Exact Solution of the Macroscopic Fluctuation Theory for the Symmetric Exclusion Process

Kirone Mallick,[1,*] Hiroki Moriya,[2,†] and Tomohiro Sasamoto[2,‡]
[1]*Institut de Physique Théorique, CEA, CNRS, Université Paris–Saclay, F–91191 Gif-sur-Yvette cedex, France*
[2]*Department of physics, Tokyo Institute of Technology, Tokyo 152-8551, Japan*

(Received 20 February 2022; accepted 24 June 2022; published 19 July 2022)

We present the first exact solution for the time-dependent equations of the macroscopic fluctuation theory (MFT) for the symmetric simple exclusion process by combining a generalization of the canonical Cole-Hopf transformation with the inverse scattering method. For the step initial condition with two densities, we obtain exact and compact formulas for the optimal density profile and the response field that produce a required fluctuation, both at initial and final times. The large deviation function of the current is derived and coincides with the formula obtained previously by microscopic calculations. This provides the first analytic confirmation of the validity of the MFT for an interacting model in the time-dependent regime.

DOI: 10.1103/PhysRevLett.129.040601

A fundamental difference between equilibrium and nonequilibrium physics is that a general law—the Boltzmann-Gibbs canonical distribution—exists in the former case. Moreover, dynamical fluctuations in the vicinity of equilibrium and linear response theory are well understood thanks to the Onsager-Machlup functional [1,2]. However, it is now widely believed that large deviation functions could play an overarching role for systems far from equilibrium [3–5] and their study has become a major focus of contemporary statistical mechanics [6–9]. In a series of seminal works starting from the early 2000s, Jona-Lasinio and his collaborators proposed a nonlinear action functional that encodes the fluctuations and the large deviations for a wide class of diffusive systems out of equilibrium. This theory, known as the macroscopic fluctuation theory (MFT) [10–15] posits a variational principle that determines, in a diffusive system, the dominant optimal evolution to produce a required fluctuation. In essence, the problem amounts to solving a set of two coupled nonlinear partial differential equations (PDEs) with mixed, nonlocal, initial and final conditions: the MFT equations.

Another field of research in nonequilibrium physics is to analyze microscopic interacting particle processes that display hydrodynamic behavior on the macroscopic scale [16–18]. One of the simplest and fundamental models is the symmetric exclusion process (SEP), in which particles on a lattice perform symmetric random walks subject to hard-core exclusion. Together with its driven version, the exclusion process plays the role of a paradigm in many domains, as the "simplest nonequilibrium model" [19,20]. Many results about exclusion processes have been obtained analytically, leading to significant information about general properties of nonequilibrium systems [6]. From the very beginning, the SEP has been a major benchmark to build and investigate the MFT [21–28]. We note that, in probability theory, a large deviation principle for SEP had already been established in 1989 by Kipnis, Olla, and Varadhan with a variational principle related to MFT [29] (See also Refs. [17,30]). We also mention that, while we focus on MFT for the diffusive SEP in this article, large deviation for the asymmetric and ballistic case is also of great interest (see, for instance, [31,32] for a recent work).

Exact results for large deviation properties of SEP in the nonstationary regime are quite limited. The large deviation of the total current through the origin was derived by Bethe ansatz [25] and the full distribution of a tagged particle position was obtained in [27,28], using techniques from integrable probabilities [33–36]. However, the extension of these approaches to time-dependent observables, such as the optimal fluctuation history of the process, appears to be out of reach. This information could be extracted from the time-dependent solutions of the MFT equations if only one could solve them: this seems to be a formidable task, since only stationary or perturbative solutions of the MFT were found for SEP [24,26,37–41].

Yet, it has been suspected for some stochastic processes that optimal path equations could be "classically" integrable: this was explicitly recognized by the authors of [42] for the Kardar-Parisi-Zhang (KPZ) equation with weak noise, a problem solved by the inverse scattering method (ISM) in 2021 [43,44]. More recently, the full statistics of nonstationary heat transfer in the Kipnis-Machioro-Presutti (KMP) model has been calculated in [45] using again the ISM: this must be hailed as the first analytical solution of the MFT equations for a specific (and not microscopically integrable) model, with very special boundary conditions.

In the meantime, Grabsch *et al.* [46] made a major breakthrough in the understanding of large deviations in single file systems such as the SEP. Without using neither integrability nor the MFT, they intuited and unveiled recursively a closed equation for the final optimal profile, allowing them to determine that profile and the corresponding large deviations (see [47–49] for precursory works in the same group).

We present here a general scheme to resolve analytically the MFT equations for SEP on the infinite line. We devise a nonlocal transformation that maps these MFT equations to the classically integrable Ablowitz-Kaup-Newell-Segur (AKNS) system, that we analyze by ISM. For the step initial condition with two densities, the calculation of the scattering amplitudes leads to a solvable Riemann-Hilbert problem, allowing us to determine analytically the density profiles and the response fields both at initial and final times. By retrieving the cumulant generating function of the current, the relevance of the MFT equations in the time-dependent regime is confirmed.

The SEP is a continuous time interacting particles Markov process in which each particle is located on a discrete site labeled by an integer $x \in \mathbb{Z}$ and can hop to its right or left nearest neighboring site with unit rate. Because of the volume exclusion, jumps to an occupied site are forbidden (see Fig. 1). We consider the time-integrated current Q_T, given by the total number of particles that have jumped from 0 to 1 minus the total number of particles that have jumped from 1 to 0 during the time interval $(0, T)$. In the long time limit, the current Q_T satisfies a large-deviation principle

$$\mathrm{Prob}\left(\frac{Q_T}{\sqrt{T}} = q\right) \simeq \exp[-\sqrt{T}\Phi(q)] \tag{1}$$

with a large deviation function $\Phi(q)$ (note the $\sqrt{T}$ scaling, which implies anomalous diffusion in single-file systems [50]). The cumulant generating function $\mu(\lambda)$ of the current Q_T is defined as

$$\langle e^{\lambda Q_T} \rangle \simeq e^{\sqrt{T}\mu(\lambda)} \quad \text{for } T \to \infty, \tag{2}$$

where λ is a real parameter (or fugacity) conditioning the total current q during the time interval $(0, T)$. The functions $\Phi(q)$ and $\mu(\lambda)$ are Legendre transforms of each other. As already mentioned above, μ was calculated explicitly for the step initial condition in [25].

The MFT describes the evolution of the system in terms of two coupled fields defined on a mesoscopic scale: the density $\rho(x, t)$ and the auxiliary response field $H(x, t)$ (that can be interpreted as a dynamically generated local drift). A dynamical action is ascribed to each history of the system.

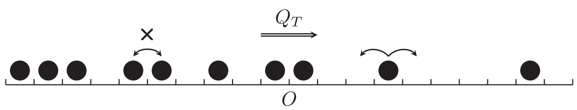

FIG. 1. The symmetric simple exclusion process.

In the long time limit, the extremal action principle determines the optimal path history that produces a required fluctuation and expresses its probability at the level of large deviations. In the hydrodynamic limit, the time-integrated current Q_T is given by

$$Q_T = \int_0^\infty [\rho(x, T) - \rho(x, 0)]dx, \tag{3}$$

and $\mu(\lambda)\sqrt{T}$ is given by the maximum of the functional

$$S[\rho, H] = \lambda Q_T - \mathcal{F}_0[\rho(x, 0)] - \int_0^T dt \int_{-\infty}^\infty dx(H\partial_t\rho - \mathcal{H}), \tag{4}$$

where $\mathcal{H}[\rho, H] = \frac{1}{2}\sigma(\rho)(\partial_x H)^2 - (\partial_x\rho)(\partial_x H)$ with $\sigma(\rho) = 2\rho(1-\rho)$ is the MFT Hamiltonian. The initial free energy is given by

$$\mathcal{F}_0[\rho(x, 0)] = \int_{-\infty}^\infty dx \int_{\bar{\rho}(x)}^{\rho(x,0)} dr \frac{2[\rho(x, 0) - r]}{\sigma(r)}, \tag{5}$$

for a Bernoulli initial state with density $\bar{\rho}(x)$ [6,38].

The dominant path maximizing the action (4) satisfies the MFT equations that couple two nonrandom optimal fields, the density $\rho(x, t)$, and the response $H(x, t)$:

$$\partial_t\rho = \partial_x[\partial_x\rho - \sigma(\rho)\partial_x H], \tag{6}$$

$$\partial_t H = -\partial_x^2 H - \frac{\sigma'(\rho)}{2}(\partial_x H)^2. \tag{7}$$

These equations must be solved with the following conditions at the initial and the final times:

$$H(x, T) = \lambda\theta(x), \tag{8}$$

$$H(x, 0) = \lambda\theta(x) + f'(\rho(x, 0)) - f'(\bar{\rho}(x)), \tag{9}$$

where $f'(\rho) = \log[\rho/(1-\rho)]$ is the derivative of the free energy with respect to the density [26,51]. The MFT equations are a set of nonlinear coupled PDEs that evolve in opposite time directions, with a major complication due to the two-time mixed boundary conditions, rendering numerical simulations arduous [41,52]. In this work, we show that the MFT equations, (6), (7) with conditions (8) and (9), are integrable in the classical sense after a nonlinear transformation and we solve them analytically. This framework can be stated for general initial conditions but we focus here on the two-sided Bernoulli initial condition: at $t = 0$ all sites are independent, a site with a negative label is occupied with probability ρ_- and a site on the positive side is occupied with probability ρ_+. As the initial condition fluctuates, this setup is said to be annealed and the mean density profile at initial time is given by $\bar{\rho}(x) = \rho_-\theta(-x) + \rho_+\theta(x)$ [where $\theta(x)$ is the Heaviside theta function].

The ISM [53,54], often presented as a nonlinear analog of the Fourier transform, lies at the heart of classical integrability and allows us to prove the existence of an infinite number of conservation laws and the integrability in the sense of Liouville [55]. It has led to exact solutions of dispersive PDEs with solitons, such as the Korteweg–de Vries equation or the nonlinear Schrödinger equation that appear in various domains of physics, such as nonlinear optics [56–58], hydrodynamics [59,60], plasma physics [61], or Bose-Einstein condensation [62–64]. As already mentioned, the ISM has also been used in recent works to solve the short time KPZ equation [43,44] and the KMP model [45]. A common feature of these works is that the optimal path equations are manifestly integrable and that the two-time boundary conditions present some symmetry property. These characteristics are not shared by the MFT equations for SEP and this poses a major challenge.

However, the novel nonlocal transformation that we have discovered,

$$u(x,t) = \frac{1}{\sigma'(\rho)}\frac{\partial}{\partial x}\left(\sigma(\rho)\exp\left[-\int_{-\infty}^{x} dy \frac{\sigma'(\rho)}{2}\partial_y H\right]\right), \quad (10)$$

$$v(x,t) = -\frac{2}{\sigma'(\rho)}\frac{\partial}{\partial x}\exp\left[\int_{-\infty}^{x} dy \frac{\sigma'(\rho)}{2}\partial_y H\right], \quad (11)$$

allows us to map the MFT equations (6) and (7) to the AKNS equations [65]:

$$\partial_t u(x,t) = \partial_{xx} u(x,t) - 2u(x,t)^2 v(x,t), \quad (12)$$

$$\partial_t v(x,t) = -\partial_{xx} v(x,t) + 2u(x,t)v(x,t)^2. \quad (13)$$

The transformation (10), (11) unveils the integrability of SEP at the hydrodynamic level (this fact was foreseen in [66] by finding solitons in the MFT equations). In the low density limit, obtained by writing $\rho \coloneqq \alpha\rho$ with $\alpha \to 0$, this change of variable reduces to the canonical Cole-Hopf transformation, i.e., $(u,v) \to (\partial_x \rho e^{-H}, -\partial_x e^{H})$ and the AKNS equations decouple into two diffusion equations, evolving forward and backward in time, that were used to investigate reflecting Brownian motions [26,39]. The transformation above is valid for general quadratic $\sigma(\rho)$. The AKNS system with the same type of boundary conditions below also appeared in the analysis of [43,45].

The initial and final conditions of the MFT equations for SEP given in Eqs. (9) and (8) are translated into the ones in terms of the AKNS variables by (10) and (11). For the step initial density, they become

$$u(x,0) = \omega\delta(x), \quad (14)$$

$$v(x,T) = \delta(x). \quad (15)$$

Indeed, we obtain $u(x,t) \propto \partial_x \rho - \rho(1-\rho)\partial_x H$ and $v(x,t) \propto -\partial_x H$ showing that $u(x,0)$ and $v(x,T)$ are proportional to the Dirac delta function. Because the AKNS equations are invariant by the rescalings $u \to Ku$ and $v \to K^{-1}v$, the amplitude of the Dirac delta function in Eq. (15) can be set to unity by duly choosing K. The parameter ω will be identified as the ubiquitous SEP parameter [25,28],

$$\omega = (e^{\lambda} - 1)\rho_-(1-\rho_+) + (e^{-\lambda} - 1)\rho_+(1-\rho_-), \quad (16)$$

once the scattering amplitudes are determined in (26) and (27) (see the Supplemental Material [67]). Using again that $\partial_x H(x,T)$ is a Dirac delta function at time T from Eq. (8), and taking the rescaling factor K into account, we observe that the transformations (10) and (11) imply

$$u(x,T) = \begin{cases} K\partial_x \rho(x,T), & x < 0, \\ Ke^{-\Lambda}\partial_x \rho(x,T), & x > 0. \end{cases} \quad (17)$$

The quantity $\Lambda = \frac{1}{2}\int_{-\infty}^{+\infty} dy\, \sigma'(\rho)\partial_y H(y,t)$ is conserved by the MFT dynamics and is given by (see [67])

$$e^{\Lambda} = e^{\lambda}\frac{1 + (e^{-\lambda} - 1)\rho_+}{1 + (e^{\lambda} - 1)\rho_-}. \quad (18)$$

Similarly, at $t = 0$, noting from Eq. (9) that $\partial_x H(x,0) = 2\sigma(\rho)^{-1}\partial_x \rho(x,0)$ for $x \neq 0$ and using that Λ is conserved, we deduce

$$v(x,0) = \begin{cases} -2K^{-1}\sigma(\rho_-)^{-1}\partial_x \rho(x,0), & x < 0, \\ -2K^{-1}\sigma(\rho_+)^{-1}e^{\Lambda}\partial_x \rho(x,0), & x > 0. \end{cases} \quad (19)$$

In the following we will determine ρ and H by solving (12), (13) with the three parameters ω, Λ, K appropriately fixed.

To solve the AKNS equations (12), (13), we follow the standard procedure of ISM [53,54]. However, there is an important difference: while conventionally one studies the initial value problem, here we must consider equations with mixed boundary condition (14), (15). First, we reformulate the equations in terms of the associated auxiliary linear problem, which for the AKNS system takes the form

$$\partial_x \Psi(x,t) = U(x,t;k)\Psi(x,t), \quad (20)$$

$$\partial_t \Psi(x,t) = V(x,t;k)\Psi(x,t), \quad (21)$$

where the vector $\Psi(x,t)$ plays the role of a wave function. The 2×2 matrix-valued functions U and V are given by

$$U = \begin{pmatrix} -ik & v(x,t) \\ u(x,t) & ik \end{pmatrix}, \tag{22}$$

$$V = \begin{pmatrix} 2k^2 + u(x,t)v(x,t) & 2ikv(x,t) - \partial_x v(x,t) \\ 2iku(x,t) + \partial_x u(x,t) & -2k^2 - u(x,t)v(x,t) \end{pmatrix}. \tag{23}$$

The compatibility of Eqs. (20) and (21) (i.e., $\partial_t \partial_x \Psi = \partial_x \partial_t \Psi$) is ensured by the zero curvature condition, $\partial_t U - \partial_x V + [U, V] = 0$, which is met if the functions u and v satisfy the AKNS system (12) and (13) [68].

Next, we solve the direct scattering problem of the linear equation (20), which resembles that of the Dirac equation where u and v, unknown in AKNS, now play the role of potentials. Assuming that $u(x,t)$ and $v(x,t)$ are rapidly decreasing at infinity, i.e., $u(x,t), v(x,t) \to 0$ as $|x| \to \infty$, and the solutions behave as plane waves for large $|x|$. The incoming or outgoing plane waves from $x \to -\infty$,

$$\phi(x;k) \sim \begin{pmatrix} e^{-ikx} \\ 0 \end{pmatrix} \quad \text{and} \quad \bar{\phi}(x;k) \sim -\begin{pmatrix} 0 \\ e^{ikx} \end{pmatrix}, \tag{24}$$

will be scattered at $x \to +\infty$ as follows:

$$\phi(x;k) \sim \begin{pmatrix} a(k)e^{-ikx} \\ b(k)e^{ikx} \end{pmatrix} \quad \text{and} \quad \bar{\phi}(x;k) \sim \begin{pmatrix} \bar{b}(k)e^{-ikx} \\ -\bar{a}(k)e^{ikx} \end{pmatrix}. \tag{25}$$

This defines the scattering amplitudes, denoted by $a(k)$, $\bar{a}(k)$, $b(k)$, $\bar{b}(k)$. The calculation of these amplitudes at $t = 0$ and $t = T$ under the initial and final conditions (14) and (15) is elementary, akin to solving the Schrödinger equation with a delta potential (see [67]). At $t = 0$, we obtain

$$\begin{aligned} &a(k,0) = 1 + \omega \hat{v}_+(k), \quad b(k,0) = \omega, \\ &\bar{a}(k,0) = 1 + \omega \hat{v}_-(k), \quad \bar{b}(k,0) = -[\hat{v}(k) + \omega \hat{v}_+(k)\hat{v}_-(k)]. \end{aligned} \tag{26}$$

Similarly, at $t = T$, we have

$$\begin{aligned} &a(k,T) = 1 + \hat{u}_+(k), \qquad b(k,T) = \hat{u}(k) + \hat{u}_+(k)\hat{u}_-(k), \\ &\bar{a}(k,T) = 1 + \hat{u}_-(k), \qquad \bar{b}(k,T) = -1. \end{aligned} \tag{27}$$

Here $\hat{u}_\pm(k)$ and $\hat{v}_\pm(k)$ are the half-Fourier transforms of $u(x,T)$ and $v(x,0)$ defined as

$$\hat{u}_\pm(k) = \int_{\mathbb{R}_\mp} u(x,T)e^{-2ikx}dx, \tag{28}$$

$$\hat{v}_\pm(k) = \int_{\mathbb{R}_\pm} v(x,0)e^{2ikx}dx, \tag{29}$$

and we use the notations $\hat{u}(k) \coloneqq \hat{u}_+(k) + \hat{u}_-(k)$ and $\hat{v}(k) \coloneqq \hat{v}_+(k) + \hat{v}_-(k)$.

On the other hand, by combining Eq. (25) with Eq. (21), the time evolution of the scattering amplitudes is obtained explicitly [53,67]

$$a(k,t) = a(k,0), \qquad b(k,t) = b(k,0)e^{-4k^2 t}, \tag{30}$$

$$\bar{a}(k,t) = \bar{a}(k,0), \qquad \bar{b}(k,t) = \bar{b}(k,0)e^{4k^2 t}. \tag{31}$$

The fact that the dynamics drastically simplifies in terms of the scattering amplitudes is a key feature of the ISM. It is remarkable that the time evolution of the MFT equations for SEP becomes so simple in terms of the scattering amplitudes. By using the evolution of $b(k,t)$ from $t = 0$ to T, we deduce a closed equation for $\hat{u}_\pm(k)$:

$$\hat{u}(k) + \hat{u}_+(k)\hat{u}_-(k) = \omega e^{-4k^2 T}. \tag{32}$$

This equation is the Fourier transform of the equation for determining the density profile at final time, conjectured by Grabsch *et al.* [46], by ingenious microscopic considerations and inspection; in this paper we have shown that it arises as a simple consequence of the AKNS equations. Note that a closely related relation also appears in the analysis of the KMP model by Bettelheim *et al.* [45]. Rewriting Eq. (32) as

$$[\hat{u}_+(k) + 1][\hat{u}_-(k) + 1] = 1 + \omega e^{-4k^2 T}, \tag{33}$$

we obtain a scalar Riemann-Hilbert factorization problem of finding two functions, analytic on the upper (lower) complex plane, with a given product along a specific contour. Note that scalar Riemann-Hilbert problems appear in many fields of physics and engineering [69–72]. The solution is standard [55], by taking the logarithm of Eq. (32) and using the Cauchy formula with an infinitesimal constant $\epsilon > 0$

$$\hat{u}_\pm(k) + 1 = \exp\left[\pm \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\log(1 + \omega e^{-4q^2 T})}{q - k \mp i\epsilon} dq\right]. \tag{34}$$

The coefficient in front of the exponential on the right-hand side of (34) is taken to be unity to ensure that $\hat{u}_\pm(k)$ vanishes for $k \to \infty$ for bounded u. Expanding the logarithm inside the integral and using the following formula (see, e.g., Eqs. 7.2.3 and 7.7.2 in [73]),

$$\pm \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{e^{-q^2}}{q - k \mp i\epsilon} dq = e^{-k^2} \mathrm{erfc}(\mp ik), \tag{35}$$

we conclude that [46]

$$\hat{u}_\pm(k) + 1 = \exp\left[-\frac{1}{2} \sum_{n=1}^{\infty} \frac{(-\omega e^{-4k^2 T})^n}{n} \mathrm{erfc}(\mp i\sqrt{4nT}k)\right], \tag{36}$$

where $\mathrm{erfc}(x)$ is the complementary error function. Since $a(k,t)$ and $\bar{a}(k,t)$ are conserved, see Eqs. (30), (31), we find from Eqs. (26)–(27) that $\omega\hat{v}_{\pm} = \hat{u}_{\pm}$.

Therefore, the density profiles at $t = 0$ and T are determined up to the factor K by integrating Eqs. (17) and (19) with $\rho(x,t) \to \rho_{\pm}$ for $x \to \pm\infty$. Finally, K is fixed by imposing the total mass conservation $\int_{-\infty}^{\infty}[\rho(x,T) - \rho(x,0)]dx = 0$ (see [67]):

$$K = -2\sinh(\lambda/2)e^{\Lambda/2}, \tag{37}$$

where Λ is given in Eq. (18).

Combining all the calculations above, we present exact formulas for the density profiles. The optimal density fluctuation at initial time, $t = 0$, is given by

$$\rho(x,0) = \begin{cases} \rho_- + A_- \int_{-\infty}^{x} v(y,0)dy, & x < 0, \\ \rho_+ + A_+ \int_{x}^{\infty} v(y,0)dy, & x > 0, \end{cases} \tag{38}$$

with $A_{\pm} = \sigma(\rho_{\pm})\frac{e^{\mp\lambda}-1}{2}\sqrt{\frac{1+(e^{\pm\lambda}-1)\rho_{\mp}}{1+(e^{\mp\lambda}-1)\rho_{\pm}}}$. At $t = T$, the form of the profile is obtained by replacing $v(y,0)$ by $u(y,T)$ and $A_{\pm}$ by $B_{\pm} = -[\sigma(\rho_{\pm})/2A_{\pm}]$ in Eq. (38). The response field H at $t = 0$ is determined thanks to (9). While the final density can be extracted from the information in [46], our scheme using ISM allows us to determine simultaneously the optimal profiles of ρ and H exactly at both initial and final time. An example for all of them is represented in Fig. 2.

Finally, the cumulant generating function is retrieved by noting that μ and Q_T are dual by Legendre transform, i.e., $d\mu/d\lambda = Q_T/\sqrt{T}$ [45,74]. Calculating the total current Q_T from the profiles at $t = 0$ and $t = T$, given in Eq. (38) and beneath, we obtain, using $\mu(0) = 0$ and (16),

$$\mu(\lambda) = \frac{1}{\sqrt{\pi}}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\omega^n}{n^{3/2}}. \tag{39}$$

This formula was first found in [25] at the microscopic level by applying the Bethe ansatz to the SEP. Here it has been deduced from the action principle of the MFT. The large deviations of a tracer particle, derived microscopically in Refs. [27,28], can be extracted along similar lines from the MFT framework [75].

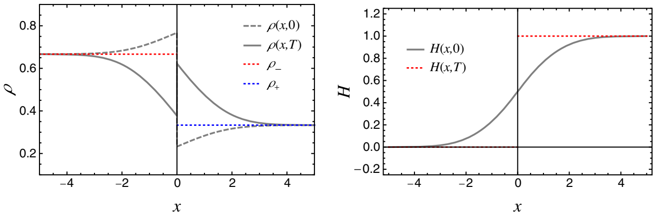

FIG. 2. Optimal profiles of ρ (left) and H (right) at $t = 0$ and at $t = T$, with $\rho_+ = 1/3$, $\rho_- = 2/3$, $\lambda = 1$, and $T = 1$.

To summarize, we have presented the first exact solution of the time-dependent MFT equations for SEP. Albeit these equations were known for a long time, their solution had remained out of reach due to their intrinsic complexity and to cumbrous boundary conditions with respect to time. A key to the solution has been our novel nonlocal change of variables given in Eqs. (10) and (11), that generalizes the canonical Cole-Hopf transformation. This enabled us to map the MFT equations to the integrable AKNS system and to use the inverse scattering method. We have derived exact expressions for the optimal density profile and the response field both at initial and final times. By retrieving the cumulant generating function of the integrated current, previously found by a microscopic calculation, we have provided a first analytic confirmation of the validity of the macroscopic fluctuation theory in the time-dependent regime.

The present work can be extended in multiple directions. Many variants of the exclusion process—different geometries, initial conditions, multiple species, asymmetry, tagged particles, defects, etc.—have been explored during the last decades and ought to be analyzed with the MFT. Besides, some diffusive interacting particle processes out of equilibrium, with identical transport coefficients, could be solvable by ISM at the macroscopic level, though the corresponding microscopic models may not be integrable. Moreover, a somewhat different relation between stochastic models and classical integrable systems has been found [76]. Understanding the connections between different scales of description and various forms of integrability poses challenging problems.

We are convinced that the analysis of the macroscopic fluctuation theory with inverse scattering, applied to the KMP model [45], to SEP in the present work and to the closely related KPZ equation subject to weak noise [43,44], opens a fascinating new perspective in the study of dynamical fluctuations in systems far from equilibrium.

K. M. is thankful to S. Mallick for a careful reading of the manuscript and to A. Grabsch, O. Bénichou, R. Dandekar, and P. L. Krapivsky for discussions. The work of K. M. has been supported by the project RETENU ANR-20-CE40-0005-01 of the French National Research Agency (ANR). The work of T. S. has been supported by JSPS KAKENHI Grants No. JP16H06338, No. JP18H01141, No. JP18H03672, No. JP19L03665, No. JP21H04432, No. JP22H01143.

*kirone.mallick@ipht.fr
†hmoriya@stat.phys.titech.ac.jp
‡sasamoto@phys.titech.ac.jp

[1] L. Onsager and S. Machlup, Fluctuations and irreversible processes, Phys. Rev. **91**, 1505 (1953).
[2] S. Machlup and L. Onsager, Fluctuations and irreversible process. II. Systems with kinetic energy, Phys. Rev. **91**, 1512 (1953).

[3] S. R. S. Varadhan, *Large Deviations and Applications* (Siam, Philadelphia, 1984).
[4] H. Touchette, The large deviation approach to statistical mechanics, Phys. Rep. **478**, 1 (2009).
[5] J. L. Lebowitz and H. Spohn, A Gallavotti-Cohen-type symmetry in the large deviation functional for stochastic dynamics, J. Stat. Phys. **95**, 333 (1999).
[6] B. Derrida, Non-equilibrium steady states: Fluctuations and large deviations of the density and of the current, J. Stat. Mech. (2007) P07023.
[7] B. Derrida, Microscopic versus macroscopic approaches to non-equilibrium systems, J. Stat. Mech. (2011) P01030.
[8] H. Touchette and R. J. Harris, Large deviation approach to nonequilibrium systems, in *Nonequilibrium Statistical Physics of Small Systems: Fluctuation Relations and Beyond*, edited by R. Klages, W. Just, and C. Jarzynski (Wiley-VCH Verlag, New York, 2013).
[9] L. Bertini, A. De Sole, D. Gabrielli, G. Jona-Lasinio, and C. Landim, Large deviation approach to non equilibrium processes in stochastic lattice gases, Bull. Braz. Math. Soc. **37**, 611 (2006).
[10] L. Bertini, A. De Sole, D. Gabrielli, G. Jona-Lasinio, and C. Landim, Fluctuations in Stationary Nonequilibrium States of Irreversible Processes, Phys. Rev. Lett. **87**, 040601 (2001).
[11] L. Bertini, A. De Sole, D. Gabrielli, G. Jona-Lasinio, and C. Landim, Macroscopic fluctuation theory for stationary non-equilibrium states. J. Stat. Phys. **107**, 635 (2002).
[12] T. Bodineau and B. Derrida, Current Fluctuations in Nonequilibrium Diffusive Systems: An Additivity Principle, Phys. Rev. Lett. **92**, 180601 (2004).
[13] T. Bodineau and B. Derrida, Distribution of current in non-equilibrium diffusive systems and phase transitions, Phys. Rev. E **72**, 066110 (2005).
[14] L. Bertini, A. De Sole, D. Gabrielli, G. Jona-Lasinio, and C. Landim, Macroscopic fluctuation theory, Rev. Mod. Phys. **87**, 593 (2015).
[15] G. Jona-Lasinio, Thermodynamics of stationary states, J. Stat. Mech. (2014) P02004.
[16] S. Katz, J. L. Lebowitz, and H. Spohn, Nonequilibrium steady states of stochastic lattice gas models of fast ionic conductors, J. Stat. Phys. **34**, 497 (1984).
[17] H. Spohn, *Large Scale Dynamics of Interacting Particles* (Springer-Verlag, New York, 1991).
[18] G. M. Schütz, Exactly solvable models for many-body systems far from equilibrium, in *Phase Transitions and Critical Phenomena* (Academic Press, 2001), Vol. 19, pp. 1–251.
[19] B. Derrida, An exactly soluble non-equilibrium system: The asymmetric simple exclusion process, Phys. Rep. **301**, 65 (1998).
[20] T. Chou, K. Mallick, and R. K. P. Zia, Non-equilibrium statistical mechanics: From a paradigmatic model to biological transport, Rep. Prog. Phys. **74**, 116601 (2011).
[21] B. Derrida, J. L. Lebowitz, and E. R. Speer, Free Energy Functional for Nonequilibrium Systems: An Exactly Solvable Case, Phys. Rev. Lett. **87**, 150601 (2001).
[22] B. Derrida, J. L. Lebowitz, and E. R. Speer, Exact Free Energy Functional for a Driven Diffusive Open Stationary Nonequilibrium System, Phys. Rev. Lett. **89**, 030601 (2002).
[23] B. Derrida, J. L. Lebowitz, and E. R. Speer, Large deviation of the density profile in the steady state of the open symmetric simple exclusion process, J. Stat. Phys. **107**, 599 (2002).
[24] L. Bertini, A. De Sole, D. Gabrielli, G. Jona-Lasinio, and C. Landim, Large deviations for the boundary driven symmetric simple exclusion process, Math. Phys. Anal. Geom. **6**, 231 (2003).
[25] B. Derrida and A. Gerschenfeld, Current fluctuations of the one dimensional symmetric simple exclusion process with step initial condition, J. Stat. Phys. **136**, 1 (2009).
[26] B. Derrida and A. Gerschenfeld, Current fluctuations in one dimensional diffusive systems with a step initial density profile, J. Stat. Phys. **137**, 978 (2009).
[27] T. Imamura, K. Mallick, and T. Sasamoto, Large Deviations of a Tracer in the Symmetric Exclusion Process, Phys. Rev. Lett. **118**, 160601, 2017.
[28] T. Imamura, K. Mallick, and T. Sasamoto, Distribution of a tagged particle position in the one-dimensional symmetric simple exclusion process with two-sided Bernoulli initial condition, Commun. Math. Phys. **384**, 1409 (2021).
[29] C. Kipnis, S. Olla, and S. R. S. Varadhan, Hydrodynamics and large deviations for simple exclusion processes, Commun. Pure Appl. Math. **42**, 115 (1989).
[30] S. Sethuraman and S. R. S. Varadhan, Large deviations for the current and tagged particle in 1D nearest-neighbor symmetric simple exclusion, Ann. Prob. **41**, 1461 (2013).
[31] J. Quastel and L.-C. Tsai, Hydrodynamic large deviations of TASEP, arXiv:2104.04444.
[32] B. Doyon, G. Perfetto, T. Sasamoto, and T. Yoshimura, Ballistic macroscopic fluctuation theory, arXiv:2206.14167.
[33] G. M. Schütz, Exact solution of the master equation for the asymmetric exclusion process, J. Stat. Phys. **88**, 427 (1997).
[34] C. A. Tracy and H. Widom, Integral formulas for the asymmetric simple exclusion process, Commun. Math. Phys. **279**, 815 (2008).
[35] A. Borodin, I. Corwin, and T. Sasamoto, From duality to determinants for q-TASEP and ASEP, Ann. Prob. **42**, 2314 (2014).
[36] A. Borodin and I. Corwin, Macdonald processes, Probab. Theory Relat. Fields **158**, 225 (2014).
[37] T. Bodineau and B. Derrida, Cumulants and large deviations of the current through non-equilibrium steady states, C. R. Phys. **8**, 540 (2007).
[38] P. L. Krapivsky and B. Meerson, Fluctuations of current in nonstationary diffusive lattice gases, Phys. Rev. E **86**, 031106 (2012).
[39] P. L. Krapivsky, K. Mallick, and T. Sadhu, Large Deviations in Single-File Diffusion, Phys. Rev. Lett. **113**, 078101 (2014).
[40] B. Meerson and P. V. Sasorov, Extreme current fluctuations in a non-stationary stochastic heat flow, J. Stat. Mech. (2013) P12011.
[41] B. Meerson and P. V. Sasorov, Extreme current fluctuations in lattice gases: Beyond nonequilibrium steady states, Phys. Rev. E **89**, 010101(R) (2014).

[42] M. Janas, A. Kamenev, and B. Meerson, Dynamical phase transition in large-deviation statistics of the Kardar-Parisi-Zhang equation, Phys. Rev. E **94**, 032133 (2016).

[43] A. Krajenbrink and P. Le Doussal, Inverse Scattering of the Zakharov-Shabat System Solves the Weak Noise Theory of the Kardar-Parisi-Zhang Equation, Phys. Rev. Lett. **127**, 064101 (2021).

[44] A. Krajenbrink and P. Le Doussal, Inverse scattering solution of the weak noise theory of the Kardar-Parisi-Zhang equation with flat and Brownian initial conditions, Phys. Rev. E **105**, 054142 (2022).

[45] E. Bettelheim, N. R. Smith, and B. Meerson, Inverse Scattering Method Solves the Problem of Full Statistics of Nonstationary Heat Transfer in the Kipnis-Marchioro-Presutti Model, Phys. Rev. Lett. **128**, 130602 (2022).

[46] A. Grabsch, A. Poncet, P. Rizkallah, P. Illien, and O. Bénichou, Exact closure and solution for spatial correlations in single-file diffusion, Sci. Adv. **8**, 5043 (2022).

[47] P. Illien, O. Bénichou, C. Mejía-Monasterio, G. Oshanin, and R. Voituriez, Active Transport in Dense Diffusive Single-File Systems, Phys. Rev. Lett. **111**, 038102 (2013).

[48] A. Poncet, O. Bénichou, and P. Illien, Cumulant generating functions of a tracer in quenched dense symmetric exclusion processes, Phys. Rev. E **103**, L040103 (2021).

[49] A. Poncet, A. Grabsch, P. Illien, and O. Bénichou, Generalized Correlation Profiles in Single-File Systems, Phys. Rev. Lett. **127**, 220601 (2021).

[50] P. L. Krapivsky, S. Redner, and E. Ben-Naim, *A Kinetic View of Statistical Physics* (Cambridge University Press, Cambridge, England, 2010).

[51] P. L. Krapivsky, K. Mallick, and T. Sadhu, Tagged particle in single-file diffusion, J. Stat. Phys. **160**, 885 (2015).

[52] A. Vilenkin, B. Meerson, and P. V. Sasorov, Extreme fluctuations of current in the symmetric simple exclusion process: A non-stationary setting, J. Stat. Mech. (2014) P06007.

[53] M. J. Ablowitz and H. Segur, *Solitons and the Inverse Scattering Transform* (SIAM, Philadelphia, 1981).

[54] L. Faddeev and L. Takhtajan, *Hamiltonian Methods in the Theory of Solitons* (Springer, Berlin, Heidelberg, 1987).

[55] M. Dunajski, *Solitons, Instantons and Twistors* (OUP, Oxford, 2009).

[56] P. Sprenger, M. A. Hoefer, and G. A. El, Hydrodynamic optical soliton tunneling, Phys. Rev. E **97**, 032218 (2018).

[57] M. J. Ablowitz, G. Biondini, and L. A. Ostrovsky, Optical solitons: Perspectives and applications, Chaos **10**, 471 (2000).

[58] A. C. Newell and J. V. Moloney, *Nonlinear Optics* (Addison-Wesley, Redwood City, 1992).

[59] M. A. Hoefer, M. J. Ablowitz, and P. Engels, Piston Dispersive Shock Wave Problem, Phys. Rev. Lett. **100**, 084504 (2008).

[60] A. Osborne, *Nonlinear Ocean Waves and the Inverse Scattering Transform* (Academic Press Inc., New York, 2006).

[61] E. A. Kuznetsov, A. M. Rubenchik, and V. E. Zakharov, Soliton stability in plasmas and hydrodynamics, Phys. Rep. **142**, 103 (1986).

[62] T. Frisch, Y. Pomeau, and S. Rica, Transition to Dissipation in a Model of Superflow, Phys. Rev. Lett. **69**, 1644 (1992).

[63] V. Hakim, Nonlinear Schrödinger flow past an obstacle in one dimension, Phys. Rev. E **55**, 2835 (1997).

[64] M. A. Hoefer, M. J. Ablowitz, I. Coddington, E. A. Cornell, P. Engels, and V. Schweikhard, Dispersive and classical shock waves in Bose-Einstein condensates and gas dynamics, Phys. Rev. A **74**, 023623 (2006).

[65] M. J. Ablowitz, D. J Kaup, A. C Newell, and H. Segur, The inverse scattering transform-Fourier analysis for nonlinear problems, Stud. Appl. Math. **53**, 249 (1974).

[66] A. P. Polychronakos, Solitons in fluctuating hydrodynamics of diffusive processes, Phys. Rev. E **101**, 022209 (2020).

[67] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.129.040601 for technical details of scattering amplitudes in order to make our paper self-consistent and calculations on the determination of the three constants Λ, ω, and K.

[68] More generally, it can be shown [53] that the zero-curvature condition is satisfied by the equations $\alpha u_t = \frac{1}{2} u_{xx} - u^2 v$ and $\alpha v_t = -\frac{1}{2} v_{xx} + v^2 u$, where α is an arbitrary scalar, which can be chosen to be real or imaginary without breaking integrability. The NLS equation is obtained by taking $v = \pm u^*$ and $\alpha = i$. The system (12), (13) considered here corresponds to the choice $\alpha = 1/2$, which can be interpreted as an imaginary time evolution.

[69] C. C. Mei, *Mathematical Analysis in Engineering* (Cambridge University Press, Cambridge, England, 1995).

[70] N. I. Muskhelishvili, *Some Basic Problems of the Mathematical Theory of Elasticity* (Noordhoff, Leyden, 1962).

[71] J. R. Rice, *Mathematical Analysis in the Mechanics of Fracture in Fracture—An Advanced Treatise*, edited by H. Liebowitz (Academic Press Inc., New-York, 1968), Vol. 2.

[72] M. F. Kanninen and C. H. Popelar, *Advanced Fracture Mechanics* (Oxford University Press, London, UK, 1985).

[73] NIST Digital Library of Mathematical Functions, edited by F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, B. V. Saunders, H. S. Cohl, and M. A. McClain, http://dlmf.nist.gov/, Release 1.1.4 of 2022-01-15.

[74] R. Dandekar (private communication).

[75] K. Mallick, H. Moriya, and T. Sasamoto (to be published).

[76] J. Quastel and D. Remenik, KP governs random growth off a 1-dimensional substrate, Forum of Mathematics, Pi **10**, e10 (2022).