Onset of Quantum Chaos in Random Field Theories

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We study the quantum Lyapunov exponent λ_L in theories with spacetime-independent disorder. We first derive self-consistency equations for the two- and four-point functions for products of N models coupled by disorder at large N, generalizing the equations appearing in SYK-like models. We then study families of theories in which the disorder coupling is an exactly marginal deformation, allowing us to follow λ_L from weak to strong coupling. We find interesting behaviors, including a discontinuous transition into chaos, mimicking classical KAM theory.

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Introduction.—Systems with disorder are ubiquitous in nature, and display a wide range of interesting physical phenomena. Disorder can sometimes be modeled by introducing random couplings, and averaging over these random couplings can lead to simplifications which allow for exact computations, at least when the number of degrees of freedom is parametrically large. A notable example is the Sachdev-Ye-Kitaev (SYK) model, in which (nearly) conformal symmetry is restored at low energies [1–4], and such averages allow for the computation of the quantum chaos exponent λ_L. The latter is defined by the fastest growing exponential mode in the double commutator $\langle[U(t), W(0)]^2\rangle_\beta \sim e^{\lambda_L t}$ for generic operators U, W, over an appropriate timescale, with temperature $1/\beta$ [5].

The SYK model consists of N free quantum mechanical fermions deformed by a relevant all-to-all spacetime-independent disordered interaction. Generalizations have appeared which consist of N copies of other free theories with similar interactions [6–13]. In this Letter we study quantum chaos in a more general setting by studying N copies of a general core model $\mathcal{Q}$ (which can be a quantum field theory or a spin system), deformed by a spacetime-independent all-to-all interaction. In particular, we eventually focus on Q being a conformal field theory (CFT) in $0+1$ or in higher dimensions, and with an exactly marginal disorder interaction. In such a setup we have better control over the space of couplings over which we are averaging, eliminating the complications of the renormalization group (RG) and without having to resort to strong coupling.

The theories we study are of the form

$$\mathcal{Q}^N + \sum_{i_1\neq\ldots\neq i_q}^{N} J_{i_1\ldots i_q}\mathcal{O}_{i_1}\ldots\mathcal{O}_{i_q}, \tag{1}$$

where $\mathcal{Q}^N$ denotes N decoupled copies of the model $\mathcal{Q}$, $\mathcal{O}_i$ with $i = 1, \ldots, N$ are the N copies of a local operator $\mathcal{O}$ in $\mathcal{Q}$, and $J_{i_1\ldots i_q}$ are Gaussian random variables with variance

$$\langle J^2_{i_1\ldots i_q}\rangle = \frac{J^2(q-1)!}{N^{q-1}}. \tag{2}$$

In the case where $\mathcal{Q}$ is a CFT, we will take $\mathcal{O}$ to be a primary operator of this core CFT, and equation (1) should be interpreted in conformal perturbation theory in J. We call such theories disordered CFTs; the simplest example of this setup is the SYK model itself, as a disordered free fermion theory.

We will first derive self-consistency equations for the two- and four-point functions of $\mathcal{O}_i$ at leading order in $1/N$ for general disordered CFTs. These extend known results for disordered free theories like the SYK model [3,4]), and for spin systems [14]. We will also discuss similar results for the double commutator [defined in (7) below]. Although these self-consistency equations are complicated, they are tractable in perturbation theory in J and allow us to establish the existence of a kernel structure from which we can extract the chaos exponent λ_L, which is the rate of growth of a double commutator. The latter is given by the fastest growing eigenvector (with eigenvalue 1) of a specific integral kernel K_R [see Eq. (9)], as long as this rate of growth is indeed positive [2,5,15]. We will denote by λ_L^{ker} the putative chaos exponent read from the diagonalization K_R.

We will be interested in computing the chaos exponent λ_L as a function of J, as it is varied from weak to strong coupling. The diagonalization of K_R is a difficult process,

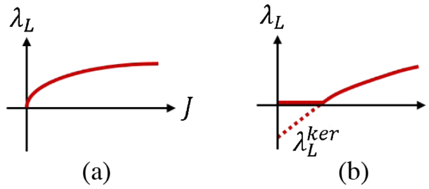

FIG. 1. The behaviors we find for the chaos exponent as a function of an exactly marginal disorder deformation J: (a) continuous and (b) discontinuous. λ_L^{ker} corresponds to the dashed line and λ_L to the solid line.

which can usually be done only when conformal invariance is restored. Normally, the disorder is a relevant deformation. To compute the chaos one first flow to the ir CFT (which is equivalent to take $J = \infty$) and find the chaos exponent there [6]. In this Letter we focus on cases where the averaged interacting theory is conformally invariant for all J. This can be done by demanding that the disorder interaction itself be exactly marginal, in which case the theory is conformally invariant for every realization of the couplings, and the space of J's forms a conformal manifold [16]. The disorder average is then simply a nonuniform average over this conformal manifold, with no RG-related complications (for recent discussions of averages over conformal manifolds see Refs. [17,18], and for the marginal case see Ref. [19]).

Surprisingly, we will find models where $\lambda_L^{\mathrm{ker}}(J)$ is negative at weak coupling. This signals a breakdown of the approximations involved in the computation, and so in this case the true chaos exponent simply vanishes, $\lambda_L = 0$. In other words, we have

$$\lambda_L = \max(0, \lambda_L^{\mathrm{ker}}). \tag{3}$$

As a result, there are two possible behaviors for the onset of chaos: either the theory undergoes a continuous transition into chaos as in Fig. 1(a) or a discontinuous transition as in Fig. 1(b), corresponding to whether λ_L^{ker} is non-negative or negative at small enough J, respectively [20].

The discontinuous transition into chaos is a surprising result, and it is tempting to compare it to similar results in classical chaos, the most famous one being the KAM theorem. In order to sharpen the comparison, we also discuss what a single core CFT should obey in order for the transition into chaos to be discontinuous. Similar works on the onset of quantum chaos include [3,8,21–24], and Refs. [13,25] are especially relevant.

We will apply our formalism to two classes of examples where the disorder interaction is exactly marginal. The first class is disordered generalized free fields in one dimension (following [12]) and in two dimensions. The second class is the disordered $\mathcal{N} = 2$ supersymmetric (SUSY) A_{q-1} minimal models. In practice, we will only discuss the simplest case of the A_2 minimal model here. We will find a discontinuous transition in the former and a continuous transition in the latter.

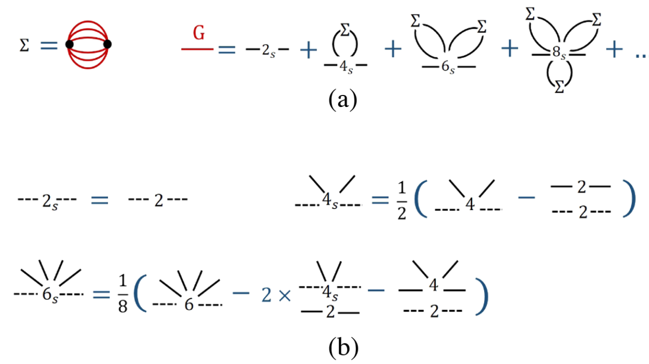

FIG. 2. (a) The SD equations. G insertions appear as red lines, and black dots denote insertions of the deformation (1). (b) Examples of n_s, the subtracted n-point functions. Dashed lines connect to the external points, and numerical factors indicate symmetry factors. The blue numbers inside the brackets denote possible ways of permuting the legs.

More details and discussions on the computations and results can be found in the companion paper [26].

Disorder around a nontrivial CFT.—The kernel structure of the four-point function: We start by writing a self-consistency equation for the averaged two-point function of (1),

$$G(x) = \langle \mathcal{O}_i(x)\mathcal{O}_i(0)\rangle. \tag{4}$$

Using the $G - \Sigma$ formalism [2,3], it can be shown that G obeys a generalized Schwinger-Dyson (SD) equation at leading order in $1/N$, which appears diagrammatically in Fig. 2(a). The equation includes subtracted n-point functions denoted by "n_s," which are combinations of the standard core CFT n-point functions with additional theory-independent subtractions which can be derived order by order in n [26]. The first few subtracted n-point functions (assuming $\mathcal{O}_i$ are real) are shown in Fig. 2(b). Generalizations to complex $\mathcal{O}_i$ exist, and also to superfields (in which case the diagrams correspond to supergraphs).

The contributions to the averaged connected four-point function

$$C = \frac{1}{N^2}\sum_{i,j}(\langle \mathcal{O}_i\mathcal{O}_i\mathcal{O}_j\mathcal{O}_j\rangle - \langle \mathcal{O}_i\mathcal{O}_i\rangle\langle \mathcal{O}_j\mathcal{O}_j\rangle) \tag{5}$$

also have a simple form, and obey an iterative ladder structure similar to the case of disordered free fields:

$$C = \sum_{n=0}^{\infty} K^n F_0, \tag{6}$$

where the kernel K and the initial diagram F_0 are defined in Fig. 3(a). The definition requires new subtracted n-point

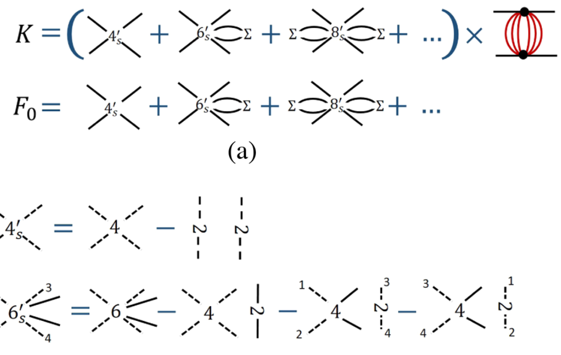

FIG. 3. (a) The kernel K and initial diagram F_0 for general disordered CFTs. Red lines denote full propagators G, and black dots denote insertions of the disorder interaction, with $q-2$ red propagators between each pair. (b) Examples of correlation functions n_s'. Dashed lines correspond to external points, while solid lines are connected via Σ's defined in Fig. 2(a).

functions called n_s', which are again theory independent [26]. The first few n_s' for real $\mathcal{O}_i$ appear in Fig. 3(b).

Some comments are in order. First, although the kernel K is very complicated, knowing that an iterative ladder structure exists for the four-point function is already an important result since it allows for a systematic computation of λ_L, as we now discuss. Second, although a full solution of the two- and four-point functions requires knowing all n-point functions of the core CFT, a solution to order J^{2n} in perturbation theory in J only requires knowing the $2m$-point functions for $m \le n+1$.

The double commutator and chaos: A similar analysis also applies to the computation of the double commutator:

$$W_R(t_1,t_2) = \frac{1}{N^2}\sum_{i,j=1}^{N}\langle[\mathcal{O}_i(\beta/2),\mathcal{O}_j(\beta/2+it_2)] \times [\mathcal{O}_i(0),\mathcal{O}_j(it_1)]\rangle, \tag{7}$$

where we suppress the spatial positions. In chaotic theories, at large Lorentzian times t_1, t_2, the double commutator is expected to grow exponentially:

$$\lim_{t_1,t_2\to\infty} W_R(t_1,t_2) \sim \frac{1}{N}\exp[+\lambda_L(t_1+t_2)/2]. \tag{8}$$

Since the double commutator can be written in terms of analytically continued four-point functions (5) and (7) satisfy a "retarded" version of the kernel structure of Fig. 3(a):

$$W_R = \sum_{n=0}^{\infty} K_R^n F_0 \Rightarrow (1-K_R)W_R = F_0, \tag{9}$$

with K_R the retarded kernel [15]. The retarded kernel is composed of the same diagrams as in Fig. 3(a), where one plugs in specific analytic continuations in time of the n-point functions.

If $\lambda_L > 0$, the ladder structure (however complicated) allows us to compute it in cases where the averaged correlator has conformal symmetry (see Ref. [6] for a review). This is done by finding the largest solution $\lambda_L^{\rm ker}$ of the equation $k_R(\lambda_L^{\rm ker}) = 1$, where $k_R(\lambda)$ are the eigenvalues of the retarded kernel K_R. If $\lambda_L^{\rm ker} > 0$ then we can identify $\lambda_L = \lambda_L^{\rm ker}$, and otherwise we learn that $\lambda_L = 0$.

Importantly, if J is exactly marginal, $k_R(\lambda_L)$ can be found perturbatively in J. As a result, one can compute $\lambda_L^{\rm ker}(J)$ in orders of J by using finitely many core CFT correlation functions at every order. The leading order of the equation is explicitly

$$k_R(\lambda) = \frac{J^2}{4}\int d^2x_3 d^2x_4 \exp[\lambda/2(t_3+t_4-t_1-t_2)] \times \langle[\mathcal{O}(\beta/2+it_2),\mathcal{O}(\beta/2+it_4)][\mathcal{O}(it_1),\mathcal{O}(it_3)]\rangle_0 \times \frac{G_{lr,\Delta(q-1)+\frac{\lambda}{2}}(3,4)}{G_{lr,\Delta+\frac{\lambda}{2}}(1,2)} + O(J^4), \tag{10}$$

where $\langle\cdot\rangle_0$ denotes an expectation value at $J=0$ (i.e., of the core CFT). The integration range of the points 3,4 is over the past light cone of the points 1,2, respectively, and $G_{lr,\Delta}$ is the analytically continued cylinder 2-point function:

$$G_{lr,\Delta}(1,2) = \frac{1}{[4\cosh(\frac{t_{12}-x_{12}}{2})\cosh(\frac{t_{12}+x_{12}}{2})]^{\Delta}}. \tag{11}$$

The expansion in J can also be used to determine whether the transition into chaos would be continuous or discontinuous (see Fig. 1). This requires finding the sign of $\lambda_L^{\rm ker}$ in the limit $J=0^+$. Using the leading contribution to k_R in orders of J^2 (10), it is easy to see that the exponent $\lambda_L^{\rm ker}(J=0^+)$ is given by the maximal λ for which the integral (10) diverges. If the integral diverges at a positive (negative) value of λ, we have a continuous (discontinuous) transition into chaos.

At $J=0$, the kernel vanishes, and so it is not clear that $\lambda_L(J=0)$ is related to chaos. Instead, at $J=0$ we find N decoupled core CFTs, and we expect

$$\lim_{t_1,t_2\to\infty} W_R(t_1,t_2)|_{J=0} \sim \frac{1}{N}\exp[+\lambda_L^0(t_1+t_2)/2] \tag{12}$$

for some λ_L^0. Note that the CFTs are decoupled and so λ_L^0 is a property of a single core CFT. However, we emphasize that it is not a chaos exponent in a single core CFT (as we take t_1, t_2 to be larger then any timescale of the core CFT); in fact, for unitary theories in two dimensions it is always nonpositive, $\lambda_L^0 \le 0$ [27].

Surprisingly, under reasonable physical assumptions it can be shown that $\lambda_L^{\rm ker}(J=0^+)$ and λ_L^0 are equal [26]. This amounts to showing that the integral (10) diverges for $\lambda \leq \lambda_L^0$ due to the large (negative) t_3, t_4 regime of the integrand. Putting these pieces together, we claim that $\lambda_L^{\mathrm{ker}}(J)$ satisfies

$$\lambda_L^{\mathrm{ker}}(J = 0^+) = \lambda_L^0. \tag{13}$$

The lhs is calculated through the kernel equation $k_R(\lambda) = 1$ in the limit $J = 0^+$, and the rhs is a property of the core CFT. We can interpret the result by noticing that λ_L^0 describes the core CFT double-commutator behavior at large timescales. At arbitrarily weak coupling this behavior seems to control the $1 \ll t \ll \log N$ behavior of the disordered theory double commutator. The result is striking: in order to determine the type of transition into chaos, it is enough to find (the sign of) λ_L^0 in a single core CFT. Below we give one example for each type of transition into chaos. In both cases we find that our conjecture (13) holds.

Examples of the onset of chaos.—Disordered generalized free fields: We now discuss our first class of examples with exactly marginal chaos, which are the disordered generalized free fields in one dimension and in two dimensions. Generalized free fields (GFFs) are nonlocal theories with no energy-momentum tensor, but can be obtained by specific large-flavor limits of local theories, and are good toy models for more complicated theories. Conformal invariance allows us to set the inverse temperature to be $\beta = 2\pi$ in the following.

In one dimension, the generalized free fermion model is called the "cSYK" model and was introduced in [12]. Explicitly, the action is

$$S = S_0 + S_{\mathrm{SYK}} \tag{14}$$

with

$$S_0 = -\Delta \sum_{i=1}^{n} \int d\tau_1 d\tau_2 \chi_i(\tau_1) \frac{\mathrm{sgn}(\tau_1 - \tau_2)}{|\tau_1 - \tau_2|^{2-2\Delta}} \chi_i(\tau_2),$$
$$S_{\mathrm{SYK}} = \frac{i^{\frac{q}{2}}}{q!} \sum_{i_1,\ldots,i_q=1}^{n} \int d\tau\, J_{i_1 i_2 \ldots i_q} \chi_{i_1} \chi_{i_2} \cdots \chi_{i_q}. \tag{15}$$

Here, $J_{i_1\ldots i_q}$ are Gaussian random variables with variance $\langle J_{i_1\ldots i_q}^2 \rangle = [J^2 (q-1)!/N^{q-1}]$. Choosing $\Delta = 1/q$, the deformation becomes classically marginal, and it is argued in [12] that it is exactly marginal at leading order in $1/N$.

The two- and four-point functions of χ_i for this model were found in [12]. It is simple to extend the results also to the double commutator. One finds that the corresponding retarded kernel K_R^{cSYK} has eigenvalues

$$k_R^{\mathrm{cSYK}}(\lambda) = \left(\frac{\bar{b}(J)}{\bar{b}(J \to \infty)} \right)^q \frac{\Gamma(3-2\Delta)\Gamma(2\Delta+\lambda)}{\Gamma(1+2\Delta)\Gamma(2-2\Delta+\lambda)}, \tag{16}$$

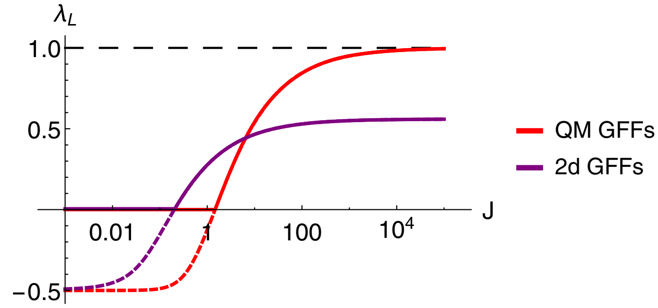

FIG. 4. The chaos exponent $\lambda_L(J)$ at $\Delta = 1/4$ for disordered GFFs in QM and for SUSY disordered GFFs in two dimensions. Dashed lines represent λ_L^{ker} and solid lines represent λ_L.

where $\bar{b}(J)$ solves the equation

$$\frac{\bar{b}^q}{1-2\bar{b}} = \frac{1}{J^2 \psi(1-\Delta)\psi(\Delta)}, \quad \psi(\Delta) \equiv 2\cos(\pi\Delta)\Gamma(1-2\Delta). \tag{17}$$

λ_L^{ker} is found by taking the largest solution to $k_R^{\mathrm{cSYK}}(\lambda) = 1$. The result for $\Delta = 1/4$ appears in Fig. 4, and similar results apply for other Δ.

At any Δ, λ_L^{ker} approaches the maximal value $\lambda_L = 1$ [28] at large J as in the SYK model. More relevant to this study, λ_L^{ker} approaches -2Δ at $J = 0$ (corresponding to the dashed red line), and so becomes negative at small J for any $\Delta > 0$. As discussed above, we cannot identify λ_L^{ker} with λ_L when the former is negative, but we immediately learn that $\lambda_L = 0$ in this regime. We thus conclude that there is no chaos at small enough J, corresponding to λ_L denoted by the solid red line. We thus find a discontinuous transition into chaos, as in Fig. 1(b). We also comment that the long time exponent λ_L^0 of a single core CFT is equal to $\lambda_L^0 = -2\Delta$, which matches $\lambda_L^{\mathrm{ker}}(J = 0^+)$, and so the conjecture (13) is obeyed.

The same analysis can be done for disordered GFFs in two dimensions. In this case we choose to work with $\mathcal{N} = 2$ SUSY GFFs, since this results in an exact conformal manifold even at finite N. The model consists of N generalized free chiral superfields Φ_i of dimension $\Delta = 1/q$, coupled via the superpotential

$$W = \sum_{i_1 \neq \ldots \neq i_q}^{N} J_{i_1\ldots i_q} \Phi_{i_1} \ldots \Phi_{i_q}. \tag{18}$$

The computation is very similar, and the results appear in Fig. 4. At large enough J, the chaos exponent for any Δ approaches the result in the two dimensional SUSY versions of the SYK model [6,9]. In addition, we again find that λ_L^{ker} approaches -2Δ as $J \to 0^+$, and so for small enough J λ_L^{ker} is negative for any $\Delta > 0$. As a result, we again find a discontinuous transition into chaos for any

$\Delta > 0$. We also find once again that the conjecture (13) is obeyed.

Disordered minimal models: We now discuss the disordered 2D $\mathcal{N} = 2$ minimal models. A single core CFT in this case consists of an $\mathcal{N} = 2$ SUSY A_{q-1} minimal model, which can be represented by a single chiral superfield Φ and superpotential $W = \Phi^q$. The full disordered theory is then

$$W = \sum_{i=1}^{N} \Phi_i^q + \sum_{i_1 \neq \ldots \neq i_q} J_{i_1 \ldots i_q} \tilde{\Phi}_{i_1} \ldots \tilde{\Phi}_{i_q}. \tag{19}$$

We emphasize that we interpret this equation in conformal perturbation theory in J around N copies of the A_{q-1} minimal model, where $\tilde{\Phi}$ is the chiral operator of dimension $1/q$, which appears in A_{q-1}. Since the CFT at $J = 0$ has no continuous non-R global symmetries, every classically marginal operator is exactly marginal [29–31], and so each deformation $J_{i_1 \ldots i_q}$ for $i_1 \neq \ldots \neq i_q$ is exactly marginal. Thus each realization of the model is conformal, and as a result averaged correlators of $\tilde{\Phi}_i$ will also be conformal (even at finite N).

We can now attempt to compute $\lambda_L(J)$ in perturbation theory. This computation is generically difficult, and so we focus on the case $q = 3$, where a core CFT has central charge $c = 1$ and so corresponds to the free compact boson at a specific radius. We can identify the operators $\tilde{\Phi}_i$ in terms of vertex operators of the $c = 1$ boson, and as a result we can read off all n-point functions of the $\tilde{\Phi}_i$. Using this result to compute the leading contribution to the retarded kernel at small J given in (10), we find that at small J the exponent λ_L^{ker} approaches zero. As a result, we expect a continuous transition into chaos in this model, as in Fig. 1(a). In particular, the value of λ_L^0 in a single A_2 minimal model is also zero, so that the conjecture (13) is again obeyed.

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