Entanglement Detection with Imprecise Measurements

Simon Morelli,* Hayata Yamasaki,* Marcus Huber, and Armin Tavakoli
Institute for Quantum Optics and Quantum Information—IQOQI Vienna, Austrian Academy of Sciences, Boltzmanngasse 3, 1090 Vienna, Austria and Atominstitut, Technische Universität Wien, Stadionallee 2, 1020 Vienna, Austria

(Received 9 March 2022; accepted 6 June 2022; published 21 June 2022)

We investigate entanglement detection when the local measurements only nearly correspond to those intended. This corresponds to a scenario in which measurement devices are not perfectly controlled, but nevertheless operate with bounded inaccuracy. We formalize this through an operational notion of inaccuracy that can be estimated directly in the lab. To demonstrate the relevance of this approach, we show that small magnitudes of inaccuracy can significantly compromise several well-known entanglement witnesses. For two arbitrary-dimensional systems, we show how to compute tight corrections to a family of standard entanglement witnesses due to any given level of measurement inaccuracy. We also develop semidefinite programming methods to bound correlations in these scenarios.

DOI: 10.1103/PhysRevLett.128.250501

Introduction.—Deciding whether an initially unknown state is entangled is one of the central challenges of quantum information science [1–3]. The most common approach is the method of entanglement witnesses, in which one hypothesises that the state is close to a known target and then finds suitable local measurements that can reveal its entanglement [4–6]. In principle, this allows for the detection of every entangled state. However, it crucially requires the experimenter to flawlessly perform the stipulated quantum measurements. This is an idealization to which one may only aspire: even for the simplest system of two qubits, small alignment errors can cause false positives [7,8]. In contrast, by adopting a device-independent approach, any concerns about the modelling of the measurement devices can be dispelled. This entails viewing them as quantum black boxes and detecting entanglement through the violation of a Bell inequality [9,10]. However, Bell experiments are practically demanding [11]. Also, many entangled states either cannot, or are not known to, violate any Bell inequality [12,13]. In addition, for the common purpose of verifying that a nonmalicious entanglement source operates as intended, a device-independent approach is to use a sledgehammer to crack a nut. In the interest of a compromise, entanglement detection has also been investigated in steering scenarios, in which some devices are assumed to be perfectly controlled and others are quantum black boxes [14]. Nevertheless, such asymmetry is often not present in nonmalicious scenarios, and the approach still suffers from drawbacks similar to both the device-independent case, albeit it milder, and the standard, fully controlled, scenario. A much less explored compromise route is to only assume knowledge of the Hilbert space dimension [15,16]. This essentially adopts the view that the experimenter has no control over the relevant degrees of freedom. Such ideas have also been used to strengthen steering-based entanglement detection [17].

Here, we introduce an approach to entanglement detection that neither assumes flawless control of the measurements nor views them as mostly uncontrolled operations. The main idea is that an experimenter can quantitatively estimate the accuracy of their measurement devices and then base entanglement detection on this benchmark. Such knowledge naturally requires a fixed Hilbert space dimension: the experimenter knows the degrees of freedom on which they operate. To quantify the inaccuracy between the intended target measurement and the lab measurement, we use a simple fidelity-based notion that can handily be measured experimentally.

In what follows, we first establish the relevance of small inaccuracies by showcasing that the conclusions of well-known entanglement witnesses can be substantially compromised. We show that the magnitude of detrimental influence associated to a small inaccuracy does not have to decrease for higher-dimensional systems. This is important because higher-dimensional entangled systems are increasingly interesting for experiments [18–22] but typically cannot be controlled as precisely as qubits. Second, we develop entanglement criteria that explicitly take the degree of inaccuracy into account. For two-qubit scenarios, we provide this based on the simplest entanglement witness and the Clauser-Horne-Shimony-Holt (CHSH) quantity. For a pair of systems of any given local dimension, we show that such criteria can be analytically established as corrections to a simple family of standard entanglement witnesses. Finally, we present semidefinite programming (SDP) relaxations for bounding the set of quantum correlations under measurement inaccuracies. We use this both to estimate the potentially constructive influence of measurement inaccuracy on entanglement-based correlations and to systematically place upper bounds for separable states on linear witnesses.

Framework.—We consider sources of bipartite states $\rho = \rho_{AB}$ of local dimension d. The subsystems are measured individually with settings x and y respectively, producing outcomes $a, b \in \{1, \ldots, o\}$. The experimenter's aim is to measure the first (second) system using a set of projective measurements $\{\tilde{A}_{a|x}\}$ ($\{\tilde{B}_{b|y}\}$). These are called target measurements. However, the measurements actually performed in the lab do not precisely correspond to the targeted measurements, but instead to positive operator-valued measures (POVMs) $\{A_{a|x}\}$ ($\{B_{b|y}\}$). These are called lab measurements and do not need to be projective. The correlations in the experiment are given by the Born rule

$$p(a,b|x,y) = \mathrm{tr}[A_{a|x} \otimes B_{b|y}\rho]. \tag{1}$$

We quantify the correspondence between each of the target measurements and the associated lab measurements through their average fidelity,

$$\mathcal{F}_x^A \equiv \frac{1}{d}\sum_{a=1}^{o} \mathrm{tr}[A_{a|x}\tilde{A}_{a|x}], \qquad \mathcal{F}_y^B \equiv \frac{1}{d}\sum_{b=1}^{o} \mathrm{tr}[B_{b|y}\tilde{B}_{b|y}]. \tag{2}$$

The fidelity respects $\mathcal{F} \in [0,1]$ with $\mathcal{F} = 1$ if and only if the lab measurement is identical to the target measurement. Importantly, the fidelity admits a simple operational interpretation: it is the average probability of obtaining outcome a (b) when the lab measurement is applied to each of the orthonormal states spanning the eigenspace of the ath (bth) target projector. Thus, the fidelities $\{\mathcal{F}_x^A, \mathcal{F}_y^B\}$ can be directly determined by probing the lab measurements with single qudits from a well-calibrated, auxiliary, source. This requires no entanglement and can routinely be achieved; see, e.g., Ref. [23]. It motivates the assumption of a bounded inaccuracy, i.e., a lower bound on each of the fidelities,

$$\mathcal{F}_x^A \geq 1 - \varepsilon_x^A, \qquad \mathcal{F}_y^B \geq 1 - \varepsilon_y^B, \tag{3}$$

where the parameter $\varepsilon \in [0,1]$ is the inaccuracy of the considered lab measurement. In the extreme case of $\varepsilon = 0$, the lab measurement is identical to the target measurement and our scenario reduces to a standard entanglement witness. In the other extreme, $\varepsilon = 1$, only the Hilbert space dimension of the measurement is known. Away from these extremes, one encounters the more realistic scenario, in which the experimenter knows the degrees of freedom, but is only able to control them up to a limited accuracy.

The simplest tests of entanglement use the minimal number of outcomes ($o = 2$). In such scenarios the fidelity constrains (3) can be simplified into

$$\mathrm{tr}(A_x\tilde{A}_x) \geq d(1 - 2\varepsilon_x^A), \qquad \mathrm{tr}(B_y\tilde{B}_y) \geq d(1 - 2\varepsilon_y^B), \tag{4}$$

where we have defined observables $A_x \equiv A_{1|x} - A_{2|x}$ and $B_y \equiv B_{1|y} - B_{2|y}$. The observables can be arbitrary Hermitian operators whose extremal eigenvalue is bounded by unity, i.e., $\|A_x\|_\infty \leq 1$ and $\|B_y\|_\infty \leq 1$.

Notice that the proposed framework immediately extends also to multipartite scenarios.

Impact of inaccuracies in entanglement witnessing.—A crucial motivating question for our approach is whether, and to what extent, small inaccuracies in the measurement devices ($\varepsilon \ll 1$) impact the analysis of a conventional entanglement witness. We discuss this matter based on several well-known witnesses.

First, consider the simplest entanglement witness for two qubits, involving two pairs of local Pauli observables: $\mathcal{W} = \langle \sigma_X \otimes \sigma_X \rangle + \langle \sigma_Z \otimes \sigma_Z \rangle$. For separable states we have $\mathcal{W} \leq \mathcal{W}_{\mathrm{sep}} = 1$ and for entangled states $\mathcal{W} \leq \mathcal{W}_{\mathrm{ent}} = 2$. Consider now that the lab observables $\{A_1, A_2\}$ and $\{B_1, B_2\}$ only nearly correspond (4) to the target observables $\{\sigma_X, \sigma_Z\}$. Since $\mathcal{W}_{\mathrm{ent}} = 2$ is algebraically maximal, it remains unchanged, but such is not the case for the separable bound $\mathcal{W}_{\mathrm{sep}}$. Thanks to the simplicity of $\mathcal{W}$, we can precisely evaluate $\mathcal{W}_{\mathrm{sep}}$ in the prevalent scenario when all measurement devices are equally inaccurate, i.e., $\varepsilon_x^A = \varepsilon_y^B = \varepsilon$. For a product state, we have $\mathcal{W} = \langle A_1\rangle\langle B_1\rangle + \langle A_2\rangle\langle B_2\rangle \leq \sqrt{\langle A_1\rangle^2 + \langle A_2\rangle^2}\sqrt{\langle B_1\rangle^2 + \langle B_2\rangle^2}$. Since the target measurements are identical on both sites and the factors are independent, they are optimally chosen equal. Then, it is easily shown that the optimal choice of Bloch vectors corresponds to aligning A_1 and A_2 (B_1 and B_2) to the extent allowed by ε. This leads to the following tight condition for entanglement detection (see Supplemental Material [24])

$$\mathcal{W}_{\mathrm{sep}}(\varepsilon) = 1 + 4(1 - 2\varepsilon)\sqrt{\varepsilon(1 - \varepsilon)}, \tag{5}$$

when $\varepsilon \leq \frac{1}{2} - (1/[2\sqrt{2}])$ and $\mathcal{W}_{\mathrm{sep}} = 2$ otherwise. Importantly, the derivative diverges at $\varepsilon \to 0^+$. Hence, a small ε induces a large perturbation in the ideal ($\varepsilon = 0$) separable bound. In the vicinity of $\varepsilon = 0$, it scales as $\mathcal{W}_{\mathrm{sep}} \sim 1 + 4\sqrt{\varepsilon}$. For example, $\varepsilon = 0.5\%$ leads to $\mathcal{W}_{\mathrm{sep}}(\varepsilon) \approx 1.28$, which eliminates over a quarter of the range in which standard entanglement detection is possible, indicating the relevance of false positives.

Second, consider the CHSH quantity for entanglement detection, namely $\mathcal{W} = \langle \sigma_X \otimes (\sigma_X + \sigma_Z)\rangle + \langle \sigma_Z \otimes (\sigma_X - \sigma_Z)\rangle$. Here, we have targeted observables optimal for violating the CHSH Bell inequality [25]. One has $\mathcal{W}_{\mathrm{sep}} = \sqrt{2}$ and $\mathcal{W}_{\mathrm{ent}} = 2\sqrt{2}$. In contrast to the previous example, the fact that all correlations from d-dimensional separable states constitute a subset of all correlations based on local hidden variables implies that entanglement can be detected for any value of ε. However, as we show in Supplemental Material

[24] through an explicit separable model that we conjecture to be optimal, this fact does not qualitatively improve the robustness of idealized ($\varepsilon = 0$) entanglement detection to small inaccuracies. We obtain

$$\mathcal{W}_{\text{sep}} = 4(1-2\varepsilon)\sqrt{\varepsilon(1-\varepsilon)} + \sqrt{2 - 16\varepsilon(1-\varepsilon)(1-2\varepsilon)^2}, \tag{6}$$

when $\varepsilon \leq \frac{1}{2} - (1/[2\sqrt{2}])$ and $\mathcal{W}_{\text{sep}} = 2$ otherwise. For small ε, we find $\mathcal{W}_{\text{sep}} \sim \sqrt{2} + 4\sqrt{\varepsilon}$. An inaccuracy of $\varepsilon = 0.5\%$ ensures $\mathcal{W}_{\text{sep}} \gtrsim 1.67$, which eliminates nearly a fifth of the range in which standard entanglement detection is possible.

Interestingly, it is *a priori* not clear how small ε should impact standard entanglement witnessing as d increases. On the one hand, the impact ought to increase due to the increasing number of orthogonal directions in Hilbert space. On the other hand, it ought to decrease due to the growing distances in Hilbert space. For instance, the ε required to transform the computational basis into its Fourier transform scales as $\varepsilon = (\sqrt{d} - 1/\sqrt{d})$, which rapidly approaches unity. To investigate the trade-off between these two effects, we consider the d-dimensional generalization of the simplest entanglement witness. Both subsystems are subject to the same pair of target measurements, namely the computational basis $\{|e_i\rangle\}_{i=1}^d$ and its Fourier transform $\{|f_i\rangle\}_{i=1}^d$, where $|f_i\rangle = \Omega|e_i\rangle$ with $\Omega_{jk} = (1/\sqrt{d})e^{(2\pi i/d)jk}$. The witness is $\mathcal{W}^{(d)} = \sum_{i=1}^d \langle e_i, e_i|\rho|e_i, e_i\rangle + \langle f_i, f_i|\rho|f_i, f_i\rangle$. Notice that for $d = 2$ this only differs from the previous, simplest, witness by a normalization term. One has $\mathcal{W}_{\text{sep}}^{(d)} = 1 + (1/d)$ and $\mathcal{W}_{\text{ent}}^{(d)} = 2$ [26]. Allowing for measurement inaccuracy, we use an alternating convex search algorithm to numerically optimize over the lab measurements and shared separable states to obtain lower bounds on $\mathcal{W}_{\text{sep}}^{(d)}(\varepsilon)$. See Supplemental Material [24] for details about the method. In order to compare the impact of measurement inaccuracy for different dimensions, we consider the following ratio between the entangled-to-separable gap in the inaccurate and ideal case, $\Delta \equiv \{[\mathcal{W}_{\text{ent}}^{(d)}(0) - \mathcal{W}_{\text{sep}}^{(d)}(\varepsilon)]/[\mathcal{W}_{\text{ent}}^{(d)}(0) - \mathcal{W}_{\text{sep}}^{(d)}(0)]\} = (d/d-1)[2 - \mathcal{W}_{\text{sep}}^{(d)}(\varepsilon)]$. Notice that the numerator features $\mathcal{W}_{\text{ent}}^{(d)}(0)$ instead of $\mathcal{W}_{\text{ent}}^{(d)}(\varepsilon)$ because ε is not in itself a resource for the experimenter. The results of the numerics are illustrated in Fig. 1 for some different choices of ε. We observe that Δ is not monotonic in d, but instead features a maximum, that shifts downward in d as ε increases. Beyond this maximum point, the impact of measurement inaccuracies grows as the dimension becomes large.

Finally, for multipartite qubit states, it is natural to expect that the detrimental influence of small ε grows with the number of qubits under consideration. The reason is that measurement inaccuracies can accumulate separately in the different subsystems. This intuition is confirmed by the models of Ref. [8], in which small alignment errors are used to spoof, with increasing magnitude, the standard fidelity-based witness of genuine multipartite entanglement for Greenberger-Horne-Zeilinger states [27]. This further confirms the need of considering measurement inaccuracies.

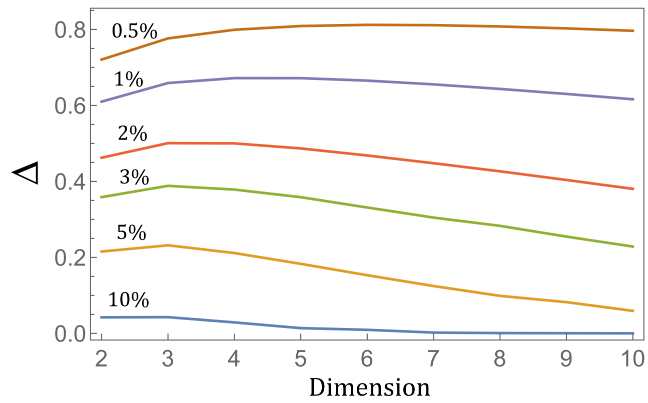

FIG. 1. Numerically obtained lower bounds on the relative magnitude of the entangled-to-separable gap Δ for entanglement witnessing based on two conjugate bases at different degrees of measurement inaccuracy $\varepsilon \in \{0.5\%, 1\%, 2\%, 3\%, 5\%, 10\%\}$.

High-dimensional entanglement criterion.—In view of the relevance of small measurement inaccuracies, it is natural to formulate entanglement criteria that take them explicitly into account beyond the simplest, two-qubit, scenario. Consider a pair of d-dimensional systems and $n \in \{1, \ldots, d^2 - 1\}$ measurements. For system A, the observables ideally correspond to (subsets of) a generalized Bloch basis $\{\lambda_i\}_{i=1}^n$ and for system B, the ideal observables are the complex conjugates $\{\bar{\lambda}_i\}_{i=1}^n$. Here, λ_i is d dimensional, traceless, and satisfies $\mathrm{tr}(\lambda_i \lambda_j^\dagger) = d\delta_{ij}$ [28]. Defining $\rho = (1/d)(\mathbb{1} + \sum_{i=1}^{d^2-1} \mu_i \lambda_i)$, one has $\|\vec{\mu}\|^2 \leq d - 1$. A simple standard entanglement witness, based on a total of n measurements, is then given by

$$\mathcal{W}^{(d)} = \sum_{i=1}^n \langle \lambda_i \otimes \bar{\lambda}_i \rangle. \tag{7}$$

Using Hölder's inequality, one finds that separable states obey $\mathcal{W}_{\text{sep}}^{(d)} = d - 1$. When the choice of Bloch basis is fixed, entangled states can achieve at most $\mathcal{W}_{\text{ent}}^{(d)} = \nu_{\max}[\sum_{i=1}^n \lambda_i \otimes \bar{\lambda}_i]$, by choosing the state as the eigenvector corresponding to the largest eigenvalue ($\nu_{\max}$). When the choice of Bloch basis is not fixed, a general upper bound for entangled states is $\mathcal{W}_{\text{ent}}^{(d)} \leq \min\{\sqrt{n(d^2-1)}, n(d-1)\}$, as shown in Supplemental Material [24]. Note that $n(d-1)$ is relevant only when $d = 2$. Notice also that the maximally entangled state $|\phi_d^+\rangle = (1/\sqrt{d})\sum_{i=0}^{d-1} |ii\rangle$ achieves $\mathcal{W}^{(d)} = n$ regardless of the choice of Bloch basis.

Consider now that the lab observables only nearly correspond to $\{\lambda_i\}$ and $\{\bar{\lambda}_i\}$, respectively. We write them as $A_i = q\lambda_i + \sqrt{1-q^2}\lambda_i^\perp$ and $B_i = q\bar{\lambda}_i + \sqrt{1-q^2}\bar{\lambda}_i^\perp$, where $q \in [-1, 1]$ is related to the inaccuracy through $q = 1{-}2\varepsilon$ and $\lambda_i^\perp$ and $\bar{\lambda}_i^\perp$ are observables orthogonal to λ_i and $\bar{\lambda}_i$, respectively, on the generalized Bloch sphere. In Supplemental Material [24], we prove that the witness $\mathcal{W}^{(d)} = \sum_{i=1}^n \langle A_i \otimes B_i \rangle$ for separable states obeys

$$\mathcal{W}_{\mathrm{sep}}^{(d)}(\varepsilon) \leq (d-1)\left(q + \sqrt{n-1}\sqrt{1-q^2}\right)^2, \qquad (8)$$

when $q \geq (1/\sqrt{n})$ and otherwise $\mathcal{W}_{\mathrm{sep}}^{(d)}(\varepsilon) \leq n(d-1)$, which is algebraically maximal. As is intuitive, the window for detecting entanglement shrinks as ε increases.

We investigate the tightness of the bound. To this end, choose the state as $|\phi^\dagger\rangle \otimes |\phi^T\rangle$, where the local Bloch vector is $\mu_i = (\sqrt{d-1}/\sqrt{n})$ and where $\lambda_i \to \lambda_i^\dagger$ ($\lambda_i \to \lambda_i^T$) for $|\phi^\dagger\rangle$ ($|\phi^T\rangle$). Choose the observables as $A_i = q\lambda_i + \sum_{j\neq i}(\sqrt{1-q^2}/\sqrt{n-1})\lambda_j$ and $B_i = q\bar{\lambda}_i + \sum_{j\neq i}(\sqrt{1-q^2}/\sqrt{n-1})\bar{\lambda}_j$. This returns the separable bound (8). However, we need to check that the Bloch vector $\vec{\mu}$ corresponds to a valid state. Curiously, for the most powerful case, namely $n = d^2 - 1$, tightness would be implied by a positive answer to the long-standing open question of whether there exists a Weyl-Heisenberg covariant symmetric informationally complete (SIC) POVM in dimension d. To see the connection, simply choose the Bloch basis as the non-Hermitian Weyl-Heisenberg basis $\{X^u Z^v\}$ for $u, v \in \{0, \ldots, d-1\}$ and $u + v > 0$, where $X = \sum_{k=0}^{d-1} |k+1\rangle\langle k|$ and $Z = \sum_{k=0}^{d-1} e^{(2\pi i k/d)}|k\rangle\langle k|$. It follows immediately that $|\langle\phi|X^u Z^v|\phi\rangle| = (1/\sqrt{d+1})$, which defines a SIC POVM. Since these SIC POVMs are conjectured to exist in all dimensions [29], and are known to exist up to well above the first hundred dimensions [30,31], our bound is plausibly tight for any d.

SDP methods.—We develop a hierarchy of SDP relaxations to bound the largest possible value of any linear witness, $\mathcal{W} = \sum_{a,b,x,y} c_{abxy} p(a, b|x, y)$, for some real coefficients c_{abxy}. The method applies both for correlations originating from entangled states and from separable states, under any given degree of measurement inaccuracy and arbitrary target measurements. Thus, we systematically establish upper bounds $\mathcal{W}_{\mathrm{ent}}^{\uparrow}(\varepsilon) \geq \mathcal{W}_{\mathrm{ent}}(\varepsilon)$ and $\mathcal{W}_{\mathrm{sep}}^{\uparrow}(\varepsilon) \geq \mathcal{W}_{\mathrm{sep}}(\varepsilon)$. This has a threefold motivation. First, $\mathcal{W}_{\mathrm{ent}}$ will generally depend on ε; cases with $\mathcal{W}^{(d)} > \mathcal{W}_{\mathrm{ent}}^{(d)}(0)$ can be observed when the inaccuracies accumulate in a constructive way (e.g., a favorable systematic error in the local reference frames). It is relevant to bound such occurrences. Second, knowledge of $\mathcal{W}_{\mathrm{ent}}^{\uparrow}(\varepsilon)$ allows an experimenter to give lower bounds on the inaccuracy of the measurement devices. Third, and most importantly, this enables a general and systematic construction of entanglement witnesses of the form $\mathcal{W} \leq \mathcal{W}_{\mathrm{sep}}^{\uparrow}(\varepsilon)$.

We discuss the main features of the method for computing $\mathcal{W}_{\mathrm{ent}}^{\uparrow}(\varepsilon)$ and then see how it can be extended to also compute $\mathcal{W}_{\mathrm{sep}}^{\uparrow}(\varepsilon)$. To this end, as is standard, the SDP relaxation method is based on the positivity of a moment matrix. This matrix consists of traces of monomials (in the spirit of, e.g., [32]) which are composed of products of the state, the lab measurements, and the target measurements (see Supplemental Material [24] for specifics). Moments corresponding to products of the first two can be used to build a generic linear witness $\mathcal{W}$ via Eq. (1). Moments corresponding to products of the final two can be used to build the constraints on the fidelities $\mathcal{F}_x^A$ and $\mathcal{F}_y^B$. Our construction draws inspiration from two established ideas. First, one can capture the constraints of d-dimensional Hilbert space, on the level of the moment matrix, by numerically sampling states and measurements [33]. Second, in scenarios without entanglement, constraints capturing the fidelity of a quantum state with a target can be incorporated into the moment matrix [34]. We adapt the latter to entanglement-based scenarios and measurement fidelities as needed for Eq. (3). Details are given in Supplemental Material [24]. We have applied this method, at low relaxation level, in several different case studies in low dimensions and frequently found that the obtained upper bounds coincide with those obtained from interior point optimization routines. We note that the computational requirements for this tool can be much reduced since sampling-based symmetrization methods of Ref. [35] can straightforwardly be incorporated.

To extend this method for the computation of $\mathcal{W}_{\mathrm{sep}}^{\uparrow}(\varepsilon)$, we must incorporate constraints on the set of quantum states. Since the set of separable states is generally difficult to characterize (see, e.g., Ref. [36]), we instead adopt an approach in which we use the ideal entanglement witness condition, $\mathcal{W} \leq \mathcal{W}_{\mathrm{sep}}(0)$, which we may realistically assume to possess, in place of the set of separable states. Then, since the probabilities associated to performing the target measurements on the state explicitly appear in our moment matrix, we can introduce it as an additional linear constraint in our SDP. Hence, the optimization is effectively a relaxation of the subset of entangled states for which the original entanglement witness holds. In fact, since the set of separable states is characterized by infinitely many linear entanglement witnesses, one can in this way continue to introduce linear standard witnesses to constrain the effective state space in the SDP and thus further improve the accuracy of the bound $\mathcal{W}_{\mathrm{sep}}^{\uparrow}(\varepsilon)$. In Supplemental Material [24] we exemplify the use of this method, in its basic version, using only a single witness constraint $\mathcal{W} \leq \mathcal{W}_{\mathrm{sep}}(0)$ on the state space, and show that it returns nontrivial, albeit not tight, bounds for two simple entanglement witnesses for relevant values of ε.

Discussion.—We have introduced and investigated entanglement detection when the measurements only nearly correspond to those intended to be performed in the laboratory. We have shown the relevance of the concept, presented explicit entanglement witnesses that take measurement inaccuracy into account, and finally shown how SDP methods can be applied to these types of problems. These results are a step toward a theoretical framework for detecting entanglement based on devices that are quantitatively benchmarked in an operationally meaningful and experimentally accessible manner.

Our Letter leaves several natural open problems. If given an arbitrary standard entanglement witness, how can we compute corrections due to the introduction of measurement inaccuracies? Our SDP method is a first step toward addressing this problem but better methods are necessary both in terms of computational cost and in terms of the accuracy of the separable bound. Moreover, for a given d, what is the smallest number of auxiliary global measurement settings needed to eliminate the diverging derivative for optimal standard entanglement witnesses under small measurement inaccuracy? In addition, can one extend our entanglement witnesses to witnesses of genuine higher-dimensional entanglement, e.g., by detecting the Schmidt number? Also, in this first work, we have focused on bipartite entanglement. It would be interesting to identify useful entanglement witnesses for multipartite states at bounded measurement inaccuracy. Finally, the framework proposed here for entanglement detection draws inspiration from ideas proposed in semi-device-independent quantum communications. Given that several frameworks for semi-device independence recently have been proposed [34,37–40], there may be other similarly inspired avenues for entanglement detection based on quantitative benchmarks.

The authors thank Mateus Araújo for discussions. This project was supported by the Wenner-Gren Foundations, the Austrian Science Fund (FWF) through Projects No. Y879-N27 (START) and No. P 31339-N27 (Stand-Alone), JSPS Overseas Research Fellowships, and JST PRESTO Grant No. JPMJPR201A.

[*]S. M. and H. Y. contributed equally to this work.

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