Universal Feature of Charged Entanglement Entropy

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(Received 30 March 2022; accepted 23 June 2022; published 6 July 2022)

Rényi entropies, S_n, admit a natural generalization in the presence of global symmetries. These "charged Rényi entropies" are functions of the chemical potential μ conjugate to the charge contained in the entangling region and reduce to the usual notions as $\mu \to 0$. For $n = 1$, this provides a notion of charged entanglement entropy. In this Letter, we prove that for a general $d(\geq 3)$-dimensional conformal field theory, the leading correction to the uncharged entanglement entropy across a spherical entangling surface is quadratic in the chemical potential, positive definite, and universally controlled (up to fixed d-dependent constants) by the coefficients C_J and a_2. These fully characterize, for a given theory, the current correlators $\langle JJ \rangle$ and $\langle TJJ \rangle$, as well as the energy flux measured at infinity produced by the insertion of the current operator. Our result is motivated by analytic holographic calculations for a special class of higher-curvature gravities coupled to a $(d-2)$ form in general dimensions as well as for free fields in $d = 4$. A proof for general theories and dimensions follows from previously known universal identities involving the magnetic response of twist operators introduced in A. Belin *et al.* [J. High Energy Phys. 12 (2013) 059.] and basic thermodynamic relations.

DOI: 10.1103/PhysRevLett.129.021601

The Rényi and entanglement entropies (EE) of spatial regions in the vacuum state of d-dimensional conformal field theories (CFTs) capture interesting universal information. This includes the Virasoro central charge c for two-dimensional theories [1,2], the Euclidean partition function on the sphere in odd dimensions [3,4], the trace-anomaly coefficients in even dimensions [5–8], the stress-tensor two-point function charge C_T [9–12], and the thermal entropy coefficient C_S [13–15], among others [16–18]. From a different perspective, it has been in fact suggested that the full CFT data might be accessible from a long-distance expansion of the mutual or N-partite information [19–25]. In this Letter, we consider a natural generalization of Rényi and entanglement entropies for theories with global symmetries [26] and add a new entry to the list of general relations satisfied by these quantities that connect them to various universal quantities.

Given a spatial bipartition, the (uncharged) Rényi entropy for some region A is defined as $S_n \equiv [1/(1-n)]\times \log \mathrm{Tr}\rho_A^n$, where ρ_A is the partial-trace density matrix associated to that region. The entanglement entropy S_{EE} is obtained as the $n \to 1$ limit of S_n. A charged notion of Rényi entropy was introduced in [26] for theories with global symmetries—see also [27–29]. This is given by

$$S_n(\mu) = \frac{1}{1-n}\log \mathrm{Tr}\left[\rho_A \frac{e^{\mu Q_A}}{n_A(\mu)}\right]^n, \qquad (1)$$

where Q_A is the total charge contained in the entangling region A, μ is the chemical potential conjugate to the charge, and $n_A(\mu)$ is a normalization factor. It is obvious from its definition that $S_n(\mu)$ reduces to S_n as $\mu \to 0$. An interesting feature of $S_n(\mu)$ is that, for spherical entangling surfaces, it admits a generalization of the conformal map of [3,30] that allows one to evaluate this quantity from the, usually simpler, thermal entropy in the hyperbolic cylinder [26]. This enables one to perform explicit holographic and free-field calculations, which we exploit below. Additional studies of charged Rényi entropies and closely related notions can be found, e.g., in [31–41].

In the uncharged case, the EE universal term across a spherical entangling surface in a d-dimensional CFT reads (see, e.g., [42,43])

$$\frac{S_{\mathrm{EE}}}{\nu_{d-1}} = a^\star, \quad \text{where } \nu_{d-1} \equiv \begin{cases} (-)^{\frac{d-2}{2}} 4 \log\left(\frac{R}{\delta}\right), \\ (-)^{\frac{d-1}{2}} 2\pi, \end{cases} \qquad (2)$$

respectively, for even and odd d. In this formula, R is the radius of the spherical region and δ a UV regulator. In even dimensions, the universal quantity $a^\star$ is nothing but the A-type trace-anomaly coefficient [5–8]. In odd d, $a^\star$ is proportional to the Euclidean partition function of the theory on the round sphere [3,4].

In this Letter, we show that the charged entanglement entropy for a spherical region is given, for general d-dimensional CFTs, by

$$\frac{S_{\rm EE}(\mu)}{\nu_{d-1}}=a^\star+\frac{\pi^d C_J}{(d-1)^2\Gamma(d-2)}\left[1+\frac{(d-2)a_2}{d(d-1)}\right](\mu R)^2 \qquad (3)$$

plus subleading corrections. Equation (3) can be alternatively formulated as a statement involving the first two derivatives of $S_n(\mu)$ with respect to μ evaluated at $n=1$, $\mu=0$ in an obvious way. In the above expression, C_J and a_2 are two constants that characterize the corresponding CFT. On the one hand, C_J is the only theory-dependent information that is not fully determined by conformal symmetry in the correlator of the current associated to the global symmetry, namely [44]

$$\langle J_a(x)J_b(0)\rangle=\frac{C_J}{|x|^{2(d-1)}}\left[\delta_{ab}-\frac{2x_a x_b}{|x|^2}\right]. \qquad (4)$$

As for a_2, its meaning can be understood from two different, albeit related, perspectives. On the one hand, the three-point function $\langle TJJ\rangle$ involves a complicated tensorial structure (which can be found in the Supplemental Material [45]) shared by all CFTs up to two theory-dependent coefficients [44]. These coefficients can be chosen to be C_J and a second one denoted a_2. The latter can also be understood from conformal collider physics. Consider a CFT in flat space in its vacuum state and the insertion of a (smeared) current operator $\epsilon_a J^a$ for certain constant polarization tensor ϵ_a. The expectation value of the energy flux measured at infinity in some direction $\vec{n}$ produced by such insertion is universally given by [53]

$$\langle\mathcal{E}(\vec{n})\rangle_J=\frac{E}{\Omega_{(d-2)}}\left[1+a_2\left(\frac{|\epsilon\cdot n|^2}{|\epsilon|^2}-\frac{1}{d-1}\right)\right], \qquad (5)$$

where $\Omega_{(d-2)}$ is the volume of the unit radius $S^{(d-2)}$ and E is the total energy. Again, the tensorial structure is fully fixed by symmetry, and all information about the corresponding CFT is in this case encoded in the coefficient a_2. Demanding the energy flux to be positive in all directions imposes the bounds $-(d-1)/(d-2)\le a_2\le(d-1)$ [53,54], which implies, given the positivity of C_J [44], that the leading correction in Eq. (3) is positive for general theories.

Equation (3) then tells us that the charged entanglement entropy across a sphere of a general CFT for small values of the chemical potential has a leading correction to the uncharged result that is quadratic in the chemical potential, positive, and universally controlled by the charges C_J, a_2, which characterize the theory as explained above.

Electromagnetic quasitopological gravities.—The realization that Eq. (3) may be a universal relation came to us from holographic calculations, so we present those first. We consider the following bulk theory for the metric field coupled to a $(d-2)$-form B with field strength $H=dB$:

$$I_{\rm EQG}=\int\frac{d^{d+1}x\sqrt{|g|}}{16\pi G}\left[R+\frac{d(d-1)}{L^2}-\frac{2H^2}{(d-1)!}+\frac{\lambda L^2\mathcal{X}_4}{(d-2)(d-3)}+\frac{2\alpha_1L^2}{(d-1)!}\mathcal{L}^{(1)}_{\rm RH^2}+\frac{2\alpha_2L^2}{(d-1)!}\mathcal{L}^{(2)}_{\rm RH^2}\right], \qquad (6)$$

where G is the Newton constant, L is a length scale, λ, α_1, α_2 are dimensionless couplings, $\mathcal{X}_4$ is the Gauss-Bonnet density, and [55]

$$\mathcal{L}^{(1)}_{\rm RH^2}\equiv H^2R-(d-1)(2d-1)R^{\mu\nu}_{\rho\sigma},$$
$$\mathcal{L}^{(2)}_{\rm RH^2}\equiv R^\mu_\nu(H^2)^\nu_\mu-(d-1)R^{\mu\nu}_{\rho\sigma}(H^2)^{\rho\sigma}_{\mu\nu}, \qquad (7)$$

where $(H^2)^{\rho\sigma}_{\mu\nu}\equiv H^{\rho\sigma\alpha_3\alpha_4\ldots\alpha_{d-1}}H_{\mu\nu\alpha_3\alpha_4\ldots\alpha_{d-1}}$ are two electromagnetic quasitopological theories (EQGs) [54,56]. These belong to a class of modifications of Einstein gravity with distinct properties, including simple black hole solutions and linearized spectrum, analytic thermodynamics, as well as providing a basis for general-order effective actions [57–67]. From an AdS/CFT perspective [68–70], Eq. (6) defines models of $(d-1)$-dimensional CFTs parametrized by the bulk action couplings. Different CFT magnitudes will involve different functions of those couplings [15,71–74], which can be used to elucidate universal patterns when some of those magnitudes in fact display the same dependence. This approach has been successfully used before, e.g., in [12,42,43,75–80].

Equation (6) can be mapped to a different theory with a vector field by dualizing the B field. The field strength of the dual vector field $F=dA$, is then identified as $F=4\pi G\ell_*^{-1}(d-1)!\star[\partial\mathcal{L}/\partial H]$, where ℓ_* is an undetermined length scale that we introduce so that A_μ has units of energy. The bulk gauge field A_μ is holographically dual to the current J^a of a global $U(1)$. The parameters C_J and a_2 associated to J^a were determined in [54], finding

$$C_J^{\rm EQG}=\frac{\Gamma(d)}{4\pi^{d/2+1}\Gamma(d/2-1)}\frac{\ell_*^2\tilde{L}^{d-3}}{\alpha_{\rm eff}G}, \qquad (8)$$

$$a_2^{\rm EQG}=-\frac{2d(d-1)[(2d-1)\alpha_1+\alpha_2]f_\infty}{(d-2)\alpha_{\rm eff}}, \qquad (9)$$

where

$$\alpha_{\rm eff} \equiv 1 - f_\infty \alpha_1 (3d^2 - 7d + 2) - f_\infty \alpha_2 (d-2), \quad (10)$$

$f_\infty \equiv L^2/\tilde{L}^2$, and $\tilde{L}$ is the $\mathrm{AdS}_{(d+1)}$ radius.

Now, the (charged) Rényi entropy across a spherical entangling surface of radius R in the vacuum state can be obtained, on general grounds, from the thermal entropy on $\mathbb{S}^1_{2\pi R} \times \mathbb{H}^{d-1}_R$ [3,26,30]. In the holographic context, the calculation amounts to computing the thermal entropy of an $\mathrm{AdS}_{(d+1)}$ hyperbolic black hole charged under the gauge field at a temperature $T_0 = 1/(2\pi R)$. For our theory, Eq. (6), this takes the form

$$ds^2 = \frac{-L^2}{f_\infty R^2}\left[\frac{r^2}{L^2} f - 1\right] dt^2 + \frac{dr^2}{\frac{r^2}{L^2} f - 1} + r^2 d\Xi^2,$$
$$H = Q\omega_{\mathbb{H}^{d-1}}, \quad (11)$$

where $d\Xi^2$ is the metric of the unit hyperbolic space $\mathbb{H}^{d-1}_1$ and $\omega_{\mathbb{H}^{d-1}}$ its volume form. The factor $(L^2/f_\infty R^2)$ has been introduced so that the boundary metric is conformal to $ds^2_{\mathbb{S}^1\times\mathbb{H}^{d-1}} = -dt^2 + R^2 d\Xi^2$. The equations of motion for $f(r)$ and its explicit form can be found in the Supplemental Material [45]. The temperature of the black holes can be written as

$$T = \frac{T_0}{2x\sqrt{f_\infty}(1-2p^2\alpha_1 - 2\lambda x^{-2})}\Bigg\{x^2 d - (d-2) + \frac{(d-4)\lambda}{x^2} - \frac{2p^2}{(d-1)}[x^2 - d(3(d-1)\alpha_1 + \alpha_2)]\Bigg\}, \quad (12)$$

where we introduced $x \equiv r_+/L$, $p \equiv Q L r_+^{-d+1}$, and r_+ is the outer horizon position. We also need the value of the chemical potential of the boundary theory. This is nothing but the asymptotic value of the electrostatic potential A_t at $r \to \infty$, which is fixed by the condition that $A_t|_{r_+} = 0$. We find

$$\mu = \frac{Lp}{\ell_* \sqrt{f_\infty} R}\left[\frac{x}{(d-2)} - \frac{\alpha_1}{x}\left(3(d-1) + \frac{T}{T_0} 2x\sqrt{f_\infty}\right) - \frac{\alpha_2}{x}\right]. \quad (13)$$

Finally, we need the Wald entropy [81,82] of the solutions. We obtain

$$S = \frac{x^{d-1} L^{d-1} V_{\mathbb{H}^{d-1}}}{4G}\left[1 + 2p^2\alpha_1 - \frac{2(d-1)\lambda}{(d-3)x^2}\right], \quad (14)$$

where $V_{\mathbb{H}^{d-1}} \equiv \nu_{d-1}\Omega_{d-1}/(4\pi)$ is the regularized volume of the unit hyperbolic space. As explained earlier, this computes the holographic charged entanglement entropy when $T = T_0$. Observe that in the above expression, the dependence on μ appears through x and p, so we would need to obtain $x(\mu)$ and $p(\mu)$ from Eqs. (12) and (13) evaluated for such temperature in order to obtain an explicit formula for $S^{\rm univ}_{\rm EE}(\mu)$. This cannot be done explicitly for arbitrary values of μ, but it is possible for small values of μR. The result for the first 2 orders reads

$$\frac{S^{\rm EQG}_{\rm EE}(\mu)}{\nu_{d-1}} = a^\star_{\rm GB} + \frac{\pi^{(d-2)/2}(d-2)^2[1 - 3d(d-1)\alpha_1 f_\infty - d\alpha_2 f_\infty]}{(d-1)8\Gamma(d/2)\alpha^2_{\rm eff}} \times \frac{\tilde{L}^{d-3}\ell_*^2}{G}(\mu R)^2 + \mathcal{O}(\mu^4), \quad (15)$$

where $\alpha_{\rm eff}$ was defined in Eq. (10). Now, the constant term is the $a^\star$ charge for our EQG theory, which reduces to the Gauss-Bonnet gravity 1, as terms involving the B form do not contribute to it. Explicitly, this reads [42]

$$a^\star_{\rm GB} = \frac{\tilde{L}^{d-1}}{8G}\frac{\pi^{(d-2)/2}}{\Gamma(d/2)}\left[1 - \frac{2(d-1)}{d-3}\lambda f_\infty\right]. \quad (16)$$

As mentioned earlier, this is the expected result for the (uncharged) entanglement entropy across a spherical surface in d dimensions. Now, the leading correction coming from the chemical potential has a complicated nonpolynomial dependence on the gravitational couplings α_1, α_2. However, this conspires to produce a linear combination of the charges $C^{\rm EQG}_J$ and $C^{\rm EQG}_J \cdot a^{\rm EQG}_2$. Indeed, using Eq. (9) it is easy to see that the above formula reduces to Eq. (3). In the Supplemental Material [45], we show that Eq. (3) in fact holds for an infinite family of EQGs of general orders.

The fact that $S^{\rm univ}_{\rm EE}(\mu)$ takes this simple form for such a large family of holographic theories leads us to think that this may actually be a relation that holds for completely general CFTs. Before proving that this is indeed the case, we can perform an additional check in a completely different context.

Free fields.—The result for the charged Rényi entropy associated to global phase rotations for a Dirac fermion and a scalar field in $d = 4$ has been computed in [26] using heat-kernel techniques. We review these calculations in the Supplemental Material [45], where we also fix a typo in the Dirac fermion result reported in [26]. The correct results read, respectively,

$$S^f_n = \frac{\nu_3}{24}\left[\frac{(1+n)(7+37n^2)}{120n^3} + \frac{(1+n)(\mu R)^2}{n}\right],$$
$$S^s_n = \frac{\nu_3}{24}\left[\frac{(1+n)(1+n^2)}{60n^3} + \frac{(1+n)(\mu R)^2}{2n} + |\mu R|^3\right]. \quad (17)$$

Interestingly, the exact dependence on μ is much simpler than for our holographic theories, for which, as we saw earlier, a completely explicit formula cannot be obtained. It is then straightforward to obtain the result of interest for the entanglement entropy expansion. One finds

$$\frac{S^f_{\rm EE}(\mu)}{\nu_3} = a^\star_f + \frac{(\mu R)^2}{12},$$
$$\frac{S^s_{\rm EE}(\mu)}{\nu_3} = a^\star_s + \frac{(\mu R)^2}{24} + \frac{|\mu R|^3}{24}, \qquad (18)$$

where $a^\star_f = 11/360$, $a^\star_s = 1/360$ are the trace-anomaly coefficients corresponding to a Dirac fermion and a real scalar field, respectively [83–85]. Now, the values of the charges C_J and a_2 for these two models are also well-known and read [44,53,86,87]

$$C^f_J = \frac{1}{\pi^4}, \qquad C^s_J = \frac{1}{4\pi^4},$$
$$a^f_2 = -\frac{3}{2}, \qquad a^s_2 = 3. \qquad (19)$$

It is then straightforward to verify that Eq. (18) satisfies the relation Eq. (3).

General CFTs.—The previous results strongly suggest that Eq. (3) holds for general CFTs. As it turns out, a proof of such universality can be easily achieved using a combination of the results presented in Ref. [26] along with some thermodynamic identities. In order to do this, we need to depart momentarily from the vacuum temperature T_0 and consider a CFT on the hyperbolic cylinder at an arbitrary temperature T. The thermal entropy of a given CFT in such state can be used to compute the Rényi entropy $S_n(\mu)$ across a spherical entangling region [26,30], the Rényi index being related to the temperature by $n = T_0/T$.

In order to proceed, we need to consider a set of related quantities: the twist operators $\sigma_n(\mu)$. In the Replica trick approach to the evaluation of Rényi and entanglement entropy, the entangling region is cut from each of the spacetime copies and consecutive copies are sewn together along the entangling surface. Such boundary conditions can be understood as produced by the insertion of $(d-2)$-dimensional operators along the entangling surface [1,2,10,30]. In the charged Rényi and EE case, the entangling surface carries a "magnetic flux," $-in\mu$, which can be understood as attaching a Dirac sheet to the twist operators [26].

The leading divergence in the correlator of $\sigma_n(\mu)$ with the current operator defines the so called "magnetic response" $k_n(\mu)$ as [26]

$$\langle J_a \sigma_n(\mu)\rangle = \frac{i k_n(\mu)}{2\pi}\frac{\epsilon_{ab} n^b}{y^{d-1}}, \qquad (20)$$

where y is the distance between the insertions, n^b is a unit vector normal to J_a from the twist operator insertion, and ϵ_{ab} is the volume form of the two-dimensional space orthogonal to the entangling surface. In the case of a spherical entangling surface, the magnetic response is given by [26]

$$k_n(\mu) = 2\pi n R^{d-1}\rho(n,\mu), \qquad (21)$$

where $\rho(n,\mu)$ is the charge density of the CFT on the hyperbolic cylinder at temperature $T = T_0/n$. As it turns out, this quantity has a universal expansion around $n = 1$ and $\mu = 0$ whose leading terms can be expressed in terms of the coefficients characterizing the $\langle TJJ\rangle$ correlator. We have [26]

$$k_n|_{n=1,\mu=0} = \partial_n k_n|_{n=1,\mu=0} = 0,$$
$$\partial_\mu k_n|_{n=1,\mu=0} = \frac{16R\pi^{d+1}}{\Gamma(d+1)}[\hat{c} + \hat{e}],$$
$$\partial_n\partial_\mu k_n|_{n=1,\mu=0} = \frac{16R\pi^{d+1}}{d\Gamma(d+1)}[2\hat{c} - d(d-3)\hat{e}], \qquad (22)$$

where the charges $\hat{c}$, $\hat{e}$ are related to C_J, a_2 by [26,87]

$$\hat{c} = \frac{C_J(d-2)\Gamma(\frac{d+2}{2})}{2\pi^{d/2}(d-1)^3}[d(d-1) - a_2],$$
$$\hat{e} = \frac{C_J\Gamma(\frac{d+2}{2})}{2\pi^{d/2}(d-1)^3}[d - 1 + (d-2)a_2]. \qquad (23)$$

Let us now consider the thermal entropy S of the CFT on the hyperbolic cylinder. In the grand canonical ensemble, the first law of thermodynamics reads

$$d\Omega = -SdT - Nd\mu, \qquad (24)$$

where Ω is the grand potential and $N = V_{\mathbb{H}^{d-1}} R^{d-1}\rho$ is the total charge. From the first law the following thermodynamic relation can be obtained:

$$\partial_\mu S = -\partial_\mu\partial_T\Omega = -\partial_T\partial_\mu\Omega = \partial_T N. \qquad (25)$$

Writing N in terms of the magnetic response $k_n(\mu)$, and using that $\partial_T = -(T_0/T^2)\partial_n$, we have

$$\partial_\mu S = -\frac{T_0 V_{\mathbb{H}^{d-1}}}{2\pi T^2}\partial_n\left[\frac{k_n(\mu)}{n}\right]. \qquad (26)$$

Expanding the derivatives, evaluating for $n = 1$ ($T = T_0$) and $\mu = 0$ and using Eq. (22), it immediately follows that the first derivative term vanishes, i.e.,

$$\partial_\mu S_{\rm EE}|_{\mu=0} = 0. \qquad (27)$$

Taking a second derivative with respect to μ in Eq. (26), we have

$$\partial^2_\mu S = -\frac{T_0 V_{\mathbb{H}^{d-1}}}{2\pi T^2}\partial_\mu\partial_n\left[\frac{k_n(\mu)}{n}\right]. \qquad (28)$$

Evaluating again for $n = 1$ ($T = T_0$) and $\mu = 0$, we have

$$\partial^2_\mu S_{\rm EE}|_{\mu=0} = R V_{\mathbb{H}^{d-1}}[\partial_\mu k_n - \partial_\mu\partial_n k_n]|_{n=1,\mu=0}. \qquad (29)$$

Using then Eq. (22), we can rewrite this as

$$\partial_\mu^2 S_{\rm EE}|_{\mu=0} = V_{\mathbb{H}^{d-1}} \frac{16(d-2)R^2\pi^{d+1}}{d\Gamma(d+1)}[\hat{c} + d\hat{e}], \tag{30}$$

which, via Eq. (23), reduces to Eq. (3). This therefore completes the proof that such relation is universally valid for arbitrary CFTs.

Final comments.—Our formula, Eq. (3), holds for general CFTs in $d \geq 3$. In $d = 2$, there are various reasons to expect a different situation. On the one hand, observe that the coefficient a_2 is not even defined in that case. Similarly, from Eq. (8) it is clear that C_J for our holographic calculations is divergent for $d = 2$ and therefore meaningless. The free-field results reported in [26] also suggest a different structure in that case, including possible linear terms in μ or jumps in $S_n(\mu)$ as n and μ vary. Additional two-dimensional counterexamples to the subleading quadratic behavior in μ have appeared in [88]. It would be interesting to investigate these features further—natural candidates would be three-dimensional holographic EQGs [89].

On a different front, it would also be interesting to rederive Eq. (3) using the techniques developed in [90]. In the case of a small perturbation by a relevant operator $\mathcal{O}$, the leading correction to the EE across a sphere was shown to be quadratic in the perturbation and proportional to a double integral of $\langle K\mathcal{O}\mathcal{O}\rangle - \langle \mathcal{O}\mathcal{O}\rangle$, where K is the modular Hamiltonian of ρ_A, which for spheres involves an integral of the stress tensor. In the present context, it would be natural to relate $\mathcal{O}$ to the charge operator, which would bring about integrals of $\langle TJJ\rangle$ and $\langle JJ\rangle$, precisely as expected from Eq. (3).

In [91], a somewhat similar universal relation for charged Rényi entropies involving the uncharged result plus an extra term was obtained in the case of discrete symmetry groups. It would be nice to study the connection between Eq. (3) and the approach developed in that paper and [92] in the case of continuous groups.

A particularly interesting application of our formula is to the case of supersymmetric CFTs (SCFTs), which come with a global R-symmetry group. For instance, for $d = 4$, $\mathcal{N} = 1$ SCFTs, one has a $U(1)_R$ current with [53,93,94]

$$C_J^{\mathcal{N}=1,U(1)_R} = \frac{4c}{\pi^4}, \qquad a_2^{\mathcal{N}=1,U(1)_R} = 3\left(1 - \frac{a}{c}\right), \tag{31}$$

and therefore, our formula, Eq. (3), yields the prediction

$$S_{\rm EE}^{\mathcal{N}=1,U(1)_R} = \nu_3\left[a + \frac{2}{3}\left(c - \frac{a}{3}\right)(\mu R)^2 + \ldots\right], \tag{32}$$

where we used $a^\star = a$ and c is the other trace-anomaly coefficient. Similarly, for $\mathcal{N} = 2$ SCFTs, the R-symmetry group is $U(1)_R \times \mathrm{SU}(2)_R$. Using the corresponding values of C_J and a_2 [95,96], one finds [97]

$$S_{\rm EE}^{\mathcal{N}=2,U(1)_R} = \nu_3\left[a + 2\left(c - \frac{a}{3}\right)(\mu R)^2 + \ldots\right],$$
$$S_{\rm EE}^{\mathcal{N}=2,\mathrm{SU}(2)_R} = \nu_3\left[a + \frac{1}{6}(2c - a)(\mu R)^2 + \ldots\right]. \tag{33}$$

It would be interesting to verify these predictions using alternative methods.

Finally, it is natural to wonder what additional relations connecting quantum information measures and universal CFT quantities may still remain to be discovered.

We would like to thank Nikolay Bobev and Javier Magán for useful discussions. The work of P. A. C. is supported by a postdoctoral fellowship from the Research Foundation—Flanders (FWO Grant No. 12ZH121N). The work of Á. M. is funded by the Spanish FPU Grant No. FPU17/04964. Á. M. is further supported by the MCIU/AEI/FEDER UE Grant No. PGC2018-095205-B-I00 and by the Spanish Research Agency (Agencia Estatal de Investigación) through the Grant Instituto de Física Teórica Centro de Excelencia Severo Ochoa No. CEX2020-001007-S, funded by Ministerio de Ciencia e Innovación/AEI. A. R. S. is supported by the Spanish MECD Grant No. FPU18/03719. The work of A. R. S. is further funded by AEI-Spain (under Project No. PID2020–114157GB-I00 and Unidad de Excelencia María de Maeztu MDM-2016-0692), by Xunta de Galicia-Consellería de Educación (Centro singular de investigación de Galicia accreditation 2019–2022, and Project No. ED431C-2021/14), and by the European Union FEDER.

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