Nonlinear Transport by Bethe Bound States

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We consider nonlinear ballistic spin transport in the XXZ spin chain and derive an analytical result for the nonlinear Drude weight $D^{(3)}$ at infinite temperatures. In contrast to the linear Drude weight $D^{(1)}$, we find that the result not only depends on anisotropy but also on the string length of the quasiparticles transporting the spin current. Our result provides further insights into transport by quasiparticles and raises questions about Luttinger liquid universality.

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Introduction.—Transport in metals is usually well described by a Boltzmann equation for long-lived quasiparticles [1,2]. Standard transport theory, though, cannot necessarily be expected to hold for one-dimensional quantum systems. Here, Fermi liquid theory breaks down and the low-energy properties are instead described by Luttinger liquid theory, which is based on collective excitations rather than quasiparticles [3]. For such systems, transport in the linear response regime can be studied using standard many-body techniques based on the Kubo formulas [4]. For a clean one-dimensional metal at zero temperature, one finds that the dc conductivity has a zero-frequency Drude peak D whose weight is determined entirely by the velocity of the collective excitations v and the Luttinger parameter K [5,6].

At finite temperatures, currents are always able to relax even in a clean one-dimensional lattice system as long as at least two noncommuting Umklapp scattering processes are present [7]. This typically leads to diffusive transport [8–10]. An exception is one-dimensional integrable models that have an extensive number of local conservation laws which can protect a current from decaying completely [10–14]. Integrable models are not just a mathematical oddity: close realizations have also attracted considerable experimental attention [15–21]. In particular, good realizations of the spin-$1/2$ Heisenberg chain—which can also be viewed as a system of interacting spinless fermions—have recently been achieved in cold atomic gases and dynamical and transport properties have been studied extensively [22–26].

Gapless integrable quantum systems show an interesting duality: Thermodynamic quantities at low energies and zero temperature transport properties can be calculated essentially exactly within the Luttinger liquid framework if integrability is understood as a fine-tuning of the parameters in the field theory fixing the velocity of the collective excitations, the Luttinger parameter, and the amplitudes of irrelevant operators [14,27–29]. At the same time, thermodynamic quantities can also be calculated using the thermodynamic Bethe ansatz (TBA), which deals with particles and their bound states. Thus, both a picture of collective excitations and of quasiparticles seems to be equally valid.

For the finite temperature transport properties of the anisotropic spin-$1/2$ Heisenberg (XXZ) chain, however, this no longer seems to be the case. It was recently demonstrated using the TBA formalism that the Drude weight of this model is fractal as a function of anisotropy not only at infinite [30,31] but also at low temperatures [32], while the Drude weight is continuous and Luttinger liquid theory does hold at zero temperature [8,9,14]. A similar fractal structure has also been obtained for the return probability following a quench from a domain-wall state [33]. More generally, it has been argued that the nonequilibrium properties of integrable models can be understood using a Bethe-Boltzmann equation that includes densities for all conserved charges leading to a generalized hydrodynamical (GHD) description [34–39]. However, while the Drude weight is fractal for all finite temperatures, it strangely does not reveal the full particle content of the model: the quantum numbers of the type of magnon bound state carrying the current for a given anisotropy do not directly enter.

In this Letter, we will extend the study of finite-temperature transport in the anisotropic spin-$1/2$ Heisenberg chain to the nonlinear response regime. We will, in particular, study the most singular part of the nonlinear response obtained when all relevant frequencies in a generalized conductivity are sent to zero, leading to the recently introduced notion of nonlinear Drude weights [40,41]. These weights can be understood as a straightforward generalization of the Kohn formula [42,43], relating the Drude weights to changes in the individual energy levels when a magnetic flux ϕ pierces a ring. Our main result is an exact formula for the nonlinear Drude weight $D^{(3)}$ at infinite temperatures, demonstrating that in this response function the full particle content of the model is revealed.

Nonlinear Drude weights.—It has been shown by Kohn [42] that the Drude weight at zero temperature can be understood as the response of the ground state energy to a magnetic flux through a ring. Alternatively, this can also be understood as the response to a twist in the boundary conditions. Later, this result was generalized to finite temperatures [43]. In general, we can express all Drude weights as

$$D^{(l)} = \frac{N^l}{Z}\sum_i \mathrm{e}^{-\beta E_i}\frac{\partial^{l+1}E_i}{\partial\phi^{l+1}}\bigg|_{\phi=0}. \quad (1)$$

Here, Z is the partition function, N the number of sites, E_i the energy levels, and $\beta = 1/T$ the inverse temperature. The linear Drude weight $D^{(1)}$ is given by the curvature of the energy levels, while nonlinear Drude weights ($l > 1$) correspond to higher order derivatives. Note that because of the $\phi \to -\phi$ symmetry of the response, all Drude weights with l even will vanish. The linear Drude weight can be obtained as $D^{(1)} = \lim_{t\to\infty}\lim_{N\to\infty}(1/2NT)\langle J(t)J(0)\rangle$, where $J(t)$ is the current operator at time t and $\langle\cdots\rangle$ denotes the thermal average [9,10]. It thus describes the nondecaying, ballistic part of the current. In a similar way, the nonlinear Drude weights can be understood as being given by more complicated multipoint current correlation functions [41].

A simplification of the generalized Kohn formula, Eq. (1), is obtained by noting that a one-dimensional system does not support a persistent current at zero flux in the thermodynamic limit. Therefore, $\lim_{N\to\infty}\partial^l f/\partial\phi^l|_{\phi=0} = 0$ where f is the free energy density. This relation can be used to express $D^{(l)}$ by flux derivatives of order l and lower only. A straightforward calculation shows, in particular, that

$$\begin{aligned} D^{(3)} &= \frac{N^3}{Z}\sum_i \mathrm{e}^{-\beta E_i}\frac{\partial^4 E_i}{\partial\phi^4}\bigg|_{\phi=0} \\ &= \frac{N^3\beta}{Z}\sum_i \mathrm{e}^{-\beta E_i}\{\beta^2\dot{E}_i^4 - 6\beta\dot{E}_i^2\ddot{E}_i + 3\ddot{E}_i^2 + 4\dot{E}_i\dddot{E}_i\}, \end{aligned} \quad (2)$$

where the dots denote flux derivatives. This formula is also useful for numerical evaluations; see the Supplemental Material [44].

Model and noninteracting limit.—In the following, we will calculate $D^{(3)}$ for the spin-1/2 XXZ chain,

$$H = \frac{1}{4}\sum_{j=1}^{N}(e^{-\mathrm{i}\frac{\phi}{N}}\sigma_j^+\sigma_{j+1}^- + e^{\mathrm{i}\frac{\phi}{N}}\sigma_j^-\sigma_{j+1}^+ + \Delta\sigma_j^z\sigma_{j+1}^z), \quad (3)$$

for anisotropies in the critical regime, $|\Delta| = |\cos(\gamma)| < 1$, with $\gamma = n\pi/m$, and $n, m \in \mathbb{Z}$. The model can be mapped to spinless fermions by a Jordan-Wigner transform with Δ then describing the strength of a nearest-neighbor interaction. For $\Delta = 0$, the model is equivalent to free spinless fermions. Like the linear Drude weight $D^{(1)}$ [5], the nonlinear Drude weights for the XXZ chain also exhibit a smooth dependence on the anisotropy at $T = 0$ [46]; however, the nonlinear Drude weights $D^{(l)}$, $l > 1$, diverge if $\gamma < ((l-1)/(l+3))\pi$, the cause being nonanalytic flux corrections, which stem from Umklapp scattering.

The $T > 0$ linear Drude weight for the XXZ chain was first calculated in Ref. [11]. Here, it was noted that it has the same structure as in the noninteracting case once the noninteracting particles are replaced by the appropriately dressed quasiparticles and their distribution functions. Such an equivalence was also later found for heat transport [12]. The same quasiparticle structure is also underlying the GHD approach that has been successfully applied to integrable systems in the linear response regime and beyond [34–36,38,47]. In particular, the analytical expression for the infinite temperature linear Drude weight first derived in Refs. [30,31,48,49] based on a set of quasilocal conserved charges has been reproduced in GHD [50] and also directly from the TBA equations [51]. An extension to the GHD framework was proposed in [52] for numerically calculating nonlinear conductivities, which was then applied to the case of $\Delta > 1$. At the moment there are, however, no nonzero temperature results for the nonlinear Drude weights in the critical regime of the XXZ spin chain.

Let us first calculate $D^{(3)}$ in the noninteracting case, $\Delta = 0$. In this case, the free energy density is given by $f = -N^{-1}T\sum_k \ln[1 + \exp(-\beta\varepsilon_k)]$ with dispersion $\varepsilon_k = -\cos(k + \phi/N)$. It is now straightforward to take derivatives with respect to the flux ϕ. Using again that $\partial^4 f/\partial\phi^4 = 0$ in the thermodynamic limit, we arrive at the following expression for the nonlinear Drude weight:

$$\begin{aligned} D^{(3)} &= N^3\sum_k n_k\varepsilon_k^{(4)} \\ &= N^3\beta\sum_k n_k\bar{n}_k\{3\ddot{\varepsilon}_k^2 + 4\dot{\varepsilon}_k\dddot{\varepsilon}_k + 6\beta(2n_k - 1)\dot{\varepsilon}_k^2\ddot{\varepsilon}_k \\ &\quad + \beta^2(1 - 6n_k\bar{n}_k)\dot{\varepsilon}_k^4\}. \end{aligned} \quad (4)$$

Here, $n_k = 1/(\exp(\beta\varepsilon_k) + 1)$ is the Dirac distribution and $\bar{n}_k = 1 - n_k$. Equation (4) can be evaluated numerically for all finite temperatures and in closed form at zero and infinite temperatures; see the Supplemental Material [44].

The interacting case.—From the Bethe ansatz equations, the quasiparticle momenta and phase shifts for the XXZ model can be obtained. Both can be characterized by so-called rapidities that arrange themselves, according to the string hypothesis, in regular patterns in the complex plane [53]. These strings describe bound states of magnons. Using the string hypothesis leads to the TBA equations that determine the Fermi weights ϑ_α and dressed energies $\tilde{\varepsilon}_\alpha$ of the quasiparticles. Remarkably, as we will see below, by simply replacing bare by dressed quantities in Eq. (4), $D^{(3)}$ in the interacting case is immediately obtained.

A mathematically more rigorous proof is provided in the Supplemental Material [44].

The flux ϕ enters the Hamiltonian, Eq. (3), in the usual Peierls construction as a phase factor $\sim\exp(\pm i\phi/N)$. It is equivalent to a twist in the boundary conditions for the wave function $\Psi(l+N)=e^{i\phi}\Psi(l)$. As such, it will not affect the properties of the system in the thermodynamic limit. We therefore need to calculate the finite-size corrections to the energy eigenvalues. Here, we follow the approach in Refs. [11,54] for the linear Drude weight.

The rapidities θ_α, which determine the quasimomenta k_α, fulfill the Bethe ansatz equations

$$\left[\frac{\sinh\frac{\gamma}{2}(\theta_\alpha+i)}{\sinh\frac{\gamma}{2}(\theta_\alpha-i)}\right]^N=-e^{i\phi}\prod_{\alpha'=1}^{M}\frac{\sinh\frac{\gamma}{2}(\theta_\alpha-\theta_{\alpha'}+2i)}{\sinh\frac{\gamma}{2}(\theta_\alpha-\theta_{\alpha'}-2i)} \tag{5}$$

for $\alpha=1,\ldots,M$, where M is the number of spins down. The rapidities arrange themselves in regular patterns (strings) in the complex plane with the type of allowed strings α determined by the anisotropy $\Delta=\cos\gamma$. For a finite system of length N, we can now for each string type expand the rapidities θ_α around the thermodynamic limit, resulting in

$$\theta_\alpha^N=\theta_\alpha^\infty+g_{1\alpha}\frac{\phi}{N}+g_{2\alpha}\left(\frac{\phi}{N}\right)^2+g_{3\alpha}\left(\frac{\phi}{N}\right)^3+\cdots. \tag{6}$$

In the thermodynamic limit, we can obtain the densities of the θ_α in terms of string densities ρ_α and hole densities ρ_α^h, which can be combined into total densities $\rho_\alpha^T=\rho_\alpha+\rho_\alpha^h$ fulfilling the standard TBA integral equation

$$\rho_\alpha^T(\theta)=\frac{k'_\alpha(\theta)}{2\pi}+\sum_\beta\int d\lambda K_{\alpha\beta}(\theta-\lambda)\rho_\beta(\lambda), \tag{7}$$

where $K_{\alpha\beta}(\theta-\lambda)$ denotes the TBA kernel; see the Supplemental Material [44] for details.

From here, we can determine the finite size corrections $g_{j\alpha}$ in Eq. (6). In an expansion to leading order in inverse temperature β we find, in particular,

$$\rho_\alpha^T g_{1\alpha}=\frac{\tilde{q}_\alpha}{2\pi}=\frac{m}{4\pi}(\delta_{\alpha,m}+\delta_{\alpha,m-1}),$$
$$g_{n\alpha}=\frac{1}{n}g_{1\alpha}\partial_\theta g_{(n-1)\alpha}\quad\text{for } n>1, \tag{8}$$

with the dressed energy given by $\tilde{\varepsilon}_\alpha=(2\pi\sin\gamma/\gamma)\sigma_\alpha\rho_\alpha^T$ and $\tilde{q}_\alpha$ denoting the dressed spin. Furthermore, $\sigma_\alpha=\pm1$ is a sign that depends on the type of string. We note that Eq. (8) reveals a very simple structure of the g functions at infinite temperature: they are given by taking recursively higher derivatives of the inverse of the dressed energy. A dressed function $\tilde{f}_\alpha$ is related to its bare quantity by the TBA integral relation

$$\tilde{f}_\alpha(\theta)-\sum_\beta\int d\lambda K_{\alpha\beta}(\theta-\lambda)\vartheta_\beta(\lambda)\tilde{f}_\beta(\lambda)=f_\alpha(\theta). \tag{9}$$

The Fermi weights ϑ_α are given by $\vartheta_\alpha=(1+\eta_\alpha)^{-1}$ with $\eta_\alpha=\rho_\alpha^h/\rho_\alpha$. The infinite temperature case is particularly simple because dressed derivatives of the energy can directly be replaced by derivatives of the dressed energy $\tilde{\varepsilon}_\alpha(\theta)$. The derivatives of the latter with respect to the flux ϕ can straightforwardly be expressed by the finite-size corrections $g_{j\alpha}$ using Eq. (6). We find, in particular,

$$\left.\frac{\partial\tilde{\varepsilon}}{\partial\phi}\right|_{\phi=0}=\partial_\theta\tilde{\varepsilon}\frac{g_1}{N},\qquad\left.\frac{\partial^2\tilde{\varepsilon}}{\partial\phi^2}\right|_{\phi=0}=2\partial_\theta\tilde{\varepsilon}\frac{g_2}{N^2}+\partial_\theta^2\tilde{\varepsilon}\frac{g_1^2}{N^2}$$
$$\left.\frac{\partial^3\tilde{\varepsilon}}{\partial\phi^3}\right|_{\phi=0}=6\partial_\theta\tilde{\varepsilon}\frac{g_3}{N^3}+6\partial_\theta^2\tilde{\varepsilon}\frac{g_1g_2}{N^3}+\partial_\theta^3\tilde{\varepsilon}\frac{g_1^3}{N^3}. \tag{10}$$

Note that we have suppressed the subscript α for the string type. We are now in a position to express the Drude weight $D^{(3)}$ in Eq. (4) by the dressed energies and their distribution functions. To do so, we replace $(1/N)\sum_k\to\sum_\alpha\int d\theta\rho^T$, $n_k\to\vartheta_\alpha$, and $\partial_\phi\varepsilon_k\to\partial_\phi\tilde{\varepsilon}$.

High-temperature asymptotics.—This then leads us to the following result for the high-temperature asymptotics with the index α suppressed and the $g_{j\alpha}$ functions given by Eq. (8):

$$D^{(3)}\approx\beta\sum_\alpha\int d\theta\rho^T\vartheta(1-\vartheta)\{(\partial_\theta\tilde{\varepsilon})^2(12g_2^2+24g_1g_3)+36(\partial_\theta\tilde{\varepsilon})(\partial_\theta^2\tilde{\varepsilon})g_1^2g_2+[3(\partial_\theta^2\tilde{\varepsilon})^2+4(\partial_\theta\tilde{\varepsilon})(\partial_\theta^3\tilde{\varepsilon})]g_1^4\}. \tag{11}$$

Note that $g_{j\alpha}\sim(\tilde{q}_\alpha)^j$ with the dressed spin given by $\tilde{q}_\alpha=(m/2)(\delta_{\alpha,m}+\delta_{\alpha,m-1})$. Therefore, only the last and second last strings will contribute to the sum over α in Eq. (11) and both can be shown to give the same contribution [48,55]. Equation (11) is one of our main results and can be evaluated numerically using standard TBA equations for η_α and $\tilde{\varepsilon}_\alpha=\partial_\beta\ln\eta_\alpha$. In order to carry out the explicit high-temperature asymptotic calculation, the underlying T and Y systems [56] for the XXZ chain are used to determine the last Fermi weight. The setup for this analysis is carried out in more detail in [32,51] for the high- and low-temperature asymptotics, respectively, and more details are given in the Supplemental Material [44]. Here, we want to demonstrate the method by first concentrating on the simple roots of unity case, $\Delta=\cos\gamma=\cos(\pi/m)$. In this case, it is straightforward to show that for the last string $\eta_\alpha=\rho_\alpha^h/\rho_\alpha=m-1$ for $\beta\to0$. For the Fermi weights it follows that $\vartheta_\alpha(1-\vartheta_\alpha)=\eta_\alpha/(1+\eta_\alpha)^2=(m-1)/m^2$. To obtain the dressed energy $\tilde{\varepsilon}_\alpha=\partial_\beta\ln\eta_\alpha$ to leading order, the first correction linear in β to the η_α function is needed. One finds for $\alpha=m-1$

$$\tilde{\varepsilon}_\alpha = \frac{m}{4(m-1)} \frac{\sin^2\gamma}{\cosh\frac{\gamma}{2}(\theta_\alpha + \mathrm{i})\cosh\frac{\gamma}{2}(\theta_\alpha - \mathrm{i})}. \quad (12)$$

Note that the prefactor $m/(m-1) = [m\vartheta_\alpha(1-\vartheta_\alpha)]^{-1}$, i.e., each factor of $\tilde{\varepsilon}_\alpha$ brings in an inverse power of the distribution functions. We will see below that this explains why the particle content is hidden in the linear Drude weight while it is fully visible in the nonlinear Drude weights. Finally, we note that the total density $\rho_\alpha^T = \rho_\alpha + \rho_\alpha^h$ can also be expressed by the dressed energy resulting in $\rho_\alpha^T = \sigma_\alpha(\gamma/2\pi\sin\gamma)\tilde{\varepsilon}_\alpha$. It is now straightforward to use the high-temperature g functions, Eq. (8), to express the nonlinear Drude weight, Eq. (11), in terms of the Fermi weights ϑ_α and dressed energies $\tilde{\varepsilon}_\alpha$. The integral over the rapidities θ is then a convergent integral over hyperbolic functions that can be evaluated in closed form. The final result can be expressed as

$$TD^{(l)} = (-1)^{\frac{l-1}{2}} \frac{m^{2l-2}}{8}\{\vartheta(1-\vartheta)\}^{l-1} \times \frac{\sin^2(\gamma)}{\sin^2(\frac{\pi}{m})}\left[1 - \frac{(3l-2)m}{2\pi}\sin\left(\frac{2\pi}{m}\right)\right], \quad (13)$$

with $l = 3$ and we have suppressed the string index, which corresponds to either the last or second last string. In the Supplemental Material [44], we explicitly show that the result above is valid for all anisotropies $\gamma = n\pi/m$ with $\gamma > \pi/3$. It is remarkable that the analytical expression for $D^{(3)}$ is quite similar to the one for the linear Drude weight $D^{(1)}$ [31], which is obtained from Eq. (13) by setting $l = 1$.

A couple of comments are in order: (i) In addition to the corrections $(\phi/N)^j$ with j integer in Eq. (6) there will also be corrections with noninteger powers. It is well known that the leading irrelevant operator for the XXZ chain stems from Umklapp scattering and has scaling dimension $2\pi/(\pi-\gamma)$. Umklapp scattering leads to a $(\phi/N)^{4\pi/(\pi-\gamma)-3}$ correction to the energies. To calculate the Drude weight $D^{(l)}$, we need an lth derivative with respect to the flux; see Eq. (2). Therefore, the expansion, Eq. (6), is no longer valid if $\gamma/\pi < (l-1)/(l+3)$ [57]. While our main result, Eq. (13), remains finite in the entire critical regime $|\Delta| < 1$, it is therefore only valid for $-1 < \Delta < 1/2$. (ii) While the result for $D^{(3)}$ in Eq. (13) looks strikingly similar to the result for the linear Drude weight, there is a very important difference: while $D^{(3)}$ does explicitly contain the Fermi weight ϑ, $D^{(1)}$ does not. Since $\vartheta \sim \bar{\mu}/m$ [44], where $\bar{\mu}$ is the length of the last string, $D^{(3)}$ reflects the full bound-state particle content. For example, for $n = m-3$ the string length can take the two values $\bar{\mu} = (m\pm 1)/3$. $D^{(3)}$ is therefore even "more fractal" than $D^{(1)}$. This statement can be made more precise: Let us consider approaching irrational anisotropies by sending $n, m \to \infty$ with n/m fixed. We find $D^{(1)} \sim (\beta/12)\sin^2\gamma$, i.e., there is a continuous lower envelope function. On the other hand, $D^{(3)} \sim (3\beta/4)[(m\bar{\mu}(m-\bar{\mu})\sin\gamma/\pi)]^2$, which means that the nonlinear Drude weight is always diverging when approaching an irrational anisotropy value. When integrability is weakly broken, leading to a broadening of the δ functions in frequency, we might therefore expect a finite diffusion constant proportional to the envelope function in linear response while the nonlinear low-frequency response is expected to be very large or even divergent.

From a phenomenological perspective, we expect that if there are stable quasiparticles that contribute to the transport, then their properties should enter the Drude weights explicitly. It is therefore surprising—an issue that does not seem to have been addressed explicitly so far—that the linear Drude weight $D^{(1)}$ does not depend on the string length $\bar{\mu}$ but rather only on n, m, which determine the anisotropy $\Delta = \cos(n\pi/m)$. Looking at the Drude weights $D^{(l)}$ more generally we see that $D^{(1)} \sim \int d\theta\vartheta(1-\vartheta)(\partial_\theta\tilde{\varepsilon})^2/\tilde{\varepsilon}$ is an exceptional case. The dressed energy contains a factor $\tilde{\varepsilon} \sim [\vartheta(1-\vartheta)]^{-1}$ so that the dependence on the Fermi weight $\vartheta \sim \bar{\mu}/m$ and thus the dependence on the string length $\bar{\mu}$ exactly cancels out in this case. More generally, we expect that the scaling with l in the first line of Eq. (13) also holds for $l > 3$ because it is a direct consequence of the scaling of the dressed spin and dressed energy. It will be interesting to see if the structure in the second line also holds for $l > 3$. We note that Eq. (13) for all odd l does give the correct result $TD^{(l)} = (-1)^{(l-1)/2}/8$ for $\Delta = 0$.

$D^{(3)}$ as a function of anisotropy and string length is shown in Fig. 1. The numerical results based on Eq. (11) and standard TBA equations for η_α are consistent with the analytical formula, Eq. (13), at $\beta \to 0$. Note also that for the

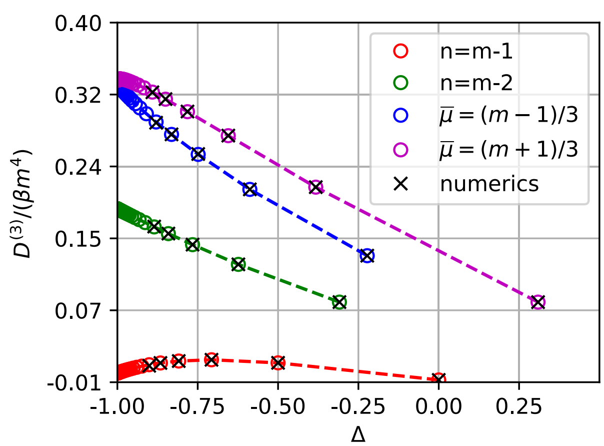

FIG. 1. The nonlinear Drude weight $TD^{(3)}$ scaled by m^4 to fit on a single scale. The lines are a guide for the eye with the note that Eq. (13) is only valid at rational points marked by an empty circle. Crosses are the result of numerically solving the nonlinear Drude weight formula, Eq. (11), at $\beta = 10^{-4}$.

case $n = m - 3$ there are two curves for the two different possible string lengths, clearly demonstrating that the current is transported by quasiparticles, the Bethe bound states.

Conclusions.—We have shown that the nonlinear transport properties of the XXZ chain are directly determined by the particle content of the theory by deriving an analytical result for the nonlinear Drude weight $D^{(3)}$ at infinite temperatures. $D^{(3)}$ exhibits a nowhere continuous dependence on the anisotropy $\Delta = \cos(n\pi/m)$ just like the linear Drude weight $D^{(1)}$. However, in contrast to $D^{(1)}$ it also explicitly depends on the Bethe string length $\bar{\mu}$ and thus shows the full particle content of the theory. Our results shed further light on transport in metals, which is typically understood using Boltzmann theory for long-lived quasiparticles. However, the validity of such an approach is usually difficult to examine except for free models. Here, integrable models can provide important benchmarks because their excitation spectra can contain complicated quasiparticles such as bound states that are exact, i.e., have infinite lifetime. Our results also shed further light on the dynamics of integrable systems by showing that Bethe strings are not merely a tool for calculations but appear indispensable. In particular, $D^{(3)}$ diverges when approaching irrational anisotropies where arbitrary string lengths are possible. This raises the interesting question of how these types of bound states can be incorporated in an effective field theory.

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