Topologically Enabled Superconductivity

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Majorana zero modes are a much sought-after consequence of one-dimensional topological superconductivity. Here, we show that, in turn, zero modes accompanying dynamical instanton events strongly enhance—in some cases even enable—superconductivity. We find that the dynamics of a one-dimensional topological triplet superconductor is governed by a θ term in the action. For isotropic triplets, this term enables algebraic charge-$2e$ superconductivity, which is destroyed by fluctuations in nontopological superconductors. For anisotropic triplets, zero modes suppress quantum phase slips and stabilize superconductivity over a large region of the phase diagram. We present predictions of correlation functions and thermodynamics for states of topologically enhanced superconductivity.

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One-dimensional topological p-wave superconductors are widely proposed as building blocks for quantum information processing [1–8]. While proximity-induced superconductivity is discussed most frequently, the prospect of intrinsic superconductivity in one-dimensional structures would certainly allow for more versatile architectures. However, despite the obvious challenge of identifying the right material, there seems to be a more fundamental limitation to such an approach. In low dimensions the influence of fluctuations is very strong. The impact of order-parameter fluctuations on one-dimensional topological superconductivity has been extensively studied for the spinless p-wave case [9–12] and it has been demonstrated that vacuum tunneling by 2π quantum phase slips (QPS) is suppressed in these systems [10,13], thus enlarging the superconducting domain in the phase diagram. However, for isotropic triplet superconductivity the role of fluctuations in the spin sector is ordinarily so strong as to completely destroy $2e$ superconductivity [14–17].

In this Letter, we show that charge-$2e$ topological triplet superconductivity in one-dimensional quantum wires becomes possible while it is not allowed for nontopological systems. The latter can only undergo vestigial charge-$4e$ pairing in a much reduced regime of the phase diagram; see Figs. 1(a) and (b). Zero modes, primarily discussed as static Majorana bound states of topological superconductivity, emerge in our analysis as dynamical events accompanying instantons of the order parameter field. They are shown to suppress order-parameter fluctuations via destructive interference due to a Berry phase, enabling the charge-$2e$ superconducting state. This Berry phase leads to a topological term in the field theory, a θ term [18,19]. For the topological angle we find $\theta = \pi$ for topological superconductors and $\theta = 0$ for nontopological ones. We obtain this result using non-Abelian bosonization and, using a physically more transparent reasoning, by demonstrating that dynamical zero modes give rise to a complex QPS fugacity.

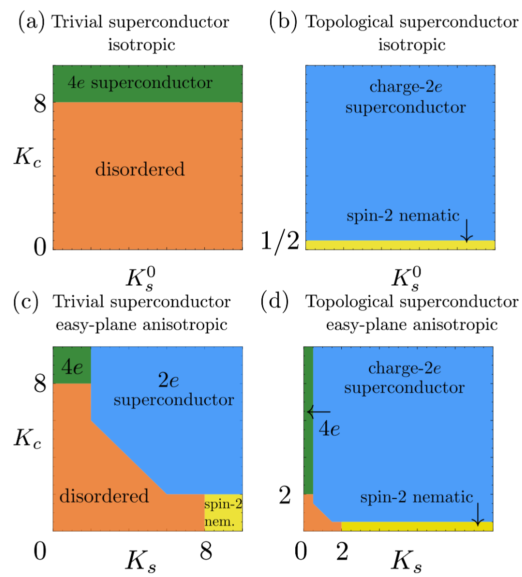

FIG. 1. Ground state phase diagram of the trivial (a), (c) and topological (b), (d) superconductor. The top (bottom) row depicts the spin-isotropic (strongly anisotropic) case. The axes correspond to the renormalized, i.e., experimentally relevant, charge (K_c) stiffness, and the bare (K_s^0) and renormalized (K_s) spin stiffness in the isotropic and the anisotropic case, respectively. All superconducting states have algebraic, quasi-long-range order. While in the isotropic case only vestigial charge-$4e$ superconductivity is allowed for topologically trivial systems, a θ term in the theory enables charge-$2e$ pairing of topological superconductors, which occurs in a much larger domain of the phase diagram. At finite spin anisotropy, disorder-inducing quantum phase slips are suppressed by zero modes, stabilizing superconductivity.

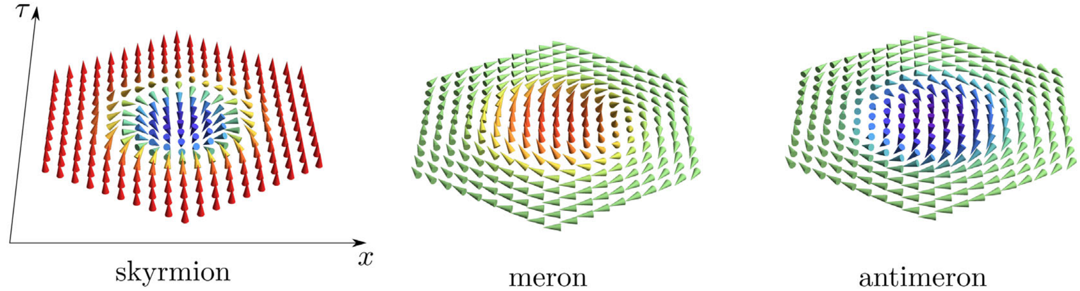

FIG. 2. Selected topologically nontrivial order-parameter configurations in space-time [27]. Left panel: skyrmion of topological charge $Q = 1$. Middle and right panel $Q = \pm\frac{1}{2}$ merons, i.e., spin vortices that differ by the orientation of the spin inside the core.

These findings occur for systems that are isotropic in spin space where we exploit a connection to Haldane's conjecture [20] for spin chains. Including an anisotropy in spin space, disorder-inducing configurations are suppressed by zero modes that are more akin to what is known for spinless systems [10,13].

Model.—We study a time reversal invariant p-wave spin triplet superconductor in one dimension at zero temperature. The Bogoliubov–de Gennes Hamiltonian for given space- and time-dependent order-parameter configuration $\boldsymbol{\Delta}(x,\tau)$ in the Nambu spinor basis $\Psi = (\psi_\uparrow, \psi_\downarrow, \bar{\psi}_\downarrow, -\bar{\psi}_\uparrow)^T$ is

$$\mathcal{H}[\boldsymbol{\Delta}] = \begin{pmatrix} -\frac{\partial_x^2}{2m} - \mu & \boldsymbol{\Delta}(x,\tau)\cdot\boldsymbol{\sigma}\partial_x \\ -\partial_x \boldsymbol{\Delta}^\dagger(x,\tau)\cdot\boldsymbol{\sigma} & \frac{\partial_x^2}{2m} + \mu \end{pmatrix}, \tag{1}$$

with m the fermion mass and μ the chemical potential. The order parameter takes the form of a real unit vector $\boldsymbol{n}$, which describes the orientation of the Cooper pair spin, times a global phase ϑ, i.e., $\boldsymbol{\Delta} = |\Delta| e^{i\vartheta}\boldsymbol{n}$, with the pairing strength $|\Delta|$. $\mu < 0$ corresponds to a trivial and $\mu > 0$ to a topological superconductor [21]. The order parameter manifold is $\mathcal{M} = (S^1 \times S^2)/\mathbb{Z}_2$, where the $\mathbb{Z}_2$ quotient stems from the equivalence of field configurations $(\vartheta, \boldsymbol{n}) \sim (\vartheta + \pi, -\boldsymbol{n})$ [22,23]. The fermionic dynamics is then governed by the action

$$S_f = \frac{1}{2}\int d\tau dx\, \Psi^\dagger(\partial_\tau + \mathcal{H})\Psi. \tag{2}$$

In low-dimensional intrinsic superconductors, order-parameter fluctuations are important and are governed by the nonlinear σ-model (NLσM)

$$S_b = \frac{K_c^0}{2\pi}\int d\tau dx \left(\frac{1}{v_c}(\partial_\tau\vartheta)^2 + v_c(\partial_x\vartheta)^2\right) + \frac{K_s^0}{2\pi}\int d\tau dx \left(\frac{1}{v_s}(\partial_\tau\boldsymbol{n})^2 + v_s(\partial_x\boldsymbol{n})^2\right). \tag{3}$$

Here, K_c^0 and K_s^0 denote the bare stiffnesses of the charge and spin sector respectively. v_c and v_s are the respective velocities. We use $K_{c,s}$ for the renormalized stiffnesses.

This model is a generalization of previous descriptions for spinless fermions [11,24,25].

Topological field configurations.—The fields ϑ, $\boldsymbol{n}$ that describe $U(1)$-phase and spin of a Cooper pair allow for several distinct QPSs, shown in Fig. 2: we call vortices of ϑ with winding ν_c in space-time $2\pi\nu_c$ "charge QPSs." Space-time "skyrmions" in $\boldsymbol{n}$ have integer winding Q, which relies on $\pi_2(S^2) = \mathbb{Z}$. The evaluation of the partition function contains a sum over all possible topological sectors, determined by a set of numbers $N = \{\nu_c, Q, \cdots\}$ [19],

$$Z = \sum_N \int \mathcal{D}\vartheta_N \mathcal{D}\boldsymbol{n}_N Z_N^f[\vartheta, \boldsymbol{n}] e^{-S_b[\vartheta,\boldsymbol{n}]}, \tag{4}$$

where $\int \mathcal{D}\vartheta_N \mathcal{D}\boldsymbol{n}_N \ldots$ denotes the integral over smooth bosonic fluctuations on top of the topological field configuration and $Z_N^f[\vartheta, \boldsymbol{n}]$ is the fermionic partition sum in a given bosonic background of fixed N. Notice, all fluctuations around the trivial state, including these instantons, must be taken into account as long as their action is finite. Estimating the core size of QPS we obtain, following Ref. [26], $r_{\mathrm{QPS}} \approx (K_c v/2\pi E_{\mathrm{cond}})^{1/2}$, and the core action $S_{\mathrm{QPS}} \approx K_c/2$.

Topologically trivial superconductivity.—To analyze the model it is tempting to argue that fermions are gapped and should not change the universal behavior of order-parameter fluctuations. Then, the NLσM of Eq. (3) implies $\langle\boldsymbol{\Delta}\rangle = 0$ with exponentially decaying correlations, caused by fluctuations in the spin sector. As last resort the system can still enter a state of algebraic vestigial order characterized by the composite $\boldsymbol{\Delta}\cdot\boldsymbol{\Delta}$ below a Berezinskii-Kosterlitz-Thouless (BKT) transition [22,23]. However, this state, where two spin triplets form a charge-$4e$ spin singlet, can be destroyed by the proliferation of π, i.e., fractional QPSs. In our units, the QPS configurations with smallest winding number ν_c become relevant at a critical stiffness of $K_c = 2/\nu_c^2$. Hence, in order to stabilize $4e$ order with $\nu_c = 1/2$ fractional vortices, the charge stiffness has to be 4 times larger compared to the usual BKT transition, which implies $K_c = 8$; see the left panel of Fig. 1. This behavior, deduced from the two-dimensional classical model, is indeed correct in the topologically trivial phase.

Yet, as we will see next, it does not apply to topological superconductors.

In the topological phase, order-parameter configurations with nontrivial topology in space-time play a key role [28–34], requiring us to carefully distinguish their effects and their interplay with the fermionic degrees of freedom.

Integrating out fermions.—In order to systematically integrate out fermions in a one-dimensional system, we employ non-Abelian bosonization [35], which manifestly preserves the symmetry properties of interacting fermion theories and is expressed in terms of the Wess-Zumino-Novikov-Witten (WZNW) action.We employ this method to effectively eliminate the massive degrees of freedom while retaining the effect of zero modes. The degrees of freedom of the WZNW theory are group valued boson fields defined on the orthogonal group $O(4)$. The matrix components of this field can be related to fermionic currents by $g_{ij} = (-i/M)\chi_R^i \chi_L^j$, where χ_R, χ_L are right- and left-moving Majorana fermions that can be constructed from the microscopic fermions, and M is a regularization dependent mass scale. The degrees of freedom described by the Bogoliubov–de Gennes Hamiltonian [Eq. (1)] deep in the topological phase are exactly four right-moving and four left-moving Majorana fermions, which explains the manifold $O(4)$. Effectively, the field g encodes the fermionic fluctuations and the opening of a gap corresponds to breaking the chiral symmetry, i.e., individual components of g develop a finite expectation value. The action with coupling constant γ reads

$$S_W = \frac{1}{4\gamma^2} \int d\tau dx \mathrm{tr}[\partial_\mu g \partial^\mu g^{-1}] + \Gamma, \tag{5}$$

where the Wess-Zumino term $\Gamma[g]$ is defined by extending the domain of the group valued field $g \in O(4)$ to a hemisphere of the three-dimensional unit sphere with S^2 as boundary:

$$\Gamma = \frac{\varepsilon^{\mu\nu\rho}}{24\pi i} \int d^3 r \, \mathrm{tr}[g^{-1}(\partial_\mu g) g^{-1}(\partial_\nu g) g^{-1}(\partial_\rho g)]. \tag{6}$$

The pairing term is bosonized and introduces a $(\vartheta, \boldsymbol{n})$ dependent mass term for g. In the adiabatic limit, g follows the variation of the background fields $(\vartheta, \boldsymbol{n})$ and thereby traces the equator of $O(4)$. The details of the subsequent analysis are given in the Supplemental Material [36]. The first term in Eq. (5) yields terms identical to the NLσM in Eq. (3), i.e., it yields a renormalization of the stiffnesses due to fermions. More interesting is the second term. When constrained to the equator, the Wess-Zumino term [Eq. (6)], which measures the solid hyperangle on $O(4)$ normalized to 2π, may only take values of π mod 2π. We find that

$$\Gamma \to S_\theta = i\frac{\theta}{4\pi} \int d\tau dx \, \boldsymbol{n} \cdot (\partial_\tau \boldsymbol{n} \times \partial_x \boldsymbol{n}), \tag{7}$$

where $\theta = \pi$. For a topologically trivial superconductor we obtain instead $\theta = 0$. Equation (7) has profound implications. According to Haldane's conjecture [20], where $\theta = \pi$ occurs in the same action for $\boldsymbol{n}$ for half-integer spins [37], S_θ leads to a critical state described by a $SU_1(2)$ WZNW theory [38,39] rather than a state with finite correlation length. Hence, algebraic charge-$2e$ superconductivity becomes possible for topological superconductors while it is forbidden in topologically trivial ones.

Before we discuss further implications of this finding, we offer an alternative and physically more transparent derivation of the θ term. We consider nontrivial skyrmion configurations, but in this context it is particularly convenient to introduce a soft easy-plane anisotropy [40]. This anisotropy yields spin-vortex configurations but allows $\boldsymbol{n}$ to escape the plane in a core region, whose size is determined by the strength of the anisotropy, avoiding a singularity. There are two kinds of 2π spin QPSs distinguished by the orientation of the vector in the core ($\pm n_z$), called "meron" and "antimeron" with skyrmion winding number $Q = \pm\frac{1}{2}$; see Fig. 2. The fermionic operator $\mathcal{K} \equiv \partial_\tau + \mathcal{H}$ acts in space and (imaginary) time. Since the Nambu spinor satisfies the reality condition $\Psi^T = \Psi^\dagger C$ (where $C = \sigma^y \tau^y$ with σ^i acting on spin and τ^i on Nambu degrees of freedom), the fermionic partition function $Z_N^f[\vartheta, \boldsymbol{n}] = \mathrm{Pf}(C\mathcal{K}[\vartheta, \boldsymbol{n}])$ follows directly from the Pfaffian of the fermionic kernel. Hence, in the case that the fermionic kernel possesses a zero eigenvalue mode the fermionic partition function vanishes. As summarized in [36], integrating out fermions yields in the limit of large anisotropy that two zero modes are shifted to finite eigenvalue in such a manner that

$$Z_N^f[\vartheta, \boldsymbol{n}] \propto e^{-i\pi Q}. \tag{8}$$

With this complex fugacity, the contribution of 2π spin QPS vanishes upon summation over Q, the internal degree of freedom of the vortex [40]. Within a field theoretical language, this behavior is precisely the effect that follows from a θ term in the action, $S_\theta = i\theta Q$, where $Q = \int d\tau dx \, \boldsymbol{n} \cdot (\partial_\tau \boldsymbol{n} \times \partial_x \boldsymbol{n})/(4\pi)$ is the associated topological charge [19]. It reveals that the θ term yields destructive interference of disordering spin configurations.

In addition to nontrivial spin textures, we can also analyze charge vortices or combinations of charge and spin vortices. We find that one can map this dynamic problem onto an effective Hermitian single-particle Hamiltonian H_{eff} in two dimensions (x, τ). H_{eff} can then be reduced to the two-dimensional Fu-Kane Hamiltonian of a 3D topological insulator surface state in contact with an s-wave superconductor [41]. Established results for zero modes due to static vortex configurations of the Fu-Kane Hamiltonian can now be used to obtain the number and character of zero modes for $\mathcal{K}$. Since one coordinate of this two-dimensional problem corresponds to (Euclidean) time, those are again dynamical instanton events. The details of this rather powerful but straightforward analogy are summarized in the Supplemental Material [36]. It yields, for example, two zero modes of $\mathcal{K}$ for a charge-2π QPS ($\nu_c = 1$) and one zero mode for a combined 2π half QPS ($\nu_c = \nu_s = \frac{1}{2}$), with winding number of a planar spin vortex ν_s. More generally we obtain a zero mode for each odd $\nu_c \pm \nu_s$. These dynamical zero modes can be understood as protected level crossings under the adiabatic variation of a parameter. Perturbations that do not destroy the level crossing also do not lift the dynamic zero modes. Hence, $Z_N^f[\vartheta, \boldsymbol{n}] = 0$ for these single-defect configurations and the corresponding vortex fugacities vanish. More importantly, vortex-antivortex pairs do not contribute to a BKT transition as the exponentially small overlap between modes gives rise to a linear, confining potential overruling the usual logarithmic interaction. Our mapping to the Fu-Kane model can also be applied to spinless p-wave superconductors where it agrees with past results on QPSs in this system [10,13].

Quantum phase diagram.—If we combine the θ term and the presence of zero modes due to charge vortices, we can determine the phase diagrams shown in Fig. 1. In the topologically trivial phase, $2e$ superconducting order is destroyed because $\boldsymbol{n}$ fields are gapped but $4e$ superconducting order may exist for $K_c > 8$. In contrast, in the topological phase the $\boldsymbol{n}$ fields are algebraically ordered and the charge-$2e$ superconductor persists. A BKT transition driven by $\nu_c = 2$, i.e., 4π, QPS disorders the phase sector for $K_c < 2/\nu_c^2 = 1/2$. The resulting state possesses vestigial order of the composites $\Delta_x^\dagger \Delta_y + \Delta_y^\dagger \Delta_x$, $\Delta_x^\dagger \Delta_x - \Delta_y^\dagger \Delta_y$, and $\Delta_z^\dagger \Delta_z$. It is a charge insulator and a spin-nematic state. Hence, the critical stiffness for the destruction of superconductivity is 16 times lower than in the trivial case; see Fig. 1(b).

Correlators.—Let us discuss experimentally and numerically measurable consequences of the topological phases in terms of bosonic and fermionic correlators. In topologically trivial superconductors, fermions are gapped and fermion correlators decay exponentially. This is qualitatively different for topological superconductors. The power-law behavior of the single-fermion correlator is again caused by dynamical zero modes. We obtain

$$\langle \psi_\sigma(r) \psi_{\sigma'}^\dagger(0) \rangle \sim \delta_{\sigma\sigma'} r^{-\frac{1}{8K_c} - \frac{K_c}{2} - \frac{1}{2}}, \tag{9}$$

where $r = \sqrt{x^2 + v_c^2 \tau^2}$ denotes the Euclidean norm of a point (x, τ) in space-time. For simplicity we assumed $v_s = v_c$. Hence, the topological superconductor possesses gapless charged fermionic excitations. This is in contrast to nodal excitations that can appear in higher dimensions and are charge neutral. However, experiments that measure the phase coherence, such as flux quantization or the ac Josephson effect are expected to detect a condensate charge of $2e$.

Equation (9) implies a power-law dependence on energy in the tunneling density of states $\rho(E)$, which can be detected in scanning tunneling microscopy measurements,

$$\rho(E) \sim E^{\frac{1}{8K_c} + \frac{K_c}{2} - \frac{1}{2}}, \qquad E > 0. \tag{10}$$

Similarly, we expect power-law dependence on temperature of various transport coefficients, such as the thermal conductivity [42].

Notice that, nonetheless, the system cannot be described as a Tomonaga-Luttinger liquid of charge-e fermions, since the behavior of two-particle correlators differs. Zero modes also nontrivially affect two-particle correlators in the topological state [36]. Specifically, we find the tunneling density of states for tunneling a singlet and triplet Cooper pair into the wire

$$\rho_{\rm sg}(E) \sim E^{2K_c + \frac{1}{2K_c} - 1}, \qquad \rho_{\rm tr}(E) \sim E^{\frac{1}{2K_c}}. \tag{11}$$

The exponent for the singlet pair-correlator differs from the Luttinger liquid result (extracting the Luttinger parameters from the one-particle correlator) for general values of K_c, $\rho_{\rm sg}^{\rm LL} \sim E^{1+(1/2K_c)}$. In the $2e$-ordered phase the singlet correlations are less singular than in a conventional Luttinger liquid. At low energies the amplitude for tunneling triplet pairs into the system is enhanced compared to the one for singlet pairs. A detailed derivation is given in [36].

Easy axis anisotropy.—We already discussed an additional easy axis anisotropy $S_{\rm an} = \lambda \int d\tau dx\, n_z^2$ when we offered an alternative derivation of the θ term in Eq. (8). The fact that topological superconductivity is stabilized over the topologically trivial case can also be seen at $\lambda > 0$. It is, however, more similar to what is known from spinless p-wave superconductors [10,13], since the spin sector is now also governed by a $U(1)$ order parameter and charge-$2e$ superconductivity becomes possible for both topological and nontopological superconductors. However, at small stiffnesses, algebraic order is now destroyed by the proliferation of three kinds of space-time topological defects: 2π charge, spin, or combined vortices; see also Refs. [43–46]. This yields the four possible phases presented in Fig. 1(c) and 1(d): a completely ordered (all QPSs expelled), a completely disordered (all QPSs proliferate), and two vestigial phases (only one kind of QPS proliferates). Physically, the resulting phases can be identified as a spin-nematic charge-$2e$ superconductor, where the two gapless excitations are spin-1 Cooper pairs, a correlated insulator, a charge-$4e$ superconductor with gapless excitations carrying charge $4e$ and spin 0, and a spin-nematic phase similar to the isotropic case. While thermodynamically the phase diagram of topological and trivial limits is similar, in the former case 2π charge, spin, or combined QPSs are suppressed by dynamic zero modes. Hence, the leading transitions are effected only by 4π QPSs, and as a result we find again that the superconducting state is stabilized for the topological phase. Similar to the isotropic case, zero modes lead to gapless fermion and two-particle excitations in the $2e$ phase. In the vestigial charge-$4e$ phase, single fermions are gapped, but the singlet pair-correlator remains gapless. There is another distinction between the topological and the trivial vestigial phases: in the topological case the bosonic operator $\phi_s(x)$ $[\phi(x)]$, which is the dual field to the phase $\vartheta_s(x)$ $[\vartheta(x)]$, develops bona fide long range $\mathbb{Z}_2$ order in the vestigial charge (spin) phase on the top left (bottom right) of the phase diagram. The reason is that the partial disorder is induced by 4π spin (charge) QPSs that preserve a remnant $\mathbb{Z}_2$ order. Physically, these order parameters correspond to the magnetization (charge) integrated up to a point x along the 1D system—thus, they are nonlocal order parameters.

Conclusion.—We find that superconducting fluctuations in topological and topologically trivial superconductors are qualitatively different, leading to distinct phase boundaries, symmetry breaking, and excitation spectra. The common theme is that superconductivity in topological systems is much more robust against fluctuations. The reason is the crucial role of dynamic zero modes. The most dramatic effect is the emergence of otherwise forbidden charge-$2e$ superconductivity for isotropic triplets caused by a topological term in the action. It yields algebraic superconducting order for a stiffness 16 times lower than charge-$4e$ superconductivity in the nontopological counterpart.

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