$ for any short-range fermionic system [28]: The covariance matrix Γ in Eq. (1) is real and antisymmetric. Hence, there exists an orthogonal transformation $O\in O(2N)$ such that $\Gamma' := O\Gamma O^T = \begin{pmatrix} & \chi \\ -\chi & \end{pmatrix}$, where the elements χ_i of the $N\times N$ diagonal matrix χ correspond to the imaginary eigenvalue pairs $\pm\mathrm{i}\chi_i$. The transformation defines an alternative set of Majorana operators $\hat{w}'_{i\mu} := \sum_{j\nu} O_{i\mu,j\nu}\hat{w}_{j\nu}$ with covariance matrix Γ' such that $\chi_i = \mathrm{i}\langle \hat{w}'_{i+}\hat{w}'_{i-}\rangle$. As each fermionic occupation number operator $\hat{a}_j^\dagger\hat{a}_j$ has eigenvalues 0 and 1, the operators $\mathrm{i}\hat{w}_{j+}\hat{w}_{j-} = 1/2 - \hat{a}_j^\dagger\hat{a}_j$ and the operators $\mathrm{i}\hat{w}'_{i+}\hat{w}'_{i-}$ have eigenvalues $\pm 1/2$. Thus, all χ_i are in the interval $[-1/2,1/2]$, and all covariance matrix elements obey $|\Gamma_{i,j}| \leq \|O^T\Gamma' O\| = \|\chi\oplus(-\chi)\| \leq \frac{1}{2}$. The Fourier transform (6) to momentum space just adds another unitary transformation. Hence, the elements of $\tilde{\gamma}(\boldsymbol{k})$ have modulus $\leq 1/2$, i.e., singularities can only occur at complex momenta k_a. Their imaginary parts provide bounds on correlation lengths as in Eq. (10), and the system is not critical.

Proposition 3 is in stark contrast to closed fermionic systems, where tight-binding models have, for example, critical Fermi-sea ground states. The situation for open bosonic systems is different. Note that bosonic open systems can be unstable in the sense that the Liouvillian can have eigenvalues with positive real parts that lead to unlimited absorption of energy and particles. In quasifree systems, however, the existence of a steady state implies stability [26]. So, stability is implied in the following.

Proposition 4: Quasifree bosonic systems with translation-invariant finite-range couplings can be critical in $D\geq 2$ dimensions. 1D systems cannot be critical.

The statement on 1D systems follows immediately from Proposition 1 and Wick's theorem. Furthermore, one can construct quasifree bosonic models that are critical for $D\geq 2$ dimensions. Specifically, consider a purely dissipative model with one Lindblad operator $\hat{L}_{\boldsymbol{j}}^{(1)} := \sqrt{2D\eta}(\hat{w}_{\boldsymbol{j}+} - \mathrm{i}\hat{w}_{\boldsymbol{j}-}) = \sqrt{2D\eta}\hat{a}_{\boldsymbol{j}}$ for every site $\boldsymbol{j}\in\mathbb{Z}^D$ of the D-dimensional square lattice as well as four Lindblad operators $\hat{L}_{\boldsymbol{j},a}^{(2\pm)} := \hat{w}_{\boldsymbol{j}+} + \mathrm{i}\hat{w}_{(\boldsymbol{j}\pm\boldsymbol{e}_a)-}$ and $\hat{L}_{\boldsymbol{j},a}^{(3\pm)} := \hat{w}_{\boldsymbol{j}+} \pm \hat{w}_{(\boldsymbol{j}\pm\boldsymbol{e}_a)-}$ for every edge, where $\boldsymbol{e}_a$ are the unit vectors for directions $a=1,\ldots,D$. One finds that $\tilde{x}(\boldsymbol{k}) = 2D(c_{\boldsymbol{k}}-\eta)\mathbb{1}_2$, where $c_{\boldsymbol{k}} := \sum_a \cos k_a/D$ [51]. The largest X eigenvalue real part determines the dissipative gap Δ [26]. Here, $\tilde{x}(\boldsymbol{k})$ has the doubly degenerate eigenvalue $\xi(\boldsymbol{k}) = 2D(c_{\boldsymbol{k}}-\eta)$ and, hence, $\Delta = -\max_{\boldsymbol{k}}\mathrm{Re}\,\xi(\boldsymbol{k}) = 2D(\eta-1)$. So the model is stable for loss rates $\eta\geq 1$ and the gap closes for $\eta=1$ at momentum $\boldsymbol{k}=\boldsymbol{0}$. Solving the Lyapunov Eq. (7) yields the covariance matrix $\tilde{\gamma}(\boldsymbol{k})$ with the diagonal and off-diagonal elements

$$\tilde{\gamma}_{\pm,\pm}(\boldsymbol{k}) = \frac{\eta+2}{2(\eta-c_{\boldsymbol{k}})} \quad \text{and} \quad \tilde{\gamma}_{\pm,\mp}(\boldsymbol{k}) = \frac{\pm\mathrm{i}s_{\boldsymbol{k}}}{2(\eta-c_{\boldsymbol{k}})}, \quad (11)$$

where $s_{\boldsymbol{k}} := \sum_a \sin k_a/D$. With a Fourier transform to $\gamma(\boldsymbol{r})$, one can assess criticality. For $D=1$ dimensions, the Fourier integral can be evaluated exactly using the residue theorem. In agreement with Propositions 1 and 2, we find an exponential decay of correlations if $\eta>1$. The correlation length diverges for $\eta\to 1$, but there is no power-law decay. For dimensions $D\geq 2$, one can expand $\tilde{\gamma}(\boldsymbol{k})$ in a multipole series over hyperspherical harmonics [63] to reduce the Fourier transformation to a radial integral, which takes the form of a Hankel transform. The leading contributions to $\tilde{\gamma}_{\pm,\pm}$ are isotropic while those to $\tilde{\gamma}_{\pm,\mp}$ are antisymmetric with respect to reflection. For $D=2$ dimensions, the diagonal correlations $\gamma_{\pm,\pm}(\boldsymbol{r})$ decay logarithmically in $|\boldsymbol{r}|$ and the off-diagonal $\gamma_{\pm,\mp}(\boldsymbol{r})$ decay as $1/|\boldsymbol{r}|$. For $D=3$, they decay as $1/|\boldsymbol{r}|$ and $1/|\boldsymbol{r}|^2$, respectively. A detailed discussion is given in the Supplemental Material [51].

Phase classification for quadratic systems.—Like quantum phase transitions in closed systems [1–3], driven-dissipative phase transitions are characterized by a nonanalytic dependence of steady-state expectation values on system parameters. This requires a nonanalytic change in the steady-state density matrix and, hence, a level crossing [64]. So, the dissipative gap Δ needs to close at the transition point [65,66]. As seen so far, there are some restrictions on criticality in quadratic open systems, but the gap can of course close. As another fundamental result, we will see why, here, closing the gap does generally not lead to phase transitions.

Proposition 5: For quadratic systems without symmetry constraints beyond invariance under single-particle basis transformations and fermionic particle-hole symmetry, all gapped systems belong to the same phase. For any pair of gapped systems $\mathcal{L}_1$ and $\mathcal{L}_2$, one can construct a continuous path of gapped Liouvillians that links the two.

In particular, we claim that for any quadratic Liouvillian $\mathcal{L}$ with gap Δ, the auxiliary Liouvillian

$$\mathcal{L} + \kappa\mathcal{D} \text{ has a gap } \Delta' \geq \Delta + \kappa. \tag{12}$$

For fermionic systems, the added dissipator $\mathcal{D}$ comprises two linear Lindblad operators $\hat{L}_{i\pm} = \hat{w}_{i\pm}$ for every mode i [67]. For bosons, $\mathcal{D}$ comprises one operator $\hat{L}_i = \hat{w}_{i+} - \mathrm{i}\hat{w}_{i-} = \sqrt{2}\hat{a}_i$ per mode. With this choice and any $\kappa_0 > 0$, the gap stays nonzero, e.g., along the path $(1-g)\mathcal{L}_1 + g\mathcal{L}_2 + \kappa\mathcal{D}$, where the parameters are tuned as $(g,\kappa){:}\,(0,0) \to (0,\kappa_0) \to (1,\kappa_0) \to (1,0)$ to connect $\mathcal{L}_1$ to $\mathcal{L}_2$. Note that this proposition does not require short-range interactions.

The Eq. (12) statement can be proven by employing the third-quantization formalism [26,68–70] as detailed in the companion paper [26]: (a) There exist ladder superoperators $a_{j\nu}$ and $a'_{j\nu}$ that obey canonical (anti)commutation relations and form a basis for the superoperator algebra. (b) One can then construct a biorthogonal operator basis

$$\langle\langle \boldsymbol{n}| \quad \text{and} \quad |\boldsymbol{n}\rangle\rangle \quad \text{with} \quad \langle\langle \boldsymbol{n}|\boldsymbol{n}'\rangle\rangle = \delta_{\boldsymbol{n},\boldsymbol{n}'}, \tag{13}$$

occupation numbers $\boldsymbol{n}^T = (n_{1+}, \ldots, n_{N+}, \ldots, n_{N-})$, and $a'_{j\nu}a_{j\nu}|\boldsymbol{n}\rangle\rangle = n_{j\nu}|\boldsymbol{n}\rangle\rangle$. The Dirac notation with superbras $\langle\langle\hat{A}|$ and superkets $|\hat{B}\rangle\rangle$, where $\hat{A}$ and $\hat{B}$ are operators on the Hilbert space, is based on the Hilbert-Schmidt inner product $\langle\langle\hat{A}|\hat{B}\rangle\rangle \equiv \mathrm{Tr}(\hat{A}^\dagger\hat{B})$. (c) The ladder superoperators can be chosen such that the matrix representation $\langle\langle\boldsymbol{n}|\mathcal{L}|\boldsymbol{n}'\rangle\rangle$ of $\mathcal{L}$ assumes a block-triangular form when ordering the basis [Eq. (13)] according to increasing eigenvalues $N_a \in \mathbb{N}$ of the number superoperator $\mathcal{N}_a \coloneqq \sum_{j\nu} a'_{j\nu}a_{j\nu}$. The spectra of the blocks $\mathcal{L}|_{N_a}$ on the diagonal determine the full Liouvillian spectrum [71]. The only terms due to $\mathcal{D}$ that affect the blocks $\mathcal{L}|_{N_a}$ are [26]

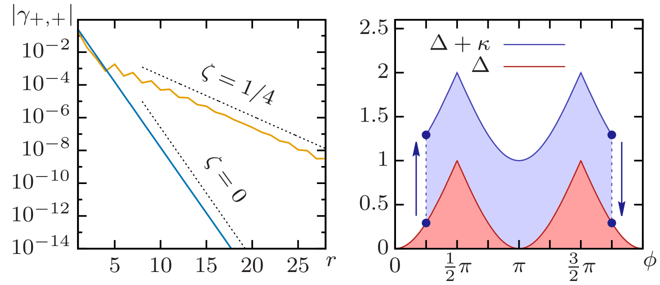

FIG. 1. The open fermionic model (15) with $\eta = 1$ and $\mu = 0$. Left: Both in the quasifree case ($\zeta = 0$) and quadratic case ($\zeta = 1/4$) with $\alpha = 1/5$ and $\phi = 2\pi/5$, correlations decay exponentially, where the asymptotic form $\sim\beta^r$ (dashed lines) is determined by an eigenvalue β of the transfer matrix in Eq. (5). Right: The dissipative gap Δ for $\zeta = 0$ and $\alpha = 1/2$ vanishes at $\phi = 0, \pi$. It can be increased using the additional dissipator $\kappa\mathcal{D}$ from Eq. (12).

$$-\boldsymbol{a}'^T\frac{B+B^*}{2}\boldsymbol{a}, \quad -\boldsymbol{a}^T\frac{B+B^*}{2}\boldsymbol{a}', \quad \boldsymbol{a}'^T U^\dagger \tau \frac{B-B^*}{2} U\boldsymbol{a} \tag{14}$$

for fermions with even N_a, fermions with odd N_a, and bosons, respectively. In Eq. (14), $\boldsymbol{a}^T = (a_{1+}, \ldots, a_{N-})$ and $\boldsymbol{a}'^T = (a'_{1+}, \ldots, a'_{N-})$ are vectors containing all ladder superoperators, U is a unitary matrix, $\tau = \left(\begin{smallmatrix} 0 & -\mathrm{i}\mathbb{1}_N \\ \mathrm{i}\mathbb{1}_N & 0 \end{smallmatrix}\right)$, and $B_{i\mu,j\nu} = \sum_s L_{s,i\mu}L^*_{s,j\nu}$ is a positive-semidefinite matrix, characterized by the expansion coefficients of the linear Lindblad operators $\hat{L}_s = \sum_{j\nu} L_{s,j\nu}\hat{w}_{j\nu}$.

For fermions, the Lindblad operators of dissipator $\mathcal{D}$ have coefficients $L_{i\pm,j\nu} = \delta_{i,j}\delta_{\pm,\nu}$ and, hence, $B = \mathbb{1}_{2N}$ such that the first two terms in Eq. (14) are simply $-\mathcal{N}_a$ and $\mathcal{N}_a - 2N$, respectively. This implies that the spectrum of block $\mathcal{L}|_{N_a}$ is shifted by $-N_a\kappa$ and $(N_a - 2N)\kappa$ for even and odd N_a, respectively. As the $N_a = 0$ block that contains the steady-state eigenvalue zero is one-dimensional, the spectral shifts due to $\kappa\mathcal{D}$ necessarily increase the gap to $\Delta' \geq \Delta + \kappa$. For bosons, we have $L_{i,j+} = \delta_{i,j}$ and $L_{i,j-} = -\mathrm{i}\delta_{i,j}$. Hence, $B = \left(\begin{smallmatrix} \mathbb{1}_N & \mathrm{i}\mathbb{1}_N \\ -\mathrm{i}\mathbb{1}_N & \mathbb{1}_N \end{smallmatrix}\right)$ and $\tau(B - B^*)/2 = -\mathbb{1}_{2N}$ such that the third term in Eq. (14) reads $-\mathcal{N}_a$. Thus, also in the bosonic case, the gap increases at least by κ. For quasifree fermionic and bosonic systems, the gap increases exactly by κ, i.e., $\Delta' = \Delta + \kappa$. Dissipator $\mathcal{D}$ is invariant under single-particle basis transformations $\hat{a}_j \leftrightarrow \sum_i U_{j,i}\hat{a}_i$ and also under particle-hole transformations $\hat{a}_j \leftrightarrow \hat{a}_j^\dagger$ for fermions. This completes the proof of Proposition 5.

Example.—To illustrate some of the above results, consider the quadratic fermionic 1D model with Hamiltonian

$$\hat{H} = \sum_j (\hat{a}_j^\dagger \hat{a}_{j+1} + \alpha\hat{a}_j^\dagger\hat{a}_{j+1}^\dagger + \mathrm{H.c.}) - \mu\sum_j \hat{a}_j^\dagger\hat{a}_j, \tag{15}$$

corresponding to a spin-1/2 XY chain, and Lindblad operators $\hat{L}_j = \sqrt{\eta}(\hat{w}_{j+} + e^{\mathrm{i}\phi}\hat{w}_{(j+1)+})$ as well as $\hat{M}_j = \sqrt{\zeta}(2\hat{a}_j^\dagger\hat{a}_j - 1)$. In accordance with Proposition 1, $\gamma(r)$ is always found to decay as β^r for an eigenvalue β of the transfer matrix in Eq. (5). Proposition 3 implies that the quasifree model ($\zeta = 0$), considered in Ref. [29], is never critical, and $\tilde{x}(k)$ determines the full many-body spectrum [26]. In particular, if the Hamiltonian is gapped and $\eta > 0$, the dissipative gap Δ closes only at $\phi = 0$ and π. The correlation length diverges at those points ($\beta \to 1$) but, at the same time, $\gamma(r) \to 0$ for all r. Furthermore, employing the additional dissipator $\kappa\mathcal{D}$ from Eq. (12), any two gapped points can always be connected by a path of gapped Liouvillians as explained by Proposition 5 and illustrated in Fig. 1 for the points $\phi = \pi/4$ and $\phi = 9\pi/4$. So, the system is neither critical at $\phi = 0$ or π, nor does it undergo phase transitions. Details are presented in the Supplemental Material [51].

Discussion.—We have found fundamental prerequisites for criticality and phase transitions in driven-dissipative many-body systems that are in stark contrast to properties of closed systems. For any number of spatial dimensions, there exist fermionic and bosonic closed systems with phase transitions and critical ground states, i.e., states featuring an algebraic decay of spatial correlations, even if the systems are quasifree. In contrast, steady states of open 1D quasifree systems as well as higher-dimensional quasifree fermionic systems are never critical. For quadratic systems, we found that, while the dissipative gap may close and the system might even be critical for certain points in parameter space, all steady states basically belong to the same phase. The only way for realizing phase transitions in such systems is to impose symmetries on the considered Liouvillians that go beyond invariance under single-particle basis transformations (e.g., lattice symmetries) and fermionic particle-hole transformations or combinations thereof. A notable example are topological transitions in quasifree systems, occurring under the (strong) restriction that the Lindblad operators from a complete anticommuting set [72,73]. The observation that dissipative phase transitions are, in the above sense, more rare than phase transitions in closed systems adds to the idea that steady states are in certain scenarios related to thermal states of closed systems [10,74–76] such that continuous symmetries cannot be broken in $D \leq 2$ dimensions according to the Mermin-Wagner theorem [77,78]. Interactions and more complex Lindblad operators can break the block-triangular Liouvillian structures [26,71] that underlie our results on quasifree and quadratic systems and can cause true phase transitions [75,79–83] as long as we are below an upper critical dimension where all systems become effectively quasifree.

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