# Symmetric Finite-Time Preparation of Cluster States via Quantum Pumps

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(Received 18 July 2021; accepted 15 July 2022; published 22 August 2022)

It has recently been established that clusterlike states—states that are in the same symmetry-protected topological phase as the cluster state—provide a family of resource states that can be utilized for measurement-based quantum computation. In this Letter, we ask whether it is possible to prepare clusterlike states in finite time without breaking the symmetry protecting the resource state. Such a symmetry-preserving protocol would benefit from topological protection to errors in the preparation. We answer this question in the positive by providing a Hamiltonian in one higher dimension whose finite-time evolution is a unitary that acts trivially in the bulk, but pumps the desired cluster state to the boundary. Examples are given for both the 1D cluster state protected by a global symmetry, and various 2D cluster states protected by subsystem symmetries. We show that even if unwanted symmetric perturbations are present in the driving Hamiltonian, projective measurements in the bulk along with feed-forward correction is sufficient to recover a clusterlike state.

DOI: 10.1103/PhysRevLett.129.090501

*Introduction.*—Symmetry-protected topological (SPT) states [1–5] are gapped states of matter that cannot be adiabatically connected to an unentangled product state without breaking the protecting symmetry. It has been recently realized that certain SPT states, and in some cases, entire SPT phases, can be leveraged to perform measurement-based quantum computation (MBQC) [6–9]. The fact that these states cannot be smoothly connected to an unentangled product state without breaking a global symmetry in many cases implies an adequate entanglement structure of the state which is sufficient to perform MBQC.

So far, it is known that certain SPT phases host computational power throughout the entire phase: any state within that SPT phase can be used as a resource state [10–20]. In one dimension, the canonical example is the cluster state, defined as the unique state which satisfies $Z_{i-1}X_iZ_{i+1}|\psi\rangle = +|\psi\rangle$ on a spin chain. The state enjoys a global $\mathbb{Z}_2^2$ symmetry and for any state within the same SPT phase, arbitrary quantum gates can be performed by choosing appropriate measurements, making the entire SPT phase universal [17,18]. In higher dimensions, certain fixed points (with possibly finite regions around those fixed points) of SPT phases with global symmetries have been found to be universal [16,19–22] for MBQC, but a SPT state with global symmetry whose entire phase is universal has yet to be found.

On the other hand, there has been increased interest in subsystem symmetries due to their connections to fracton topological order in three spatial dimensions [23–30]. Unlike global symmetries, subsystem symmetries only act on a rigid subdimensional region, such as lines, planes or even fractals. It was recently realized that if one instead considers states protected by such symmetries, called subsystem SPTs states[31–38], then there are indeed examples where the entire phase can be used as a universal resource state [39–42]. Serendipitously, these examples are again cluster states on various 2D lattices.

However, there seems to be a drawback for such a convenient property. Although cluster states are easily created by evolving a product state with, for instance, an Ising Hamiltonian for a certain time [7], because the initial and final states belong to different SPT phases, any Hamiltonian evolution that relates the two in finite time necessarily breaks the global(subsystem) symmetry [43,44]. Therefore, in experimental setups, unless the Hamiltonian is prepared exactly, the resulting entangled state does not need to be a SPT state, and its use as a resource state is not guaranteed. We seem to come to the conclusion that in order to exploit the universality of the entire SPT phase, one must instead adiabatically prepare the resource state without breaking the symmetry. Such preparation time scales at least linearly in the system size.

In this Letter, we present a method to get around the above argument. Our motivation can be traced back to the seminal work of Thouless [45], where an evolution of a 1D system under a symmetric Hamiltonian leaves the bulk invariant after a certain period of time, but can “pump” quantized amounts of charge from one boundary to another. More recently, higher dimensional generalizations of such a construction have been realized in the field of Floquet SPTs phases [46–51], where in fact entire (stationary) phases of matter in one lower dimension can be pumped to the boundary under a finite time evolution while leaving the bulk invariant. Applying this concept, we are able to start with a product state and evolve the system with a Hamiltonian which respects the global(subsystem)

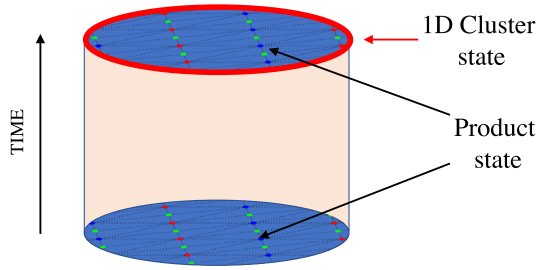



FIG. 1. Time evolution by a symmetric three-body Hamiltonian in the 2D bulk pumps a 1D cluster state to the boundary while leaving the bulk invariant. Similarly, 2D cluster states can be prepared at the boundary of a 3D bulk respecting the corresponding subsystem symmetries. In both cases the preparation only takes a finite time, independent of system size.

symmetry of a 2D(3D) system in such a way that after a fixed finite time—independent of the system size—a 1D (2D) cluster state is created on the boundary, completely uncoupled from the bulk (Fig. 1). To summarize, the previous no-go argument only holds when the cluster state is assumed to live strictly in the dimension of the defining lattice. Because of additional ancillae coming from the extra dimension of the bulk, the constraint is lifted, and we are able to prepare cluster states both symmetrically and in finite time.

With this setup, we can now take full advantage of the universality of the entire phase. Conceptually, as long as the driving Hamiltonian is modified by any small perturbation that preserves the symmetry, the entangled state on the boundary would still be symmetric and belongs to the same phase as the cluster state. It is therefore still a universal resource state. More realistically, it is possible that symmetric perturbations to the Hamiltonian leave the boundary state coupled to the bulk after the evolution, but we further demonstrate that by performing measurements and feed-forward correction, we can recover a completely decoupled boundary resource state.

The remainder of the Letter is organized as follows. We first review the notion of cluster states and how they can be viewed as SPT phases. Then, we show how a symmetric 2D Hamiltonian can be used to pump a 1D cluster state to the boundary. This procedure is generalized to pump 2D cluster states to the boundary of a 3D system using a 3D Hamiltonian that respects subsystem symmetry. Lastly, we discuss how to recover a clusterlike resource state in the case that small but symmetric perturbations are added to the Hamiltonian.

*Cluster states.*—Let $|0\rangle$ and $|1\rangle$ be $Z$ basis states. Given a graph $\mathcal{G} = (V, E)$, a graph state [52] is the entangled state $|\psi\rangle = \prod_{ij\in E} CZ_{ij} \otimes_{i\in V} |+\rangle_i$, constructed from initializing with qubits in the $X = 1$ eigenstate $|+\rangle \sim |0\rangle + |1\rangle$ at each vertex and applying the controlled-$Z$ operator $CZ_{ij} = (-1)^{n_i n_j}$, where $n_i = [(1 - Z_i)/2]$ is the number operator, to every edge of the graph. Equivalently, the graph state is the unique ground state of the stabilizer Hamiltonian $H = -\sum_{i\in V} X_i \prod_{j|(ij)\in E} Z_j$, which is obtained by conjugating $X_i$ on each vertex by the circuit $\prod_{ij\in E} CZ_{ij}$.

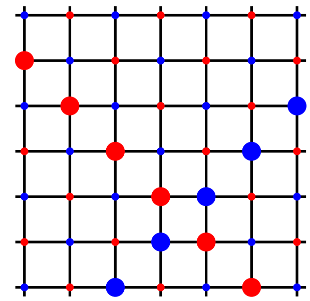


FIG. 2. The computational power of the 2D cluster state on a square lattice is protected by spin-flip symmetries along individual diagonal lines of the lattice.

When the graph $\mathcal{G}$ also forms a lattice, the state is called a cluster state. Cluster states are resource states that are universal for MBQC in two or greater spatial dimensions [8,53]. It was later realized that cluster states are examples of SPT phases [31,32,54–56]. The 1D cluster state is protected by a $\mathbb{Z}_2^2$ global symmetry, which flips the spins on even and odd sites of the chain, respectively. On the other hand, 2D cluster states are SPTs states protected by subsystem symmetry. For example, on a square lattice, the cluster state can be protected by symmetries which flip spins on individual diagonal lines (Fig. 2), while on a honeycomb lattice, it can be protected by fractal symmetries which only flip certain spins in the shape of Sierpinski triangles [32,34] (Fig. 3). Furthermore, any state in the same (subsystem) SPT phase as these cluster states (called clusterlike states) can also be used as a universal resource state [17,18,39–41].

*Pumping SPTs protected by global symmetries.*—To design Hamiltonians whose time evolution pumps SPT states to the boundary, we take inspiration from Floquet SPTs phases for bosonic systems with unitary symmetry $G$. The classification of Floquet SPTs phases

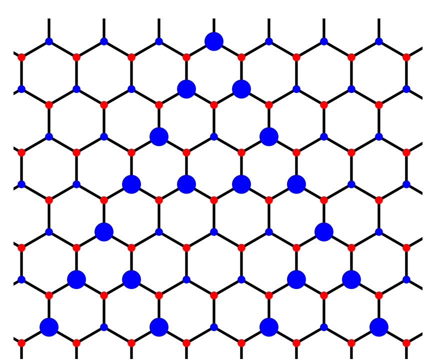


FIG. 3. The 2D cluster state on the honeycomb lattice. A generator of the fractal symmetries flips all the enlarged blue spins which form a Sierpinski triangle. There is also another set of fractal symmetry generators for the red spins.

protected by $G$ can be thought of as that of a static system with symmetry $G \times \mathbb{Z}$ where $\mathbb{Z}$ denotes time translation [46–48]. In the language of group cohomology, we can use the Künneth formula to write [57]

$$H^{d+1}[G \times \mathbb{Z}, U(1)] = H^{d+1}[G, U(1)] \times H^{d}[G, U(1)]. \quad (1)$$

The first factor classifies static $G$-SPTs phases, while the latter attaches the time-translation symmetry action with $(d-1)$-dimensional $G$-SPTs states. It can therefore be interpreted as a drive which pumps such $G$-SPT state to the boundary per driving period. We can devise a Hamiltonian to generate this Floquet unitary, which acts as the identity in the bulk, but pumps the SPT phase to the boundary while commuting with the symmetry. The idea is similar to a coupled-layer construction: dividing our $d$-dimensional system (hosting a boundary) into volume-filling "cells," the Floquet unitary is obtained by evolving a local Hamiltonian that creates, in one Floquet period, a bubble of the $d-1$-dimensional SPT along the boundary of each cell. The SPTs states cancel in the bulk, leaving only a $(d-1)$-dimensional SPT state on the boundary. Without restrictions to the number of interactions required, the pump for a general bosonic SPT state can be constructed [49,59]. Here, we build on these works by reducing the weight of the interactions required, focusing on $1+1$D SPT phases (since it is undetermined whether there exist SPT phases protected by a global symmetry in higher dimensions whose entire phase is universal). Subsequently, we will turn to pumps for subsystem SPTs phases in higher dimensions, which are new, and for which a formal classification has not been put forward.

*Pumping the 1D cluster state.*—Let us demonstrate how to prepare the 1D cluster state on the boundary of the Union-Jack lattice respecting a global $\mathbb{Z}_2^2$ symmetry using only three-body interactions, an improvement over previous setups using four-body interactions [49,60,61]. We place qubits on the vertices on the Union-Jack lattice, which is three-colorable as red, blue, and green as shown in Fig. 4. The global $\mathbb{Z}_2^2$ symmetry is defined via the action of its three $\mathbb{Z}_2$ subgroups, which flip spins in the $Z$ basis on two of the three colors. Starting with the product state $\otimes_{v\in V} |+\rangle_v$, we will evolve our system with the following Hintermann-Merlini Hamiltonian [63] for time $\pi/4$:

$$H = -\sum_{\Delta_{123}} Z_1 Z_2 Z_3, \quad (2)$$

where the sum is over triangles $\Delta_{123}$ of all orientations. This Hamiltonian commutes with the $\mathbb{Z}_2^2$ symmetry.

To see the action of the resulting unitary, we expand using $Z_i = 1–2n_i$, and find

$$\exp -i\frac{\pi}{4} Z_1 Z_2 Z_3 = e^{-i\frac{\pi}{4}} e^{i\frac{\pi}{2}(n_1+n_2+n_3)} e^{i\pi(n_1 n_2 + n_2 n_3 + n_3 n_1)}$$
$$\propto S_1 S_2 S_3 CZ_{12} CZ_{23} CZ_{31}. \quad (3)$$

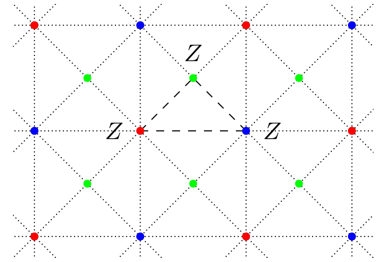



FIG. 4. The Union-Jack lattice, with a global $\mathbb{Z}_2^2$ symmetry defined as flipping spins on two of the three colors. The Hintermann-Merlini three-body interaction Eq. (2) commutes with this symmetry.

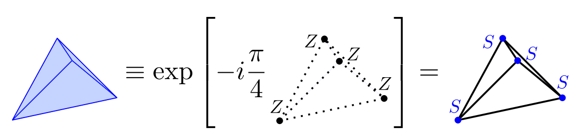
Hence, the local three-body term exponentiates to a product of $S = e^{(\pi i/2)n}$ gates for each vertex and $CZ$ gates for each edge of the triangle. Taking the product of such local unitaries for all triangles, each vertex is always acted by either four or eight $S$ gates, which cancel both in the bulk and on the boundary. On the other hand, the $CZ$ gates cancel pairwise in the bulk, leaving (up to an overall phase) $\exp -iH(\pi/4) \propto \prod_{ij\in\partial M} CZ_{ij}$, where $\partial M$ denotes the boundary spins of the lattice. Therefore this unitary pumps the cluster state to the boundary.

*Pumping subsystem SPTs.*—We will now generalize the results to prepare 2D cluster states. Note that such states can already be prepared strictly in 2D in the presence of only global symmetries. Nevertheless, they belong to the trivial phase under such symmetries and thus we cannot exploit the universality of the phase. Thus, we need the presence of subsystem symmetries, which requires a 3D pump.

First, consider the FCC lattice with planar subsystem symmetries defined as flipping spins in individual (100), (010), or (001) planes. These planar symmetries terminate as line symmetries on the boundary. Our driving Hamiltonian will be a four-body tetrahedral Ising interaction [27]

$$H = -\sum_{\triangle_{1234}} Z_1 Z_2 Z_3 Z_4 \quad (4)$$

where each tetrahedron consists of a vertex along with three adjacent face centers within the same cube. This Hamiltonian commutes with the planar symmetries. Evolving the product state with the above Hamiltonian for time $\pi/4$, a similar calculation to Eq. (3) shows that visually,




(5)

where each dense edge of the tetrahedron denotes a $CZ$ gate. Taking the product over all tetrahedra in the bulk

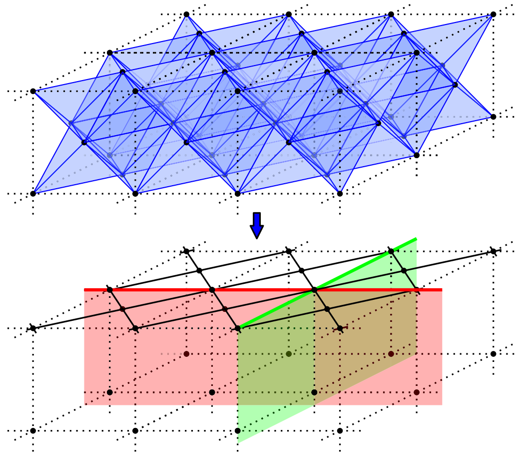
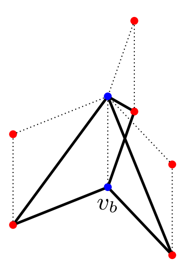
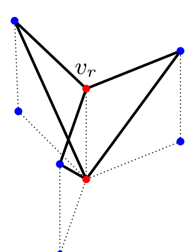


FIG. 5. A unitary evolution of the tetrahedral Ising interaction on the FCC lattice pumps the 2D cluster state to the entire boundary (shown here for the (001) boundary). The local unitary (blue tetrahedron) is generated by a four-body term in Eq. (5). Planar symmetries (red and green) terminate on the boundary as line symmetries.

(Fig. 5), we are left with $CZ$ gates acting only along the boundary. The cluster state on the (rotated) square lattice (Fig. 2) can therefore be prepared on the (100), (010), and (001) boundaries. In the Supplemental Material, we show how to prepare the triangular lattice cluster state by instead choosing the (111) boundary, and an alternative method to prepare it on the boundary of the cubic lattice.

For our second example, we will prepare the 2D cluster state on the honeycomb lattice. Our 3D bulk is a stack of 2D honeycombs with the two sites per unit cell labeled red and blue, as in Fig. 3. The fractal symmetry action is defined as acting the fractal symmetry of Fig. 3 simultaneously for every layer. Consider the following gates defined for each blue and red vertex, respectively:

$V_{v_b} =$  , $V_{v_r} =$ 

where each solid line denotes a $CZ$ gate. This can be expanded using $CZ_{ij} = \frac{1}{2}(1 + Z_i + Z_j - Z_iZ_j)$ to a sum of at most five $Z$ operators. The blue fractal symmetries trivially commute with $V_{v_b}$, while the red fractal symmetries around any blue site only flip zero or two of the three adjacent red sites within in each layer. Therefore, $V_{v_b}$ commutes with the all the fractal symmetries and similarly for $V_{v_r}$. The product $\prod_v V_{v_b} V_{v_r} = e^{-i(\pi/2)H}$ over all vertices $v$ in the 3D lattice creates two cluster states on the top and bottommost honeycomb layers, where $H = \sum_v V_{v_b} + V_{v_r}$.

Generalizing this, it is possible to similarly prepare any 2D fractal cluster state [32] generated by some 1D cellular automaton. Here, we will give the underlying argument, and prove it rigorously in the Supplemental Material [64]. For each blue site $v_b$, define $V_{v_b}$ to be a product of $CZ$ operators connecting $v_b$ and the blue site directly above it to its nearest neighbor red sites. Any symmetry generated by the cellular automaton will only flip an even number of the nearest neighbor red sites, so these gates are symmetric. Analogously, for each red site $v_r$, $V_{v_r}$ is a product of $CZ$ operators connecting $v_r$ and the red site directly below it to its nearest neighbor blue sites. A product of such gates over all vertices creates the cluster state at the top and bottommost layers, so the sum of these gates is exactly our desired driving Hamiltonian.

*Recovering clusterlike states in practical setups.*—Finally, we discuss how to take into account possible undesirable perturbations that could be introduced into the driving Hamiltonian when implemented in practice. These perturbations could entangle the boundary state with the bulk, rendering it useless as a resource state. However, we will show that as long as these perturbations are small and respect the symmetry, measurements in the bulk followed by feed-forward correction can recover a resource state in the same phase as the cluster state.

The basic idea is as follows. Suppose the driving Hamiltonian is perturbed by symmetric local terms, whose operator norms are bounded above by $\epsilon\|H\|$ for some small $\epsilon$. To prepare the resource state, we choose a bulk which is much larger than the support of possible perturbations and initialize all qubits to the all $|+\rangle$ state. We now consider how the terms possibly affect the cluster state on the boundary after the evolution. (1) If the perturbation acts purely in the bulk, then our resource state on the boundary is not affected. (2) If the perturbation acts purely on the boundary, then the state is perturbed symmetrically, which will still be a valid resource state as long as $\epsilon$ is small enough to keep it in the SPT phase. (3) If the perturbation acts both in the bulk and on the boundary, then this term could break the symmetry restricted to the boundary or bulk separately, while preserving the total symmetry of the whole system. In that case, the term will flip an odd number of $|+\rangle$ states in the bulk to $|-\rangle$. Therefore, we can eliminate this error by performing a measurement in the $X$ basis for all qubits in the bulk, and if we measure an odd number of $|-\rangle$ states along any bulk symmetry operator, we apply single-spin flips $X$ on the boundary to recover a symmetric state.

*Discussion.*—Inspired by quantum pumps and Floquet SPTs phases, we devised a 2D(3D) Hamiltonian which respects the global(subsystem) symmetry extended into the bulk, and showed that a product state driven by this Hamiltonian for a fixed time independent of system size prepares a 1D(2D) cluster state on the boundary. Then, exploiting the universality of the entire symmetry-protected phase, we were able to guarantee the preparation of a resource state even when the Hamiltonian is not implemented exactly as long as perturbations are small and symmetric. This was achieved by followup measurements and feed-forward correction. We find it remarkable that topology proves itself useful in methods beyond topological quantum computing.

We conclude with prospects for future work.

From the point of view of topological phases, our results entail that intrinsically interacting Floquet SPTs phases protected by subsystem symmetry are at least classified by subsystem SPTs phases in one lower dimension, identical to the global symmetry case. It would be interesting to see whether this classification is complete. Furthermore, gauging Floquet subsystem SPT phases can give rise to Floquet fracton orders, where gapped excitations with restricted mobility are dynamically enriched into non-Abelian excitations via the Floquet drive [60]. There is also an intriguing connection between pumps and transversal logical gates of the gauged topological codes that deserve exploration in the case of subsystem symmetries [64].

For future prospects for MBQC, we have presented three(four)-body interactions to symmetrically prepare one(two)-dimensional cluster states. It would be interesting if this number can be further lowered given that universal resource states can arise as ground states of two-body Hamiltonians [10,11,13]. In addition, current computational schemes implicitly assume the clusterlike states possess translation invariance [17,39,41,42,56,77], which might not hold for the states prepared using this method. It would be crucial to devise a computational scheme which relaxes such an assumption. Finally, we hope to investigate whether there are experimental platforms where such global or subsystem symmetries are inherent or arise as an approximate symmetry. Ultimately, finding a Hamiltonian that can be faithfully implemented experimentally, and a way to limit perturbations to ones that respect the symmetry, would provide a scalable and reliable method to create universal resource states.

We thank Yanzhu Chen, Ruihua Fan, Dominic Else, Andrew Potter, Abhishodh Prakash, Robert Raussendorf, Ryan Thorngren, Ruben Verresen, Sagar Vijay, and Tzu-Chieh Wei for helpful discussions. We also thank the referees for suggestions that were crucial to improving the manuscript. N. T. is grateful to the Yukawa Institute for Theoretical Physics at Kyoto University. Discussions during the workshop YITP-T-19-03 “Quantum Information and String Theory 2019” were useful to complete this work. N. T. acknowledges the support of NSERC. A. V. was supported by the DARPA DRINQS program (Award No. D18AC00033) and by Simons Investigator award (A. V.) and by the Simons Collaboration on Ultra-Quantum Matter, which is a grant from the Simons Foundation (No. 651440, A. V.).

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