Thermodynamic Signatures of Genuinely Multipartite Entanglement

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The theory of bipartite entanglement shares profound similarities with thermodynamics. In this Letter we extend this connection to multipartite quantum systems where entanglement appears in different forms with genuine entanglement being the most exotic one. We propose thermodynamic quantities that capture a signature of genuineness in multipartite entangled states. Instead of entropy, these quantities are defined in terms of energy—particularly the difference between global and local extractable works (ergotropies) that can be stored in quantum batteries. Some of these quantities suffice as faithful measures of genuineness and to some extent distinguish different classes of genuinely entangled states. Along with scrutinizing properties of these measures we compare them with the other existing genuine measures, and argue that they can serve the purpose in a better sense. Furthermore, the generality of our approach allows us to define suitable functions of ergotropies capturing the signature of k nonseparability that characterizes qualitatively different manifestations of entanglement in multipartite systems.

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Introduction.—Thermodynamics is a framework that deals with the ordering of abstract states connected by abstract processes. Due to their metatheoretic character, thermodynamic laws have withstood several paradigm shifting scientific revolutions and evolved to encompass general relativity and quantum mechanics. Although it started as a phenomenological theory of heat engines, a rigorous axiomatic framework, motivated by the seminal work of Carathéodory [1], has been formulated initially by Giles [2], and more recently by Lieb and Yngvason [3–5]. As thermodynamics has a deep-rooted connection with information theory [6–11], its axiomatic formulation finds profound similarities with the theory of quantum entanglement [12–14]. Like the second law of thermodynamics that prohibits the complete conversion of heat (disordered form of energy) to work (ordered form of energy) in a cyclic process, the theory of entanglement is also governed by a no-go that forbids the creation of entanglement among spatially separated quantum systems under local operations and classical communication (LOCC). This qualitative analogy goes even deeper—in accordance with thermodynamic reversibility, the interconversion among pure bipartite entangled states is reversible under LOCC in the asymptotic limit [15–17]. Furthermore, the rate of interconversion is quantitatively determined by von Neumann entropy, which has a direct relation with thermodynamic entropy [18–21]. For such states, the von Neumann entropy of the reduced marginal, in fact, serves as the unique quantifier (measure) of entanglement [17]. Although the reversibility of entanglement theory breaks down for mixed states [22–25], it does not cancel the analogy between entanglement theory and thermodynamics; rather, it acts as a constitutive element [23].

In this Letter we ask the question how far the analogy between thermodynamics and entanglement theory can go when multipartite systems are considered. This question is quite pertinent, since for such systems classification of quantum states becomes much richer as compared with the separable vs entangled dichotomy of bipartite scenario. Depending on how different subsystems are correlated with each other, qualitatively different classes of entangled states are possible when more than two subsystems are involved. Among these, the most exotic one is the genuinely entangled state that first appears in the seminal Greenberger–Horne–Zeilinger (GHZ) Version of the Bell test [26,27]. Subsequently, it has been shown that genuinely entangled states can also be of different types [28–30]. Identification, characterization, and quantification of genuine entanglement are of practical relevance, as they find several applications [31–39], and accordingly different quantifiers have been suggested [40–46].

In this Letter, we propose thermodynamic quantities that capture a signature of genuineness in multipartite states. Unlike the bipartite pure states, where entanglement is captured through entropic quantity, our proposed measures are defined in terms of internal energy of the system. In particular, the *ergotropic gap*—the difference between the extractable works from a composite system under global and local unitary operations, respectively—plays a crucial role to define these measures. We show that suitably defined functions of this quantity—minimum ergotropic gap, average ergotropic gap, ergotropic fill, and ergotropic volume—can serve as good measures of genuineness for multipartite systems. In fact, one can come up with measures that capture the notion of k separability for arbitrary multipartite systems [13]. Apart from theoretical curiosity these measures are of special interest as there are several proposals for quantum batteries to store ergotropic work [47–53]. By comparing strengths and weaknesses of these newly proposed measures with the other existing genuine measures, we show that the ergotropic measures show superiority.

Preliminaries.—A pure state of a multipartite system consisting of n subsystems is described by a vector $|\psi\rangle_{A_1\cdots A_n} \in \otimes_{i=1}^n \mathcal{H}_{A_i}$, where $\mathcal{H}_{A_i}$ be the Hilbert space associated with the i^{th} subsystem, and for finite dimensional cases they are isomorphic to complex Euclidean space $\mathbb{C}^{d_i}$. Such a state is called k separable if it can be expressed as $|\psi\rangle^{[k]} = |\psi\rangle_{X_1}|\psi\rangle_{X_2}\cdots|\psi\rangle_{X_k}$, where $X_j s$' are nonzero disjoint partitioning of n parties, i.e., $X_j \cap X_{j'} = \emptyset \& \cup_{j=1}^k X_j = \{A_1, \ldots, A_n\}$. A mixed state $\rho \in \mathcal{D}(\otimes_{i=1}^n \mathcal{H}_{A_i})$ is called k separable if it can be expressed as the convex mixture of k-separable pure states, i.e., $\rho^{[k]} = \sum_l p_l |\psi_l\rangle^{[k]}\langle\psi_l|$; $\mathcal{D}(\star)$ denotes the set of density operators acting on the Hilbert space. Note that partitionings of the pure states appearing in the convex decomposition of $\rho^{[k]}$ need not to be fixed. Denoting the convex set of k-separable states as $\mathcal{S}^{[k]}$, we have the set inclusion relations $\mathcal{S}^{[n]} \subsetneq \mathcal{S}^{[n-1]} \subsetneq \cdots \subsetneq \mathcal{S}^{[2]} \subsetneq \mathcal{D}$. The set of states $\mathcal{D}\backslash\mathcal{S}^{[k]}$ are called k nonseparable, and 2 nonseparable states, i.e., states in $\mathcal{D}\backslash\mathcal{S}^{[2]}$, are also called n-partite genuinely entangled. To be a good quantifier or measure of k non-separability, a function $E^{[k]}: \mathcal{D} \to \mathbb{R}_{\geq 0}$ is supposed to satisfy the following properties: (i) $E^{[k]}(\rho) = 0,\ \forall\ \rho \in \mathcal{S}^{[k]}$; (ii) $E^{[k]}(\rho) > 0,\ \forall\ \rho \in \mathcal{S}\backslash\mathcal{S}^{[k]}$; (iii) $E^{[k]}(\sum_i p_i \rho_i) \leq \sum_i p_i E^{[k]}(\rho_i)$, where $\{p_i\}$ is a probability distribution and $\rho_i \in \mathcal{S}^{[k]}$; and (iv) $E^{[k]}(\rho) \geq E^{[k]}(\sigma)$ whenever the state σ can be obtained from the state ρ under the LOCC operation with all of the subsystem being spatially separated. Condition (ii) can be relaxed as $E^{[k]}(\rho) \geq 0$, and in such a case the measure is not faithful. For $k = 2$ we will use the notation $E^{[k]} \equiv E^G$, an it captures genuine entanglement. Two such measures E and E' are said to be equivalent whenever $E(\rho) \geq E(\sigma) \Leftrightarrow E'(\rho) \geq E'(\sigma)$ for all pairs of ρ, σ. For a detailed review of such measures we refer to Refs. [54–58]. In the following we rather briefly recall the concept of ergotropy that will be relevant for us to define thermodynamic measures of multipartite entanglement.

The study of work extraction from an isolated quantum system under a cyclic Hamiltonian process dates back to the late 1970s [59,60]. The aim is to transform a quantum system from a higher to a lower internal energy state, extracting the difference in internal energy as work. The optimal work, termed as ergotropy, is obtained when the system evolves to the passive state [61]. Given the system Hamiltonian $H = \sum_{i=1}^d e_i|\epsilon_i\rangle\langle\epsilon_i|$, with $|\epsilon_i\rangle$ being energy eigenstate having the energy eigenvalue e_i, and given the initial state $\rho \in \mathcal{D}(\mathbb{C}^d)$ of the system, ergotropic work extraction is given by

$$W_e(\rho) = \text{Tr}(\rho H) - \min_U \text{Tr}(U\rho U^\dagger H) = \text{Tr}[(\rho - \rho^p)H]$$

where the passive state ρ^p, being the minimum energetic state, takes the form $\rho^p = \sum_{i=1}^d \lambda_i|\epsilon_i\rangle\langle\epsilon_i|$, with $\lambda_i \geq \lambda_{i+1}$ where $e_i \leq e_{i+1}\ \forall\ i \in \{1, \ldots, d\}$. During the recent past the study of ergotropy received renewed interest for multipartite quantum systems [62–73]. For such systems different kinds of ergotropic works can be extracted. For instance, from a bipartite state $\rho_{AB} \in \mathcal{D}(\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B})$ one can extract global and local ergotropic works $W_e^g(\rho_{AB})$ and $W_e^l(\rho_{AB})$ respectively by applying joint unitaries and product unitaries on the system. The difference of these two ergotropic works is termed as ergotropic gap $\Delta_{A|B}(\rho_{AB})$, which for pure bipartite states has been established as an *independent LOCC monotone* rather than von Neumann entropy, and furthermore it has been shown to satisfy the criteria of a bipartite entanglement measure [68,70,71].

Ergotropy and multiparty entanglement.—For multipartite systems different subgroups of the parties can come together, and accordingly different types of ergotropic works can be extracted from the system (see Fig. 1). For an n-party system we can define the fully separable ergotropic gap $\Delta^{(n)}_{A_1|\cdots|A_n}$ which is the difference between global ergotropy W_e^g obtained when the parties can apply joint unitary all together and fully local ergotropy $W_e^{A_1|\cdots|A_n}$ obtained through local unitaries on the respective subsystems. For a system governed by the Hamiltonian

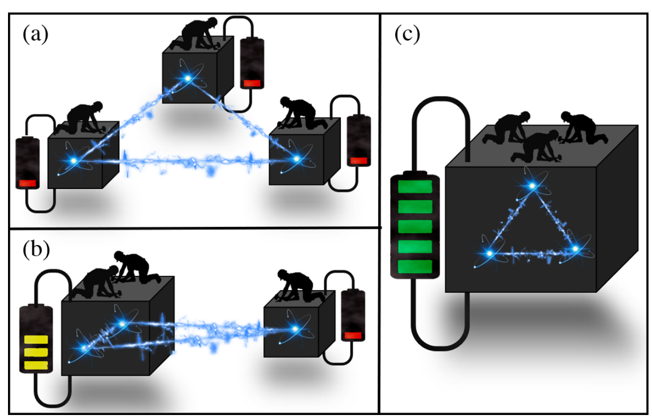

FIG. 1. Different amounts of ergotropic work can be extracted from a multipartite entangled quantum state: (a) local ergotropic work $W_e^{A|B|C} \equiv W_e^l$, (b) biseparable ergotropic work $W_e^{X|X^{\mathsf{C}}}$, with $X \in \{A, B, C\}$, and (c) global ergotropic work W_e^g. In general, $W_e^l \leq W_e^{X|X^{\mathsf{C}}} \leq W_e^g$, where strict inequalities hold for genuinely entangled states.

$H_{A_1\cdots A_n} = \sum_{i=1}^n \tilde{H}_{A_i}$ ($\tilde{H}_{A_i} \equiv \mathbb{I}_{d_1\cdots d_{i-1}} \otimes H_{A_i} \otimes \mathbb{I}_{d_{i+1}\cdots d_n}$) and prepared in a pure state $|\psi\rangle_{A_1\cdots A_n} \in \otimes_{i=1}^n \mathbb{C}^{d_i}$, it turns out that

$$\Delta^{(n)}_{A_1|\cdots|A_n}(|\psi\rangle) = \sum_{i=1}^n \mathrm{Tr}(\rho^p_{A_i} H_{A_i}), \tag{1}$$

where $\rho^p_{A_i}$ is the passive state of the corresponding subsystem. Here and throughout the Letter, without loss of any generality, we associate *zero* energy to the lowest energetic state $|\epsilon_0\rangle$. Passive state energy being the LOCC monotone makes the quantity $\Delta^{(n)}_{A_1|\cdots|A_n}$ a LOCC monotone (see the Supplemental Material [74]). Furthermore, it can be defined as a measure of multipartite entanglement by generalizing it for the mixed state through convex roof extension. For instance, expressing a 3-qubit pure state in generalized Schmidt form [29], $|\psi\rangle_{ABC} = \lambda_0|000\rangle + \lambda_1 e^{\iota\varphi}|100\rangle + \lambda_2|101\rangle + \lambda_3|110\rangle + \lambda_4|111\rangle$, with $\lambda_i \geq 0$ & $\sum_i \lambda_i^2 = 1; 0 \leq \varphi \leq \pi$, we obtain

$$\Delta^{(3)}_{A|B|C}(|\psi\rangle) = \frac{1}{2}\left(\Delta^{(2)}_{A|BC} + \Delta^{(2)}_{B|CA} + \Delta^{(2)}_{C|AB}\right).$$

Here $\Delta^{(2)}_{A|BC}$ denotes the biseparable ergotropic gap across the $A|BC$ cut, i.e., $\Delta_{A|BC} \coloneqq W_e^g - W_e^{A|BC}$ (*mutatis mutandis* for the other terms). The explicit expressions read as

$$\Delta^{(2)}_{A|BC} = 1 - \sqrt{1 - 4\lambda_0^2[1 - (\lambda_0^2 + \lambda_1^2)]}, \tag{2a}$$

$$\Delta^{(2)}_{B|CA} = 1 - \sqrt{1 - 4[\lambda_0^2(\lambda_3^2 + \lambda_4^2) + \alpha]}, \tag{2b}$$

$$\Delta^{(2)}_{C|AB} = 1 - \sqrt{1 - 4[\lambda_0^2(\lambda_2^2 + \lambda_4^2) + \alpha]}, \tag{2c}$$

where $\alpha \coloneqq |(\lambda_1\lambda_4 e^{\iota\varphi} - \lambda_2\lambda_3)|^2$. It is also immediate that $\Delta^{(3)}_{A|B|C}$ is zero for fully product state.

Ergotropy and genuine entanglement.—The quantity $\Delta^{(3)}$ for tripartite systems and more generally $\Delta^{(n)}$ for n-partite systems, despite capturing the signature of multipartite entanglement, does not capture genuineness. In fact, $\Delta^{(n)}$ can take a maximum value for some nongenuine entangled states (see the Supplemental Material [74]). At this point the biseparable ergotropic gap $\Delta^{(2)}$ becomes crucial which for an arbitrary n-partite system can be defined as $\Delta^{(2)}_{X|X^{\mathsf{C}}} \coloneqq W_e^g - W_e^{X|X^{\mathsf{C}}}$, where $W_e^{X|X^{\mathsf{C}}}$ denotes the ergotropic work obtained across $X|X^{\mathsf{C}}$ cut with X being a nonzero subset of the parties and X^{C} denoting the complement set of parties. For a pure state $|\psi\rangle_{A_1\cdots A_n} \in \otimes_{i=1}^n \mathbb{C}^{d_i}$ a straightforward calculation yields

$$\Delta^{(2)}_{X|X^{\mathsf{C}}}(|\psi\rangle) = \mathrm{Tr}(\rho^p_X H_X) + \mathrm{Tr}(\rho^p_{X^{\mathsf{C}}} H_{X^{\mathsf{C}}}), \tag{3}$$

where $H_\star$ and $\rho^p_\star$ denote the Hamiltonian and the passive state of the corresponding partition. Although the biseparable ergotropic gap turns out to be a LOCC monotone, it does not capture genuineness, as $\Delta^{(2)}_{X_1|X_1^{\mathsf{C}}}$ can take a nonzero value even when the state is separable across some $X_2|X_2^{\mathsf{C}}$ partition, where $X_1 \neq X_2$. However, this quantity leads us to define several genuine entanglement measures as listed below.

(i) *Minimum ergotropic gap* ($\Delta^G_{\min}$): It is defined as the minimum among all possible biseparable ergotropic gaps, i.e., for $|\psi\rangle_{A_1\cdots A_n} \in \otimes_{i=1}^n \mathbb{C}^{d_i}$

$$\Delta^G_{\min}(|\psi\rangle) \coloneqq \min\{\Delta^{(2)}_{X|X^{\mathsf{C}}}(|\psi\rangle)\},$$

where minimization is over all possible bipartitions $\{X|X^{\mathsf{C}}\}$ of the parties. Clearly, this is for any pure biseparable state $\Delta^G_{\min} = 0$, whereas for pure genuine entangled states it takes nonzero values. For a 3-qubit system, by analyzing the expressions in Eq. (2) it turns out that $\Delta^G_{\min}$ yields the maximum value for the maximally entangled GHZ (in short ME-GHZ) state and distinguishes it from the W class states [28]. Therefore according to the criterion imposed in Ref. [46] this measure can be called a "proper" measure of genuineness. In fact this result is quite generic. For any n-qubit system the canonical GHZ state $|GHZ_n\rangle = (|0\rangle^{\otimes n} + |1\rangle^{\otimes n})/\sqrt{2}$ gives a maximum value for $\Delta^G_{\min}$ (see the Supplemental Material [74]) indicating superiority of this state over the other classes of genuine entangled states. Furthermore, for the $(\mathbb{C}^2)^{\otimes 3}$ system, it can be shown that $\mathcal{C}_{X|X^{\mathsf{C}}} = \Delta^{(2)}_{X|X^{\mathsf{C}}}(2 - \Delta^{(2)}_{X|X^{\mathsf{C}}})$, where $\mathcal{C}_{X|X^{\mathsf{C}}}$ is the concurrence across $X|X^{\mathsf{C}}$ cut for $X \in \{A, B, C\}$. In this case $\Delta^G_{\min}$ is equivalent to another genuine measure called "genuinely multipartite concurrence" (GMC) defined as the minimum of $\mathcal{C}^2_{X|X^{\mathsf{C}}}$ [55].

Importantly, the minimum ergotropic gap carries a physical meaning as it quantifies the least collaborative advantage in ergotropic work extraction when all the three parties instead of any two of them come together. A drawback of this measure is that the ordering imposed by it is not ideal, as two states with an equal minimum value can have different ergotropies in other bipartitions which evidently tells us that the genuine entanglement of the states must be different.

The measure $\Delta^G_{\min}(|\psi\rangle)$ can be extended to mixed states via convex roof extension and accordingly for a state $\rho_{A_1\cdots A_n} \in \mathcal{D}(\otimes_{i=1}^n \mathbb{C}^{d_i})$, the expression for minimum ergotropic gap becomes

$$\Delta^G_{\min}(\rho_{A_1\cdots A_n}) \coloneqq \min_{\{p_j,\rho_j\}}\left\{\sum_j p_j \Delta^G_{\min}(\rho_j)\right\},$$

where each $\{p_j,\rho_j\}$ is a decomposition of the state $\rho_{A_1\cdots A_n}$ and the minimization is over all possible decompositions. A similar convex roof extension applies for all the measures introduced hereafter.

(ii) *Genuine average ergotropic gap* (Δ^G_{avg}): Instead of the minimum, one can consider the average of all biseparable ergotropic gaps. However it is not a genuine measure as biseparable states can yield a nonzero value. To define a genuine measure for an n-party pure state $|\psi\rangle_{A_1\cdots A_n} \in \otimes_{i=1}^n \mathbb{C}^{d_i}$ we consider the following quantity:

$$\Delta^G_{\mathrm{avg}}(|\psi\rangle) \coloneqq \frac{\Theta(\prod_X \Delta^{(2)}_{X|X^{\mathrm{C}}}(|\psi\rangle))}{2^{(n-1)}-1}\sum_X \Delta^{(2)}_{X|X^{\mathrm{C}}}(|\psi\rangle),$$

where X ranges over all possible bipartitions ($2^{(n-1)}-1$ in number for the n-party system) and $\Theta(Z)=0$ for $Z=0$; otherwise, $\Theta(Z)=1$. Once again, for the three-qubit system this measure is "proper" as it distinguishes ME-GHZ from the W class. Importantly, this measure is inequivalent to $\Delta^G_{\min}$ (see the Supplemental Material [74]). In fact, two genuine states having the same value for one measure can have different values for the other one. This is a quite important observation, as for such a pair of states the measure yielding different values puts a nontrivial restriction on their interconvertibility under LOCC, while the other remains silent.

(iii) *Ergotropic fill* (Δ^G_F): Motivated by the genuine measure of "concurrence fill" recently introduced for three-qubit systems [46], we can define ergotropic fill for such systems as follows:

$$\Delta^G_F(|\psi\rangle) \coloneqq \frac{1}{\sqrt{3}}\left[\left(\sum\nolimits_X \Delta^{(2)}_{X|X^{\mathrm{C}}}\right)^2 - 2\left(\sum\nolimits_X (\Delta^{(2)}_{X|X^{\mathrm{C}}})^2\right)\right]^{\frac{1}{2}},$$

where $X\in\{A,B,C\}$. Ergotropic fill turns out to be independent from concurrence fill [46], GMC [55], and genuine average ergotropic gap (see the Supplemental Material [74]). However, there is ambiguity regarding the monotonicity of this measure under LOCC [58]. Although this measure might be generalized for a four-qubit system, presently we have no idea regarding its generalization for arbitrary multipartite systems.

(iv) *Ergotropic volume* (Δ^G_V): For an n-party state $|\psi\rangle_{A_1\cdots A_n} \in \otimes_{i=1}^n \mathbb{C}^{d_i}$ we can define the normalized volume Δ^G_V of an N-edged hypercuboid with sides $\Delta^{(2)}_{X|X^{\mathrm{C}}}(|\psi\rangle)$ as a genuine measure of entanglement, i.e.,

$$\Delta^G_V(|\psi\rangle) \coloneqq \left(\prod_{X=1}^N \Delta^{(2)}_{X|X^{\mathrm{C}}}(|\psi\rangle)\right)^{\frac{1}{N}}; \quad N=2^{(n-1)}-1.$$

Each edge of the hypercuboid being a LOCC monotone makes the normalized volume the same. While for any pure genuine entangled states the volume is nonvanishing, it is zero for the nongenuine state as at least one of the edges is zero.

Although ergotropic volume has no direct physical meaning, it turns out to be the lower bound of the average ergotropic gap, i.e., $\Delta^G_V \le \Delta^G_{\mathrm{avg}}$ for an arbitrary multipartite state (see the Supplemental Material [74]). Interestingly, for an arbitrary multipartite system among the different states having the same average ergotropic gap the state with equal entanglement across all possible bipartite cuts yield the maximum ergotropic volume and is rated as the most entangled one. This follows from the fact that the geometric mean of a set of numbers with a constant arithmetic mean is the maximum when each number is equal to the arithmetic mean.

For a 3-qubit system Δ^G_V takes a maximum value of 1 for the ME-GHZ, whereas it takes a value of $\frac{2}{3}$ for the maximally W state. In fact, we have obtained the expression Δ^G_V for a generic 3-qubit pure state. It turns out that for some particular ranges of Schmidt coefficients generalized GHZ is more genuinely entangled than maximal W states and hence all the W class states. On the other hand, for a certain range of Schmidt coefficients Δ^G_V of extended GHZ states becomes less than that of the maximal W state. Furthermore, with some examples we also find that Δ^G_V is an independent genuine measure than GMC, minimum ergotropic gap, genuine average ergotropic gap, ergotropic fill, and concurrence fill (see the Supplemental Material [74]). More interestingly, we discuss examples of three-qubit entangled states where concurrence fill cannot order their genuineness but ergotropic volume can. Although for the three-qubit system ergotropic volume is not a smooth function with respect to the generalized Schmidt coefficients, it turns out to be smooth with respect to the biseparable ergotropic gaps.

Ergotropy and k nonseparability.—So far we have seen that a fully ergotropic gap captures the signature of multipartite entanglement, whereas suitably defined functions of a biseparable ergotropic gap turn out to be a good measure of genuine entanglement. As already mentioned, for multipartite systems manifestations of quantum entanglement can be most generally described by k nonseparability with multipartite entanglement and genuine entanglement being the two extreme cases. To capture the notion of k nonseparability here we propose the concept of a k-separable ergotropic gap $\Delta^{(k)}_{X_1|\cdots|X_k} \coloneqq W^g_e - W^{X_1|\cdots|X_k}_e$, where $W^{X_1|\cdots|X_k}_e$ denotes the ergotropic works when n different parties are partitioned as $X_1|\cdots|X_k$. For an arbitrary state $|\psi\rangle_{A_1\cdots A_n}$ the expression for the k-separable ergotropic gap reads as

$$\Delta^{(k)}_{X_1|\cdots|X_k}(|\psi\rangle) = \sum_{j=1}^k \mathrm{Tr}(\rho^p_{X_j} H_{X_j}), \tag{4}$$

where $\rho^p_{X_j}$ and H_{X_j} are the passive state and the Hamiltonian of the associated partition, respectively. Clearly $k=n$ and $k=2$ correspond to Eq. (1) and Eq. (3) respectively. Note that, if the k-separable ergotropic gap for $|\psi\rangle$ is zero for a given k partition, say the partition $X_1|\cdots|X_i|X_{i+1}|\cdots|X_k$, then the $(k-1)$-separable ergotropic gap for the state will be zero if any two subgroups of the parties are together, i.e., $\Delta^{(k)}_{X_1|\cdots|X_i|X_{i+1}|\cdots|X_k}(|\psi\rangle)=0$ implies $\Delta^{(k-1)}_{X_1|\cdots|X_iX_{i+1}|\cdots|X_k}(|\psi\rangle)=0$. However, a zero k-separable ergotropic gap for a given k partition need not imply a zero k-separable ergotropic gap for a different k partition. In other words, $\Delta^{(k)}_{X'_1|\cdots|X'_k}(|\psi\rangle)$ can take a nonzero value even when $\Delta^{(k)}_{X_1|\cdots|X_k}(|\psi\rangle)=0$. This fact restrains the quantity $\Delta^{(k)}_{X_1|\cdots|X_k}$ to be a good measure of k nonseparability. However, following the same technique as done for genuineness it is possible to define suitable functions of the k-separable ergotropic gap that turn out to be good measures of k nonseparability. In the Supplemental Material [74] we discuss a few such quantities—$\Delta^{[k]}_{\mathrm{min}}$, $\Delta^{[k]}_{\mathrm{avg}}$, and $\Delta^{[k]}_{V}$ that for $k=2$ boil down to $\Delta^{G}_{\mathrm{min}}$, $\Delta^{G}_{\mathrm{avg}}$, and Δ^{G}_{V}, respectively.

Discussion.—Genuine entanglement represents prototypical features of multipartite quantum systems. Apart from their foundational importance [26] they find several applications [31–39], and they are also crucial for the emerging technology of quantum Internet [79,80]. Here we have proposed several measures of genuine entanglement based on thermodynamic quantities. The correspondence between thermodynamics and entanglement theory is not new as information theory makes a link between bipartite entanglement and thermodynamics through the abstract concept of entropy. Importantly, the connection established between genuine entanglement and thermodynamics in this Letter is much more direct as it does not invoke entropy; rather it is based on internal energies or ergotropic works of the system. Ergotropic work being an experimentally measurable quantity, even under ambient conditions, makes this connection more interesting. In particular, we have introduced four different measures for genuine entanglement among which ergotropic volume has been inferred to perform better than the other three as well as the previously existing measures. Importantly, ergotropic volume also captures a physical meaning up to some degree while still maintaining the integrity as a genuine multipartite entanglement measure without any ad-hoc conditions. Furthermore, we have shown that based on ergotropic quantities it is also possible to define measures of k nonseparability that signify qualitatively different manifestations of entanglement for multipartite systems.

As for the future, possible experimental realizations of the proposed measures would be quite interesting. It would be instructive to explore the multiqubit systems, more particularly the three-qubit system, first. Another possible study would be to see how the ergotropic gap decreases when more and more restrictions are imposed on the collaboration among the parties, as this would give an idea whether or not the cost of coming together pays off significant increment in work extraction. It would also be interesting to capture the signature of *entanglement depth* [81] of an multipartite state through the ergotropic approach explored in this Letter. Finally, it would also be interesting to see how our approach can be generalized to study other forms of correlation in mutipartite systems; a closely related study was recently made for bipartite systems in Ref. [73].

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