Parisi-Sourlas Supersymmetry in Random Field Models

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By the Parisi-Sourlas conjecture, the critical point of a theory with random field (RF) disorder is described by a supersymmeric (SUSY) conformal field theory (CFT), related to a $d-2$ dimensional CFT without SUSY. Numerical studies indicate that this is true for the RF ϕ^3 model but not for the RF ϕ^4 model in $d<5$ dimensions. Here we argue that the SUSY fixed point is not reached because of new relevant SUSY-breaking interactions. We use a perturbative renormalization group in a judiciously chosen field basis, allowing systematic exploration of the space of interactions. Our computations agree with the numerical results for both cubic and quartic potential.

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Introduction.—Emergent symmetries are a frequent theme in modern theoretical physics. Such a symmetry is present at long distances but is not visible in the microscopic description of the system. A beautiful example is furnished by the physics of disordered systems, namely by the random field Ising model (RFIM) and its cousins. Parisi and Sourlas suggested long ago [1,2] that the critical points of these models obey emergent supersymmetry. While supersymmetry plays a prominent role in high-energy physics, its appearance in the statistical physics context came as a major surprise. A dramatic consequence of supersymmetry is dimensional reduction [3]: the critical exponents of a disordered system in d dimensions should be the same as those of the pure (i.e., nondisordered) system in $d-2$ dimensions.

Unfortunately, after 40 years of work, there is still no complete understanding whether, when, and how Parisi-Sourlas supersymmetry actually emerges. Most work has focused on the random field ϕ^4 and ϕ^3 field theories, describing respectively the phase transition in RFIM and the statistics of branched polymers (BP) in a solution [4–6]. Numerical studies of microscopic models suggest that supersymmetry and dimensional reduction are present in any dimension for the ϕ^3 case [7] but only in sufficiently high d for the ϕ^4 case [8–11]. Why does this happen? One possibility is that some supersymmetry (SUSY)-breaking perturbations are *dangerously irrelevant*, i.e., irrelevant for high d, and become relevant at lower d and break supersymmetry [12,13] [14]. In this Letter we will report the first systematic exploration of this scenario. We will show that it gives a satisfactory unified description of phenomenology in agreement with all available numerical results [15].

The model and prior work.—A random field (RF) model describes a statistical field theory with quenched disorder coupled to a local order parameter. We consider RF models of the type

$$\mathcal{S}[\phi,h]=\int d^dx\left[\frac{1}{2}(\partial_\mu\phi)^2+V(\phi)+h(x)\phi(x)\right],\qquad(1)$$

where $h(x)$ is drawn from a Gaussian distribution with zero mean and $\overline{h(x)h(0)}=H\delta(x)$. Parisi-Sourlas (PS) conjecture [1] about the critical points of these theories can be naturally divided in two parts: (1) *Emergence of SUSY.* The critical point of a RF theory is described by a special SUSY conformal field theory (CFT) (PS CFT). (2) *Dimensional reduction.* A large class of observables of the PS CFT (e.g., its critical exponents) is described by an ordinary CFT living in $\hat{d}\equiv d-2$ dimensions.

While perturbatively valid for d infinitesimally close to the upper critical dimension $d_{\rm uc}$ (see below), this remarkable conjecture is known to sometimes fail for the physically interesting cases of integer $d<d_{\rm uc}$.

As mentioned, the two most studied RF models are with ϕ^4 (RFIM) and ϕ^3 (BP) potentials. The RF ϕ^4 model has a critical point in $3\le d<d_{\rm uc}=6$. PS conjecture would relate it to the usual Ising model in $\hat{d}$ dimensions. Numerical studies [8–11] show that while both SUSY and dimensional reduction hold in $d=5$, the conjecture

fails in $d = 4$. It also fails trivially for $d = 3$, as the $\hat{d} = 1$ Ising model has no phase transition.

Similarly, the critical point of the RF ϕ^3 model with imaginary coupling should be described by the usual Lee-Yang fixed point in $\hat{d}$ dimensions [22]. BP critical exponent simulations suggest that this instance of PS conjecture works perfectly for any $2 \le d < d_{uc} = 8$ [7,23].

Let us come back to the central question of why PS conjecture sometimes works and sometimes fails. Many perturbative and nonperturbative arguments were given for Part 2 of the conjecture [1,25–29]. On the other hand Part 1 appears to be on less solid ground. Here we will focus on the scenario [12,13] that Part 1 may fail due to dangerously invariant SUSY-breaking interactions.

From replicas to Cardy fields.—We start by using the usual replica method where we take n copies of the action [Eq. (1)] and average out the disorder. This gives the replica action

$$\mathcal{S}_n = \int d^d x \left[\sum_{i=1}^{n} [(\partial_\mu \phi_i)^2 + V(\phi_i)] - \frac{H}{2} \left(\sum_{i=1}^{n} \phi_i \right)^2 \right] \quad (2)$$

from which one can get quenched averaged correlations functions $\overline{\langle A(\phi) \rangle}$ in a $n \to 0$ limit by simply computing $\langle A(\phi_1) \rangle$ having a single replica field.

We next apply Cardy's linear field transform [30],

$$\varphi = \frac{1}{2}(\phi_1 + \rho), \qquad \omega = \phi_1 - \rho, \qquad \chi_i \overset{i \neq 1}{=} \phi_i - \rho, \quad (3)$$

with $\rho = (1/n - 1) \sum_{i=2}^{n} \phi_i$ and the condition $\sum_{i=2}^{n} \chi_i = 0$. Turning off interactions for now ($V = 0$), the transformed Lagrangian takes the form

$$\mathcal{L}^{\text{free}} = \partial_\mu \varphi \partial_\mu \omega - \frac{H}{2} \omega^2 + \frac{1}{2} \sum_{i=2}^{n} (\partial_\mu \chi_i)^2. \quad (4)$$

Here and below, because of the replica limit $n \to 0$, we are dropping all terms proportional to powers of n.

From Eq. (4) we read off the classical scaling dimensions of the Cardy fields: $[\varphi] = (d/2) - 2$, $[\chi_i] = (d/2) - 1$, $[\omega] = (d/2)$. In contrast, the original replica fields ϕ_i do not even have a well-defined scaling dimension [31]. Although not manifest in the Cardy field basis, the S_n symmetry is still present and in particular not spontaneously broken [33]. It will play an important role below.

While RF criticality is often described in terms of special "zero-temperature fixed points" [35,36], Cardy transform puts it on the same footing as the more familiar non-disordered criticality. Using Cardy fields, we will be able to perform the RG analysis for the RF models borrowing the standard Wilsonian methodology [37,38].

Leaders and followers.—Let us now turn the interactions back on, and see how the theory renormalizes. Lagrangian [Eq. (2)] contains the interaction $\sum_{i=1}^{n} V(\phi_i)$. This can be written as a sum of basic S_n singlet interactions $\sigma_k \equiv \sum_{i=1}^{n} \phi_i^k$. In an exhaustive analysis, we will have to consider further interaction terms respecting the replica permutation symmetry S_n, since they will be generated by RG evolution [12]. Examples of such allowed interactions are products of σ_k's as well as interactions containing derivatives. We will classify S_n singlet interactions in the original fields of Eq. (2) and then transform them to the Cardy fields.

The simplest interaction is the mass term σ_2 which in Cardy fields reads $2\varphi\omega + \chi_i^2$ and has classical dimension $d - 2$. Continuing at the cubic level, the operator σ_3 under Cardy transform becomes

$$\sigma_3 = (3\varphi^2 \omega + 3\chi_i^2 \varphi) + (\chi_i^3) - \left(\frac{3}{2} \chi_i^2 \omega \right) + \left(\frac{1}{4} \omega^3 \right), \quad (5)$$

where different terms have unequal classical dimensions: $(3d/2) - 4$ for the first term, while the successive ones sit 1, 2, and 3 units higher. This new effect is generic: any singlet operator $\mathcal{O}$ in Cardy fields can be written as

$$\mathcal{O} = \mathcal{O}_L + \mathcal{O}_{F_1} + \mathcal{O}_{F_2} + \cdots, \quad (6)$$

where $[\mathcal{O}_{F_i}] = [\mathcal{O}_L] + i$, $i = 1, 2, \ldots$. We call the lowest dimension part $\mathcal{O}_L$ the "leader" and $\mathcal{O}_{F_i}$ "followers."

In the first part of a Wilsonian RG step, integrating out a momentum shell and lowering the momentum cutoff $\Lambda \to \Lambda/b$ ($b > 1$), a singlet operator $\mathcal{O}$, if present in the effective action, renormalizes as a whole, i.e., only through the change of the overall coefficient: $g\mathcal{O} \to \tilde{g}\mathcal{O}$ [39]. This is guaranteed by S_n symmetry. On the other hand, in the second part of a RG step, bringing the cutoff back up to its original value, which rescales the fields φ, χ_i, ω according to their classical dimensions, the followers rescale by different coefficients from the leader, suppressing their relative effect in the IR (i.e., at large b):

$$\mathcal{O}_L + \sum_i \mathcal{O}_{F_i} \to b^{-[\mathcal{O}_L]} \left(\mathcal{O}_L + \sum_i b^{-i} \mathcal{O}_{F_i} \right). \quad (7)$$

Hence, the RG flow in the IR is controlled by the leaders. This drastically reduces the number of interactions to consider: only operators in Cardy fields which can be written as a leader of an S_n singlet interaction are of interest. The RG relevance or irrelevance of the leader determines the fate of the whole interaction [40].

Keeping the free massless Lagrangian [Eq. (4)], the mass term, and the leader parts $(\sigma_2)_L$ or $(\sigma_3)_L$ of the ϕ^3 or ϕ^4 interactions, we get the two Lagrangians relevant for the description of the RF ϕ^3 and ϕ^4 models:

$$\mathcal{L}_L^{\phi^3} = \mathcal{L}^{\text{free}} + m^2(2\varphi\omega + \chi_i^2) + \frac{g}{2}(\varphi^2\omega + \chi_i^2\varphi),$$
$$\mathcal{L}_L^{\phi^4} = \mathcal{L}^{\text{free}} + m^2(2\varphi\omega + \chi_i^2) + \frac{g}{12}(2\varphi^3\omega + 3\chi_i^2\varphi^2). \quad (8)$$

The mass term m^2 is strongly relevant and should be tuned to reach the IR fixed point. The upper critical dimension in this approach is fixed simply from the marginality of the leading nonquadratic interaction, which gives the well-known values cited above: $d_{\rm uc} = 8$ for the ϕ^3 and 6 for the ϕ^4 models.

Equation (8) gives the correct effective theory for the two models close to their upper critical dimension, i.e., for $d = d_{\rm uc} - \epsilon$, $\epsilon \ll 1$. Indeed, one can check that in this case, no other S_n singlet interactions exist whose leaders would be relevant (and, for the ϕ^4 case, respecting the extra $\mathbb{Z}_2$ symmetry). However, we should keep an open mind about what may happen for $\epsilon = \mathrm{O}(1)$, as some irrelevant interactions may become relevant. This will be investigated below.

Emergence of SUSY.—It is easy to see that both Lagrangians [Eq. (8)] have emergent SUSY [30]. Note that the $n-2$ fields χ_i appear quadratically in the Lagrangians. The associated partition function is given by a Gaussian integral which at $n \to 0$ is equal to that of 2 anticommuting scalars $\psi, \bar{\psi}$. So we are allowed to replace $\chi_i\chi_i \to 2\psi\bar{\psi}$. Then both the above theories can be compactly written as

$$\mathcal{S}_{\rm susy} = \int d^d x d\theta d\bar{\theta}\left[-\frac{1}{2}\Phi\partial^a\partial_a\Phi + V(\Phi)\right], \qquad (9)$$

where $V(\phi) = m^2\phi^2 + (g/6)\phi^3$ for the cubic theory and $V(\phi) = m^2\phi^2 + (\lambda/4!)\phi^4$ for the quartic. Here $\Phi(x,\theta,\bar{\theta}) = \varphi + \theta\bar{\psi} + \bar{\theta}\psi + \theta\bar{\theta}\omega$ is a superfield depending on coordinates $x, \theta, \bar{\theta}$ parametrizing the superspace $\mathbb{R}^{d|2}$ with $\mathrm{OSp}(d|2)$ supergroup symmetry (PS supersymmetry), and $\partial^a\partial_a$ is the super-Laplacian (index a takes values $1, \ldots, d, \theta, \bar{\theta}$). In the IR, we get a further enhancement to a PS superconformal symmetry $\mathrm{OSp}(d+1,1|2)$ [41]. The fixed point of this theory is therefore a PS CFT.

We now briefly describe basic properties of PS CFTs and how they undergo dimensional reduction, as shown in the Supplemental Material [42]. Local operators in such theories are classified according to their superconformal dimension Δ and their $\mathrm{OSp}(d|2)$ spin ℓ. They are grouped in superconformal multiplets containing a superprimary operator $\mathcal{O}^{\mathbf{a}}_{\Delta\ell}$ (where $\mathbf{a}$ stands for $a_1a_2\ldots$), annihilated by the special superconformal generator K^a, and its super-descendants such as $\partial_a\mathcal{O}^{\mathbf{a}}_{\Delta\ell}$ and higher superderivatives. $\mathcal{O}^{\mathbf{a}}_{\Delta\ell}$ can be expanded in components which have different conformal dimensions:

$$\mathcal{O}^{\mathbf{a}}(y) = \underbrace{\mathcal{O}^{\mathbf{a}}_0(x)}_{\Delta} + \theta\underbrace{\mathcal{O}^{\mathbf{a}}_{\theta}(x)}_{\Delta+1} + \bar{\theta}\underbrace{\mathcal{O}^{\mathbf{a}}_{\bar{\theta}}(x)}_{\Delta+1} + \theta\bar{\theta}\underbrace{\mathcal{O}^{\mathbf{a}}_{\theta\bar{\theta}}(x)}_{\Delta+2}. \qquad (10)$$

Dimensional reduction restricts correlators of a PS CFT to a $(d-2)$-dimensional bosonic subspace $\mathcal{M}_{\hat{d}} \equiv \{y = (\hat{x}^\alpha, 0,0,0,0), \hat{x} \in \mathbb{R}^{\hat{d}}\}$. In addition, one only considers PS CFT operators invariant under the subgroup $\mathrm{OSp}(2|2)$ (super)rotating the directions orthogonal to $\mathcal{M}_{\hat{d}}$. In general, restricting to a subspace gives a nonlocal theory. The nontrivial fact is that by restricting the $\mathrm{OSp}(2|2)$-singlet sector of the SUSY theory, we get a local $\hat{d}$-dimensional CFT living on $\mathcal{M}_{\hat{d}}$ [29]. The local conserved $\mathrm{CFT}_{\hat{d}}$ stress tensor appears in this setup as the $\mathcal{T}_0$ component of the PS CFT superstress tensor $\mathcal{T}$.

The dimensionally reduced $\mathrm{CFT}_{\hat{d}}$ has the global symmetry of the original PS CFT: trivial in the ϕ^3 case and $\mathbb{Z}_2$ for ϕ^4. We will naturally assume that this $\mathrm{CFT}_{\hat{d}}$ is nothing but the $\hat{d}$-dimensional critical point of the same theory without disorder [45]: the Wilson-Fisher fixed point for ϕ^4 [47] and the Lee-Yang fixed point for ϕ^3 [22]. Dimensions of many operators in these familiar theories being well known both perturbatively and, sometimes, nonperturbatively, we can then use dimensional reduction to infer dimensions of operators in the PS CFT.

The central question is whether any S_n singlet perturbation, while irrelevant for $\epsilon \ll 1$, may become relevant for $\epsilon = O(1)$ and destabilize the SUSY IR fixed point. As discussed above, this may be answered by perturbing the Lagrangians $\mathcal{L}_L$ in Eq. (8) by the leader terms of S_n singlet interactions, and computing their scaling dimensions (restricting to $\mathbb{Z}_2$ singlets for the ϕ^4 case). *A priori* there are many leaders to consider, which moreover may mix under RG. Below we will divide them into three classes: SUSY writable (SW), SUSY null, (SN), and non-SUSY writable (NSW), with a triangular mixing matrix. Namely SN operators can generate only SN under RG flow, SW can generate SW and SN, while NSW can generate all three classes.

SW leaders.—These are invariant under $O(n-2)$ acting on the indices of the χ_i fields. These operators can be transformed to the SUSY field bases by the substitution $\chi_i \to \psi$ (hence the name). With abuse of language we will also refer to the resulting Sp(2)-invariant operators as SW. In addition, we require that the operator does not vanish after the substitution (if so it will be classified below as SUSY null). Most low-lying leaders turn out to be SW. For example, the leader of any S_n singlet $\sum_{i=1}^n A(\phi_i)$ has the form $A'(\varphi)\omega + \frac{1}{2}A''(\varphi)\chi_i^2$ which is SW. This can be written as the highest component $A_{\theta\bar{\theta}}(\Phi)$ of a scalar composite superfield $A(\Phi)$. More generally, SW leaders are always in the highest component of a superfield [48]. They do not have to be scalars of $\mathrm{OSp}(d|2)$, but only singlets of the subgroup $\mathrm{SO}(d)\times\mathrm{Sp}(2)$. These are obtained from a highest component $\mathcal{O}^{\mathbf{a}}_{\theta\bar{\theta}}$ by contracting all its $\mathbf{a}$ indices with the Sp(2) metric, i.e., by setting the indices to θ and $\bar{\theta}$ [42].

The $\mathrm{OSp}(d|2)$ tensor representations of $\mathcal{O}^{\mathbf{a}}$ are associated to the Young tableaux (YT) $(\ell_1, \ell_2, \cdots)$ with ℓ_i boxes in the ith row. Indices along the rows (columns) are graded (anti)symmetrized, and all supertraces are removed. Graded symmetry and antisymmetry respectively mean $\mathcal{O}^{ab} = (-1)^{[a][b]}\mathcal{O}^{ba}$ and $\mathcal{O}^{ab} = -(-1)^{[a][b]}\mathcal{O}^{ba}$ where $[a] = 0(1)$ if a is bosonic (fermionic). These general facts combined with the above procedure of setting the indices to θ and $\bar{\theta}$

shows that SW leaders can only be obtained from operators in representations labeled by YT of the form $(2, 2, \ldots, 2)$. SW leaders are thus in correspondence with the superfields $\mathcal{S}_{\theta\bar{\theta}}, \mathcal{J}^{\theta\bar{\theta}}_{\theta\bar{\theta}}, \mathcal{B}^{\theta\bar{\theta},\theta\bar{\theta}}_{\theta\bar{\theta}}, \ldots$ where $\mathcal{S}$ is a scalar, $\mathcal{J}^{ab}$ a spin two, and $\mathcal{B}^{ab,cd}$ a "box" operator in the YT (2,2) representation where (a, b) and (c, d) are the graded-symmetric pairs. Representations with a higher number of rows can also appear in generic d but we do not consider them since they have large classical dimensions.

The above formal considerations have a neat practical consequence: dimensions of SW leaders $\mathcal{O}_{\theta\bar{\theta}}$ can be obtained by studying the respective operators $\hat{\mathcal{O}}$ in the dimensionally reduced model using $\Delta_{\mathcal{O}_{\theta\bar{\theta}}} = \Delta_{\mathcal{O}} + 2 = \Delta_{\hat{\mathcal{O}}} + 2$ from Eq. (10). From here we see immediately that SW leaders originating from scalar and spin two PS CFT operators cannot destabilize the SUSY fixed point. Indeed in both dimensionally reduced models all scalars (besides the mass term which we tune to reach the fixed point) are irrelevant. Similarly all the spin two operators should not cross the stress tensor and thus are expected to remain irrelevant in any d [49].

Separate analysis is needed for operators in the box representation. In the dimensionally reduced models, an infinite family of such operators can be written in terms of $\hat{d}$-dimensional scalar field $\hat{\phi}$ as

$$\hat{B}^{(k)}_{\alpha\beta,\gamma\delta} \equiv \hat{\phi}^{k-3}\left(\hat{\phi}_{,\alpha\beta}\hat{\phi}_{,\gamma\delta}\hat{\phi} - \frac{2\hat{d}}{\hat{d}-2}\hat{\phi}_{,\alpha}\hat{\phi}_{,\beta}\hat{\phi}_{,\gamma\delta}\right)_Y, \tag{11}$$

with $k \geq 3$. Greek letters denote $\mathbb{R}^{\hat{d}}$ indices, and Y indicates the box YT symmetrization, the two symmetric rows being $\alpha\beta$ and $\gamma\delta$. These are the lowest dimensional operators made of k fields in such representation.

We computed their perturbative one-loop dimensions for the ϕ^3 case [46], following the standard ϵ-expansion methodology [37,38], while the ϕ^4 case was considered previously in Ref. [50]. The results (classical dimension plus one-loop correction) are

$$\Delta_{\mathcal{B}^{(k)}_{\theta\bar{\theta}}} = \begin{cases} \left(2k + 6 - \frac{k}{2}\epsilon\right)_{\text{cl}} + \frac{2k^2-5k-2}{6}\epsilon & (\phi^3), \\ \left(k + 2 - \frac{k}{2}\epsilon\right)_{\text{cl}} + \frac{(k-3)(3k+2)}{18}\epsilon & (\phi^4). \end{cases} \tag{12}$$

Importantly, all anomalous dimensions are positive (excluding the $k = 3$ ϕ^4 case which, as all odd k for ϕ^4, is unimportant since it does not respect $\mathbb{Z}_2$ symmetry).

SN leaders.—These are singlets under $O(n-2)$ (like the SW operators) and satisfy the property of vanishing under the $\chi \to \psi$ map by the Grassmann nature of ψ. A typical example is $(\chi_i^2)^2 \to (\psi\bar{\psi})^2 = 0$. These operators have restrictive mixing properties and can only generate operators of the same class under RG. We identified an infinite class of S_n singlets [42],

$$\mathcal{N}_k = \frac{2}{k-3}\left(\frac{\sigma_2\sigma_{k-2}}{k-2} - \frac{2\sigma_1\sigma_{k-1}}{k-1}\right), \tag{13}$$

for $k = 4, 5, 6, \ldots$, which have SN leaders $(\mathcal{N}_k)_L = \varphi^{k-4}(\chi_i^2)^2$. The $k = 4$ operator is the lowest dimensional SN leader overall, while $(\mathcal{N}_k)_L$ is the lowest dimensional SN leader made of k fields.

Unlike for SW leaders, we cannot use SUSY theory and dimensional reduction to infer the scaling dimensions of SN operators (since they vanish identically in SUSY fields). We compute them directly from action [Eq. (8)]. Our Cardy field approach makes these computations methodologically straightforward, being analogous to the standard ϵ expansion [37,38]. We thus computed the leading anomalous dimension of operators [Eq. (13)]. The resulting scaling dimensions (classical plus one-loop) are given by

$$\Delta_{(\mathcal{N}_k)_L} = \begin{cases} \left(2(k+2) - \frac{\epsilon}{2}k\right)_{\text{cl}} + \frac{6k^2-7k-48}{18}\epsilon & (\phi^3), \\ \left(k + 4 - \frac{\epsilon}{2}k\right)_{\text{cl}} + \frac{(k-4)(k+3)}{6}\epsilon & (\phi^4). \end{cases} \tag{14}$$

The one-loop correction is positive except for the $k = 4$, ϕ^4 case when it vanishes. Then, the first nonzero correction appears at two loops, and it is negative [40]:

$$\Delta_{(\mathcal{N}_4)_L} = (8 - 2\epsilon)_{\text{cl}} - \frac{8}{27}\epsilon^2 \quad (\phi^4). \tag{15}$$

NSW leaders.—These operators are singlets under the S_{n-1} that permutes the fields χ_i, but not under $O(n-2)$, and therefore they cannot be mapped to $\psi, \bar{\psi}$ fields. A typical example would be any leader involving $\sum_{i=2}^n \chi_i^3$. In the RG flow, leader perturbations belonging to this class can generate perturbations from the other two classes, while the opposite mixing is forbidden by SUSY.

We investigated two infinite families of S_n singlets having NSW leaders, as shown in the Supplemental Material [42]. The first family, first discussed by Feldman [13] and in Ref. [40], is given by

$$\mathcal{F}_k = \sum_{i,j=1}^{n}(\phi_i - \phi_j)^k = \sum_{l=1}^{k-1}(-1)^l\binom{k}{l}\sigma_l\sigma_{k-l}, \tag{16}$$

with $k = 6, 8, 10, \ldots$ [51]. They give rise to NSW leaders made only of χ fields, of the form

$$(\mathcal{F}_k)_L = \sum_{l=2}^{k-2}(-1)^l\binom{k}{l}(\chi_i^l)(\chi_j^{k-l}). \tag{17}$$

The first leader of this family, $(\mathcal{F}_6)_L$, is the lowest dimensional NSW leader overall.

The second family consists of S_n singlets given by

$$\mathcal{G}_k \equiv \frac{\sigma_3\sigma_{k-3}}{3(k-5)} + \frac{\sigma_1\sigma_{k-1}}{k-1} - \frac{(k-4)\sigma_2\sigma_{k-2}}{(k-5)(k-2)}, \qquad (18)$$

for $k = 6, 7, 8, \ldots$. These have NSW leaders

$$(\mathcal{G}_k)_L = \frac{(k-4)(k-3)}{36}\varphi^{k-6}[2(\chi_i^3)^2 - 3(\chi_i^2)(\chi_j^4)]. \qquad (19)$$

The two families start from the same operator ($\mathcal{G}_6 \propto \mathcal{F}_6$), but the higher operators are different. In fact $(\mathcal{G}_k)_L$ is the lowest NSW leader made of k fields and, in particular, sits lower than $(\mathcal{F}_k)_L$ for $k > 6$.

Like for the SN class, we computed NSW scaling dimensions by the ϵ-expansion methodology adapted to action [Eq. (8)]. Starting with the $\mathcal{F}_k$ family, the scaling dimension (classical plus the leading correction) is given by

$$\Delta_{(\mathcal{F}_k)_L} = \begin{cases} \left(3k - \frac{k}{2}\epsilon\right)_{\rm cl} + \frac{2k^2-3k}{18}\epsilon & (\phi^3), \\ \left(2k - \frac{k}{2}\epsilon\right)_{\rm cl} - \frac{k(3k-4)}{108}\epsilon^2 & (\phi^4). \end{cases} \qquad (20)$$

Notably, the leading anomalous dimension is one loop and positive in the ϕ^3 case [46] while it is two loop and negative for ϕ^4 [13,40].

Considering next the $\mathcal{G}_k$ family, we obtained

$$\Delta_{(\mathcal{G}_k)_L} = \begin{cases} \left(2(k+3) - \frac{k}{2}\epsilon\right)_{\rm cl} + \frac{6k^2-7k-120}{18}\epsilon & (\phi^3) \\ \left(k+6 - \frac{k}{2}\epsilon\right)_{\rm cl} + \frac{(k-6)(k+5)}{6}\epsilon. & (\phi^4) \end{cases} \qquad (21)$$

The one-loop correction is therefore always positive, except in the $k = 6$, ϕ^4 case when it vanishes. In the latter case, using $\mathcal{G}_6 \propto \mathcal{F}_6$ and Eq. (20), we see that the leading, negative correction appears at two loops.

Does SUSY emerge at $\epsilon = O(1)$?—The analysis leading to SUSY was based on the effective Lagrangians [Eq. (8)]. It would be invalidated if a new relevant leader interaction were found in the IR. Allowed by symmetry, such a growing perturbation will be generated by the RG, destabilizing the flow and leading it away from the SUSY fixed point. Let us see if this scenario is realized.

Above we discussed several infinite families of leader interactions from three different classes (SW, SN, NSW). We will now focus on the lowest dimensional operators for each class. We expect them to be most important to decide on the stability of the SUSY fixed point. First of all, ϵ-expansion computations of lowest-dimensional operators should be more reliable than for higher-dimensional ones [52]. Second, we expect crossing of operator dimensions (within the same mixing class) to be avoided nonperturbatively.

With this in mind we find that the SUSY IR fixed point of the RF ϕ^3 theory should always be stable, since the lowest leader perturbations $\mathcal{B}^{(3)}_{\theta\bar\theta}$, $\mathcal{N}_4$, $\mathcal{F}_6$ never become relevant. To see this we take their one-loop dimensions given in Eqs. (12), (14), and (20) and use these expressions in the full range of interest $2 \le d < 8$ [54].

However, the same argument for the ϕ^4 case reaches a different conclusion [40]. While $\mathcal{B}^{(4)}_{\theta\bar\theta}$ remains irrelevant [55], both $(\mathcal{N}_4)_L$ and $(\mathcal{F}_6)_L$ become relevant at some critical dimension d_c between four and five, namely $\Delta_{(\mathcal{N}_4)_L} = d$ at $d = d_c \approx 4.6$ while $\Delta_{(\mathcal{F}_6)_L} = d$ when $d_c \approx 4.2$. The precise value of d_c, and which of the two operators crosses marginality first, should be taken with a grain of salt coming from a two-loop computation. We may estimate the uncertainty replacing the expressions in Eqs. (15) and (20) by their Padé$_{[1,1]}$ rational approximants. We then find that $(\mathcal{N}_4)_L$ crosses marginality at $d_c \approx 4.7$, while $(\mathcal{F}_6)_L$ crosses it at $d_c \approx 4.5$.

NSW interaction $(\mathcal{F}_6)_L$ clearly breaks SUSY. Operator $(\mathcal{N}_4)_L$ is also potentially SUSY breaking, by affecting NSW coupling evolution (while being SN it does not directly affect SW sector). We thus conclude that SUSY will be present in the RF ϕ^4 model for $d_c < d < 6$, while it will be lost for $d < d_c$ [56].

Remarkably, our findings exactly match the expectations from numerical studies mentioned at the beginning, for both universality classes. It is encouraging that already the leading order ϵ-expansion results lead to this agreement. In the future, it would be interesting to determine our d_c more accurately. This can be done systematically, increasing the perturbative order and using Borel resummation techniques, as is standard for the usual Wilson-Fisher fixed point [38,57–59].

Finally, we wish to compare our results to functional renormalization group studies of the RF ϕ^4 model, which also predict the loss of SUSY for $d < d_c^{\rm FRG} \approx 5.1$ [60]. While their d_c is similar, their mechanism is quite different from ours, being attributed to fixed point annihilation [61], so that below d_c the SUSY fixed point does not exist. On the contrary, our SUSY fixed point exists for any d, being simply RG unstable for $d < d_c$. If so, one should be able to detect SUSY in lattice simulations for $d = 4$, by performing additional tuning [62]. This would be a decisive confirmation for our scenario.

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