Dynamics and Entanglement in Quantum and Quantum-Classical Systems: Lessons for Gravity

Viqar Husain,[1,*] Irfan Javed,[1,†] and Suprit Singh[2,‡]
[1]*Department of Mathematics and Statistics, University of New Brunswick, Fredericton, New Brunswick E3B 5A3, Canada*
[2]*Department of Physics, Indian Institute of Technology Delhi, Hauz Khas, New Delhi 110016, India*

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Motivated by quantum gravity, semiclassical theory, and quantum theory on curved spacetimes, we study the system of an oscillator coupled to two spin-$1/2$ particles. This model provides a prototype for comparing three types of dynamics: the full quantum theory, the classical oscillator with spin backreaction, and spins propagating on a fixed oscillator background. From calculations of oscillator and entanglement entropy evolution, we find the three systems give equivalent dynamics for sufficiently weak oscillator-spin couplings but deviate significantly for intermediate couplings. These results suggest that semiclassical dynamics with backreaction does not provide a suitable intermediate regime between quantum gravity and quantum theory on curved spacetime.

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A quantum theory of gravity (QG) is expected to provide a unification of gravity with the other forces of nature (for a recent review, see, e.g., Ref. [1]). The literature abounds with attempts to quantize gravity or simplified models of it [2] with no clear consensus so far on the approach to a final theory. If the QG turns out to be a conventional quantum theory, it will be a system with a Hilbert space $\mathcal{H} = \mathcal{H}_{\text{gravity}} \otimes \mathcal{H}_{\text{matter}}$. The matter component is in general a "multipartite" system representing several species of matter. Thus, quantum states can have matter-gravity entanglement, and the corresponding entanglement entropy would be an evolving observable.

If a QG theory were available, there would be several questions to pose. The Universe we observe is well described by quantum fields on either a background of an expanding cosmology on large scales or a flat spacetime on smaller scales. One of the important questions is how such an approximation emerges dynamically from quantum gravity [3,4]. In between quantum gravity and quantum fields on a classical background spacetime, there is the intermediate regime of classical gravity coupled to quantum matter with backreaction. A proposal for this intermediate regime is the much studied semiclassical Einstein equation [5–7]:

$$G_{ab}(g) = 8\pi G \langle \Psi | \hat{T}_{ab}(g, \hat{\phi}) | \Psi \rangle. \tag{1}$$

If this equation can be properly defined and solved, it would provide an association of a quantum state $|\psi\rangle$ of matter with a classical metric g (viewed in the Heisenberg picture). This is a nonperturbative hybrid classical-quantum equation; it raises many questions, such as what is the physical interpretation of the metric corresponding to a linear or entangled combination of matter states, and how exactly the right-hand side is to be defined if the metric is not known explicitly [8]. There are other hybrid models of this type: the so-called Newton-Schrödinger equation [9,10], which is a Friedmann-Schrödinger generalization to cosmology and related work [11,12]; and linear state evolution models using generalizations of the Lindblad equation [13].

In a gravity-matter system, it is of interest to study and compare three types of dynamics. These are the full quantum evolution, a suitably defined hybrid quantum-classical evolution with backreaction, and quantum evolution with no backreaction on the classical system. In the weak gravity regime, the gravitational field may be viewed as "heavy" and slowly varying, and it is weakly coupled to much lighter and faster moving matter. In this regime of couplings, it is natural to expect that gravity behaves classically. On the other hand, in the deep QG regime, matter-gravity coupling would be strong and could produce highly entangled states. Although a study of such comparative dynamics is technically challenging at the field theoretic level, it is relatively accessible in simpler models of gravity, such as cosmologies coupled to matter and nongravitational systems.

In this Letter, we study this set of questions in a model that has been a mainstay for work in atomic physics and quantum optics: the system of an oscillator coupled to a particle with spin, known as the Jaynes-Cummings model. We consider a slightly more general model of an oscillator coupled to two spin-$1/2$ particles together with a spin-spin coupling. In addition to the full quantum case, we utilize this model in a new way by defining a coupled classical-quantum model with spin backreaction on the oscillator; and we utilize another without backreaction, where the two spins propagate on an "oscillator background." The former

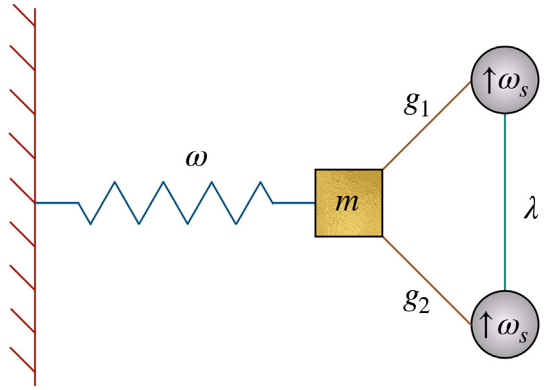

FIG. 1. Oscillator coupled to two spin-1/2 particles.

case resembles a Hamiltonian version of the semiclassical Einstein equation, whereas the latter may be viewed as a simple case of quantum theory on curved spacetime. We study the comparative dynamics numerically for a variety of initial states in the quantum-quantum (QQ) case and compare it with the dynamics in the semiclassical (SC) and classical background (CB) cases. (A related hybrid model with different dynamics was studied in [14].) Our main result is that the dynamics in the three models agrees for sufficiently small spin-oscillator coupling, but the SC case deviates significantly as this coupling is increased. Furthermore, initial product spin states can become maximally entangled in the SC and CB cases, and the SC case has unusual static solutions not present in the other cases. We discuss implications of these results for gravitational systems.

The system we consider is shown schematically in Fig. 1. The oscillator takes the place of gravity, and the spins correspond to matter. The first case is quantum-quantum, where the entire system is quantized; the second is the coupled classical oscillator-spin system with backreaction; and the third is (quantum) spin dynamics on a fixed classical oscillator background. (Generalizations of such models to gravity may be achieved by extending, e.g., the scalar-cosmology case discussed recently [15].)

Quantum oscillator-spin (QQ).—The Hilbert space of the model for this case is the tensor product of the individual Hilbert spaces of the oscillator and the two spins, $\mathcal{H} = \mathcal{H}_o \otimes \mathcal{H}^{(1)}_{\frac{1}{2}} \otimes \mathcal{H}^{(2)}_{\frac{1}{2}}$; and the Hamiltonian is

$$\begin{aligned} H = &\left(\frac{p^2}{2m}+\frac{1}{2}m\omega^2x^2\right)\otimes(I^{(1)}\otimes I^{(2)}) + I\otimes\frac{\omega_S}{2}(\sigma_z^{(1)}\otimes I^{(2)} \\ &+ I^{(1)}\otimes\sigma_z^{(2)}) + \frac{g_1}{2}(a\otimes\sigma_+^{(1)} + a^\dagger\otimes\sigma_-^{(1)})\otimes I^{(2)} \\ &+\frac{g_2}{2}(a\otimes I^{(1)}\otimes\sigma_+^{(2)} + a^\dagger\otimes I^{(1)}\otimes\sigma_-^{(2)}) \\ &+\frac{\lambda}{2}I\otimes(\sigma_+^{(1)}\otimes\sigma_-^{(2)}+\sigma_-^{(1)}\otimes\sigma_+^{(2)}) \\ \equiv\ & h_o + h_s + h_{\rm os} + h_{\rm ss}, \end{aligned} \tag{2}$$

where σ_z and $\sigma_\pm$ are the Pauli diagonal and ladder operators, and $a = x\sqrt{m\omega/2} + ip/\sqrt{2m\omega}$. The first two terms in Eq. (3) are the Hamiltonians of the noninteracting oscillator and spins (h_o and h_s), the second two are the interactions of the oscillator with each of the spins with coupling constants $g_1/2$ and $g_2/2$ ($h_{\rm os}$), and the third is the spin-spin interaction with coupling $\lambda/2$ ($h_{\rm ss}$).

We restrict our attention to the 4×4 truncation of the oscillator Hamiltonian and consider initial states that are linear combinations of the ground and first excited states. This ensures that the coupled quantum dynamics remains in the 16-dimensional Hilbert space $\mathcal{H} = \mathcal{H}_o \otimes \mathcal{H}^{(1)}_{1/2} \otimes \mathcal{H}^{(2)}_{1/2}$. Thus, the time dependent Schrödinger equation (TDSE) in this truncation is a set of 16 coupled ordinary differential equation (ODEs).

Semiclassical oscillator-spin (SC).—In this case, the oscillator is classical with orbits in the $\mathbb{R}^2$ phase space with coordinates (x, p), and the spin state is given by a vector in the Hilbert space $\mathcal{H}^{(1)}_{1/2} \otimes \mathcal{H}^{(2)}_{1/2}$. The Hamiltonian is

$$\begin{aligned} H = &\left(\frac{p^2}{2m}+\frac{1}{2}m\omega^2x^2\right)(I^{(1)}\otimes I^{(2)}) \\ &+\frac{\omega_S}{2}(\sigma_z^{(1)}\otimes I^{(2)} + I^{(1)}\otimes\sigma_z^{(2)}) + \frac{g_1}{2}(a\sigma_+^{(1)} + a^*\sigma_-^{(1)})\otimes I^{(2)} \\ &+ I^{(1)}\otimes\frac{g_2}{2}(a\sigma_+^{(2)} + a^*\sigma_-^{(2)}) \\ &+\frac{\lambda}{2}(\sigma_+^{(1)}\otimes\sigma_-^{(2)}+\sigma_-^{(1)}\otimes\sigma_+^{(2)}) \equiv h_o^{\rm SC} + h_s^{\rm SC} + h_{\rm os}^{\rm SC} + h_{\rm ss}^{\rm SC}, \end{aligned} \tag{3}$$

where each component is defined as for the fully quantum case. However x, p, a, and a^* are now classical variables. We define the coupled dynamics with the TDSE for the spins and the Hamilton equations for the oscillator:

$$i\frac{d}{dt}|\Psi\rangle = (h_s^{\rm SC} + h_{\rm os}^{\rm SC} + h_{\rm ss}^{\rm SC})|\Psi\rangle \tag{4}$$

$$\dot{q} = \{q, H_{\rm eff}\}, \qquad \dot{p} = \{p, H_{\rm eff}\}, \tag{5}$$

where $|\Psi\rangle$ is a spin state, and

$$H_{\rm eff}(x, p) \equiv \langle\Psi|H|\Psi\rangle. \tag{6}$$

This is a set of six coupled ODEs to be solved with initial dataset $\{x_0, p_0, |\psi\rangle_0\}$. The quantum dynamics is unitary by definition, and it is readily verified using the evolution equations in which the Hamiltonian $H_{\rm eff}$ is a constant of motion.

Spins on classical (oscillator) background (CB).—This case is the simplest of the three. We define it by fixing an classical oscillator "background" solution $[x_c(t), p_c(t)]$ and $a_c = x_c\sqrt{m\omega/2} + ip_c/\sqrt{2m\omega}$ with parameters m and ω, as well as the time dependent spin Hamiltonian

$$H = \frac{\omega_S}{2}(\sigma_z^{(1)} \otimes I^{(2)} + I^{(1)} \otimes \sigma_z^{(2)}) + \frac{g_1}{2}(a_c(t)\sigma_+^{(1)} + a_c^*(t)\sigma_-^{(1)}) \otimes I^{(2)} + I^{(1)} \otimes \frac{g_2}{2}(a_c(t)\sigma_+^{(2)} + a_c^*(t)\sigma_-^{(2)}) + \frac{\lambda}{2}(\sigma_+^{(1)} \otimes \sigma_-^{(2)} + \sigma_-^{(1)} \otimes \sigma_+^{(2)}). \quad (7)$$

Dynamics is defined solely by the TDSE of the spin state. Thus, this is a system of four coupled ODEs for the spin state.

Comparing dynamics.—All three oscillator-spin cases defined above (QQ, SC, and SB) have dimensional parameters m, ω, ω_S, g_1, g_2, and λ. The oscillator provides "fundamental" time and length scales $1/\omega$ and $1/\sqrt{m\omega}$, respectively (with $\hbar = 1$). We set these equal to unity and measure the remaining four parameters in these units.

Comparing dynamics in the three systems is accomplished by first fixing initial data $\{x(0), p(0), |\Psi\rangle_s(0)\}$ for the SC system, and then (i) using the same initial spin state $|\Psi\rangle_s(0)$ for the QQ and CB systems, and (ii) matching initial data for the oscillator. The latter is accomplished for the CB case by using the oscillator solution that goes through the phase space point $[x(0), p(0)]$, and for the QQ case by using the product oscillator-spin state

$$|\Phi\rangle(0) = \left[\cos\left(\frac{\theta}{2}\right)|0\rangle + \sin\left(\frac{\theta}{2}\right)e^{i\phi}|1\rangle\right] \otimes |\Psi\rangle_s(0), \quad (8)$$

(where $|0\rangle$ and $|1\rangle$ are, respectively, the ground and first excited states of the oscillator) and then fixing θ and ϕ such that the expectation values of $\hat{x}$ and $\hat{p}$ match:

$$x(0) = \frac{\sin(\theta)\cos(\phi)}{\sqrt{2m\omega}}; \quad p(0) = -\sqrt{\frac{m\omega}{2}}\sin(\theta)\sin(\phi). \quad (9)$$

This ensures the closest possible initial data for the three cases and enables comparison of (x, p) and $(\langle\hat{x}\rangle, \langle\hat{p}\rangle)$ phase space trajectories as well as evolution of spin entanglement entropy and energy in each subsystem.

For all cases, we integrated the coupled differential equations numerically for a variety of initial data. The method we used ensured that the probability and H_{eff} are conserved at least to order 10^{-8}. We computed several solutions of the three cases with comparable initial data (as described above) with the aim of studying the parameter ranges where the oscillator dynamics looks similar. A representative sample is shown in Fig. 2 with parameter values $m = \omega = 1$ and $\omega_S = \lambda = 2$, as well as oscillator-spin coupling parameters $g \equiv g_1 = g_2 = 0.0001$, 0.1, and 1.5. The latter are chosen to highlight how the dynamics in phase space, spin entanglement entropy S_{ent}, and energies in the spin subsystem $E_{\mathrm{ss}} = \langle h_s + h_{\mathrm{os}} + h_{\mathrm{ss}}\rangle$ and oscillator $E_{\mathrm{osc}} = \langle H\rangle - E_{\mathrm{ss}}$ change with oscillator-spin couplings.

The initial data for the SC and CB cases in Fig. 2 are $\{q(0)=0.1, p(0)=0, |\Psi(0)\rangle=|++\rangle\}$. The corresponding initial state for the QQ system is obtained by using Eq. (9) with $\phi = 0$ and $\sin\theta = 0.1\sqrt{2}$; these are very close to the (truncated) oscillator coherent states.

We highlight the following features evident in Fig. 2: (i) for $g = 0.0001$, the oscillator phase space trajectories and subsystem energies are indistinguishable in all three cases, and the spin entanglement entropy remains nearly zero; (ii) as g increases to 0.1, differences start to appear in each of the variables plotted and (in particular) spin entanglement entropy increases from zero, attains its maximum value of $\log 2$, and oscillates notably, even for the SC and CB cases; (iii) spin entanglement entropy and subsystem energy oscillations have higher frequencies for larger g values; and (iv) the SC phase space trajectory is more expansive, with the range of the oscillator extending an order of magnitude more than that for the QQ and CB cases. Similar features are evident for other parameter values and initial data. Although it is gratifying to see that all cases approximately agree for sufficiently small g values, point (iv) above is especially noteworthy: it provides evidence that the backreaction SC model, which is very similar in form and spirit to the semiclassical Einstein equation, may not provide a reasonable transition between the QQ and CB systems. Further evidence for this view is provided in Fig. 3, which presents a measure of phase space spread in each of the three systems: $\Delta x = x_{\max} - x_{\min}$ and $\Delta p = p_{\max} - p_{\min}$ are plotted as a function of coupling g (for time runs to 100 units), where $x_{\max}$, $x_{\min}$, etc. are the largest and smallest values in the phase space plots in Fig. 2. The drastic deviation of the SC from the QQ and CB cases at $g \sim 0.1$ is evident. The plateau for large g is due to conservation of $\mathcal{H}_{\mathrm{eff}}$, which limits phase space extent.

The SC equations provide an additional curious feature not present in the QQ and CB systems: static solutions for the oscillator. These are obtained by considering eigenstates of the spin subsystem $h_s + h_{\mathrm{os}} + h_{\mathrm{ss}}$ and setting $\dot{x} = \dot{p} = 0$. The SC equations then reduce to

$$\dot{x} = \frac{p}{m} + \{x, E(a, a^*, g, \lambda)\} = 0 \quad (10)$$

$$\dot{p} = -m\omega^2 x + \{p, E(a, a^*, g, \lambda)\} = 0, \quad (11)$$

where $E(a, a^*, g, \lambda)$ are the corresponding eigenvalues. For $\lambda = 0$, there are particularly simple static solutions: any point on the circle $x^2 + p^2 = g^2/2 - 2\omega_s^2/g^2$ with $m\omega = 1$. Thus, we must have $g^2 > 2\omega_s$. The physical interpretation of the solution with $p = 0$ and $x \neq 0$ is that the stationary spin state "holds" the stretched spring of the oscillator. But, the physical interpretation of solutions with $x \neq 0$ and $p \neq 0$ are unusual: the spring is held stretched from equilibrium (because $x \neq 0$) and the

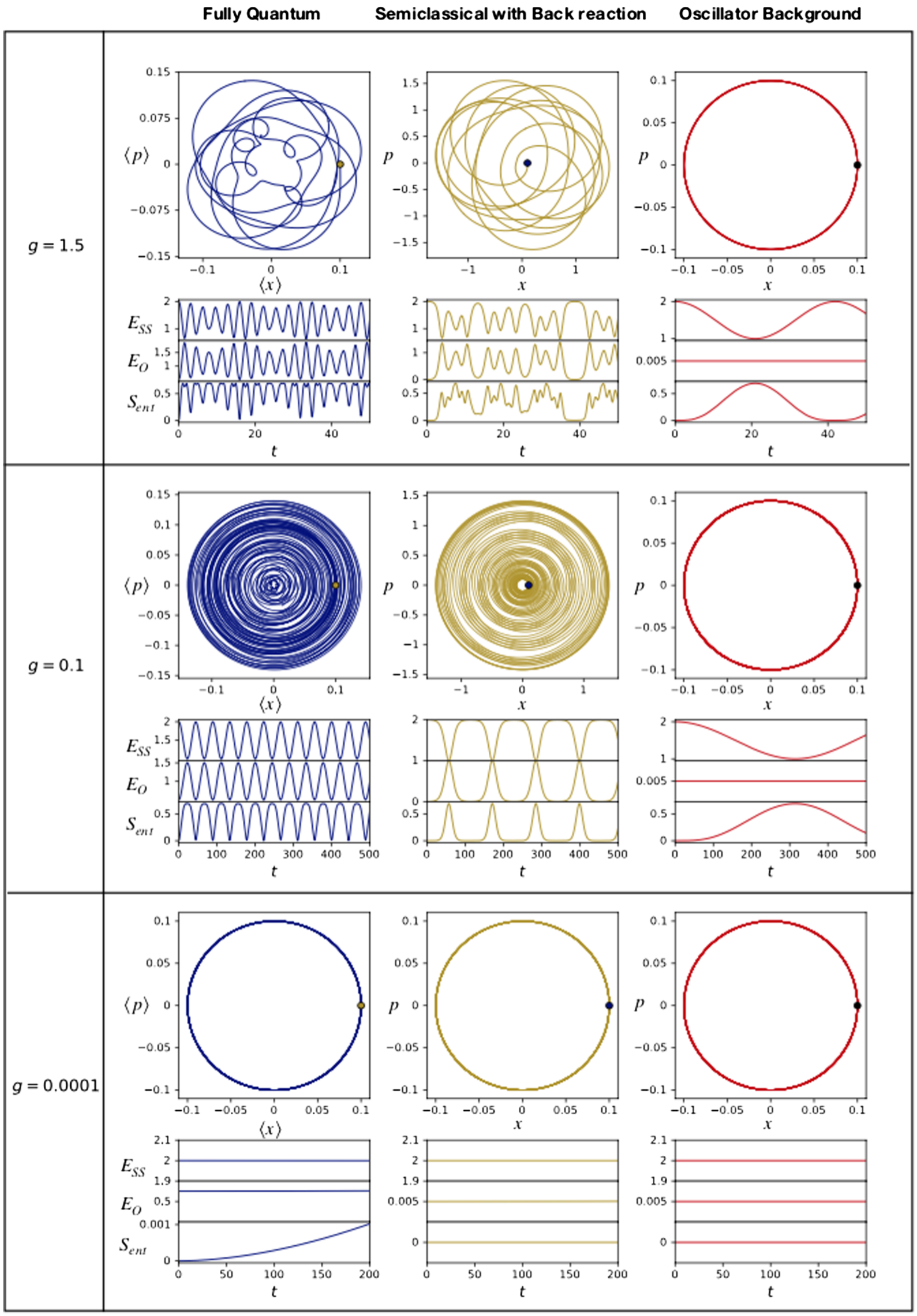

FIG. 2. Phase space, spin entanglement entropy S_{ent}, and subsystem energies E_{osc} and E_{ss} for $m = \omega = 1$ and $\omega_S = \lambda = 2$ for the g values indicated. Initial data for semiclassical cases are $x = 0.1$, $p = 0$ (indicated with a black dot), and spin state $|++\rangle$; corresponding data for the fully quantum case from Eq. (8) are $(0.99748420879|0\rangle + 0.07088902028|1\rangle)|++\rangle$. (Phase space plot axes represent $\langle x\rangle$ and $\langle p\rangle$ for QQ, as well as x and p for SC and CB cases.)

mass is in uniform motion because $p \neq 0$. Other static solutions are readily computed numerically. These solutions may be compared with static solutions of the Newton-Schrödinger equation mentioned in [16]; although the system is different, the underlying reason for their appearance is similar—the equations are

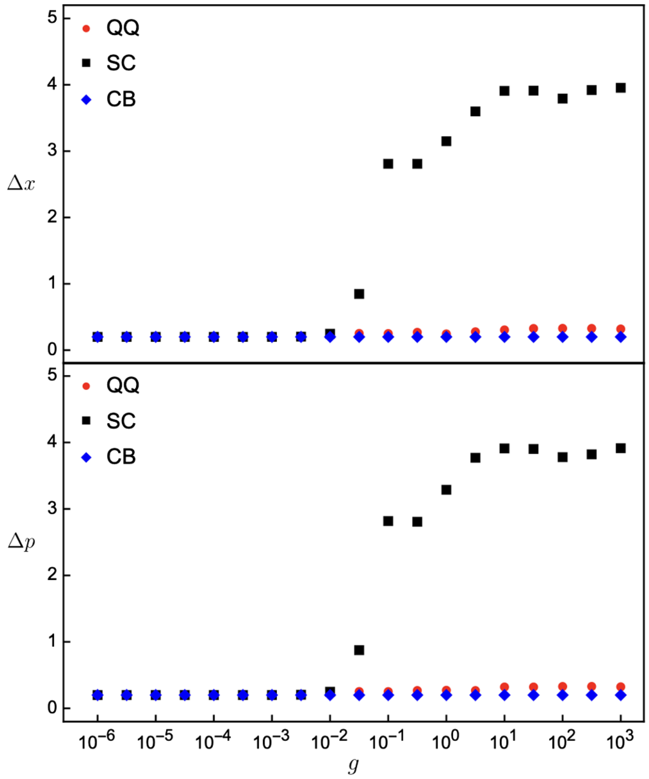

FIG. 3. A comparison of phase space spread of QQ, SC, and CB systems as a function of coupling g; deviation of SC case from others is evident. Δx and Δp are differences of maximum and minimum values in phase space plots in Fig. 2.

nonlinear in the quantum state, and so they permit a richer class of static solutions.

Discussion.—We described in detail three versions of the dynamics of the oscillator coupled to two spin-$1/2$ particles (QQ, SC, and CB) with a truncation of the oscillator to a four-level system. Our aim was to compare the dynamics of the oscillator, spin entanglement entropy, and subsystem energies for the same initial conditions. We highlight several results and comment on some implications: (1) For sufficiently small oscillator-spin couplings, the dynamics of the three systems is identical; this lends support to the idea that similar results would hold for other systems, including gravity coupled to matter. (2) Of particular note is that the SC system gives oscillator trajectories that are substantially different from the QQ system for larger oscillator-spin couplings. This may be attributed to the fact that the SC equations are nonlinear in the state, unlike the QQ and CB systems. Because the same (but more consequential) nonlinearity holds for the semiclassical Einstein equation, our results suggest that the latter does not provide the appropriate transition between quantum gravity and quantum fields on curved spacetime. (3) Spin entanglement is induced in the SC and CB systems for nonzero spin-spin coupling λ; e.g., $g = 0.1$ and $\lambda = 2$ in Fig. 2. (The CB case is similar to entanglement generation in Floquet dynamics [17].) The implication for gravity is similar: initial product states of matter can get entangled thorough the semiclassical Einstein equation, or even by propagating on a fixed but time dependent background spacetime, provided matter is self-interacting through any local field. (4) In the proposed experiments [18–20] for detecting quantization of linearized gravity through entanglement generation in mass states, it is posited that interaction between masses is not action at a distance (as it is here for spin-spin), but it is instead generated via a mediating quantum gravitational field. However, if the masses are sufficiently close in a laboratory setting, a point interaction may be a good approximation; and any entanglement generated through nongravitational quantum interactions, whether local or not, could be significant, as demonstrated in the model discussed here. (See Ref. [21] for related discussion.) In the final analysis, a quantum interaction is of course necessary to generate entanglement between masses, whatever its origin, and the spin-spin interaction in the present model is a stand-in for that. (5) There is a curious case for the SC model where, for the $|++\rangle$ or $|--\rangle$ initial states, there is no entanglement induced for $g = 0$ and $\lambda \neq 0$. Then, increasing g from zero (i.e., turning on the classical coupling) induces entanglement; this special case is an exception to the proof in [22], which covers the $\lambda = 0$ case.

What are the lessons for gravity that can be inferred from the models we present? The "coupling constant" between gravity and matter in the Hamiltonian formulation is $\sqrt{q}$, which is the square root of the determinant of the spatial metric; e.g., for a scalar field, the Hamiltonian density is $\mathcal{H} = p_\phi^2/(2\sqrt{q}) + \sqrt{q}V(\phi)$. Thus, in the semiclassical approximation of the type we consider here, the coupling constant evolves with the classical gravitational dynamics. In an expanding universe, the kinetic term decreases and the potential term increases; hence, the metric-matter coupling increases. This is precisely the regime where our model indicates significant deviation from the QQ and CB cases. And, similarly, the gravitational SC equations would be nonlinear in the state, unlike the QQ and SC cases, and would give rise to static solutions, as in the present model.

Our results suggest several areas for further investigation. These include considering in the same spirit cosmological and other gravitational models with scalar and/or spinorial fields by extending the work in [15]; a field theoretic version of the SC model for studying backreaction in gravity coupled to a scalar field in spherically symmetric gravity (where matter-gravity entanglement is a potentially important feature [23]); and linear alternatives to the semiclassical Einstein equation, as discussed in [13], applied to similar model systems: the latter may address the issue of the significant difference between the QQ and SC systems for the results presented here.

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*vhusain@unb.ca
†i.javed@unb.ca
‡suprit@iitd.ac.in

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