Energy Spectrum of Two-Dimensional Acoustic Turbulence

Adam Griffin,[1] Giorgio Krstulovic,[2,*] Victor S. L'vov,[3] and Sergey Nazarenko[1]
[1]*Université Côte d'Azur, Institut de Physique de Nice (INPHYNI), Parc Valrose, 06108 Nice, France*
[2]*Université Côte d'Azur, Observatoire de la Côte d'Azur, CNRS, Laboratoire Lagrange, Boulevard de l'Observatoire CS 34229–F 06304 NICE Cedex 4, France*
[3]*Department of Chemical and Biological Physics, Weizmann Institute of Science, Rehovot 76100, Israel*

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We report an exact unique constant-flux power-law analytical solution of the wave kinetic equation for the turbulent energy spectrum, $E(k) = C_1\sqrt{\varepsilon a c_s}/k$, of acoustic waves in 2D with almost linear dispersion law, $\omega_k = c_s k[1 + (ak)^2]$, $ak \ll 1$. Here, ε is the energy flux over scales, and C_1 is the universal constant which was found analytically. Our theory describes, for example, acoustic turbulence in 2D Bose-Einstein condensates. The corresponding 3D counterpart of turbulent acoustic spectrum was found over half a century ago, however, due to the singularity in 2D, no solution has been obtained until now. We show the spectrum $E(k)$ is realizable in direct numerical simulations of forced-dissipated Gross-Pitaevskii equation in the presence of strong condensate.

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Waves in nonlinear systems interact and transfer energy along scales in a cascade process with a constant flux, creating an out-of-equilibrium state known as wave turbulence. When nonlinearity is small, the weak-wave turbulence (WWT) theory provides a mathematical description of the system [1,2]. The most common applications of this theory are capillary-gravity waves [3], Alfvén waves in magnetohydrodynamics [4], Langmuir waves in plasmas [5], inertial and internal waves in rotating stratified fluids [6,7], Kelvin waves in vortices [8], elastic plates [9], gravitational waves [10], and density waves in Bose-Einstein condensates [11].

Note that acoustic waves in ideal compressible fluids, one of the most common examples in nature, are nondispersive: their frequency $\omega_{\boldsymbol{k}} = c_s k$ is linear in wave number $k \equiv |\boldsymbol{k}|$. Accordingly, the resonance conditions

$$\omega_{\boldsymbol{k}} = \omega_{\boldsymbol{k}_1} + \omega_{\boldsymbol{k}_2}, \qquad \boldsymbol{k} = \boldsymbol{k}_1 + \boldsymbol{k}_2, \tag{1}$$

allow for interaction of the waves with parallel wave vectors only: $\boldsymbol{k}\|\boldsymbol{k}_1\|\boldsymbol{k}_2$ and in the same direction [2]. Therefore, in the reference frame moving with the speed of sound c_s in the direction of $\boldsymbol{k}$, $\boldsymbol{k}_1$, and $\boldsymbol{k}_2$ all wave packets are at the rest and their overlapping time $\tau_{\rm ovr}$ goes to infinity, allowing for wave steepening and breaking effects, which requires finite shock creation time $\tau_{\rm sh}$. In other words, the main assumption of the WWT theory, roughly formulated as $\tau_{\rm ovr} \ll \tau_{\rm sh}$, fails for dispersionless acoustic waves even even at small nonlinearity.

This has caused a long-standing debate whether WWT theory is applicable for their description [12], or alternatively, if acoustic waves should be viewed as a random collection of weak shocks leading to the Kadomtsev-Petviashvilli spectrum [13]. There is a hand waving argument—yet unsupported by rigorous proof—that the theory applies to 3D acoustic turbulence because the divergence of wave packets in 3D space plays a role similar to the wave dispersion in preventing wave breaking. It was argued first in [14] and later in [15], that the WWT description is indeed possible for 3D weak acoustic turbulence. However, the respective kinetic equation for the spectrum has to be modified so that interactions of noncollinear waves are described correctly. Concerning the 2D case, it is evident that WWT theory is not applicable in its classical form. Indeed, the main equation of the theory, the wave-kinetic equation, is singular and meaningless in the 2D case. The possibility of an alternative statistical description of weak nondispersive acoustic 2D sound was claimed in [14], but it remains an unfinished task. Fortunately, in some important physical applications, 2D sound is regularized by weak dispersion effects, and the use of the classical WWT theory becomes again possible. One such example, which we will use in the present Letter, is the acoustic turbulence in 2D Bose-Einstein condensates (BEC).

Recent experiments with 3D BECs have succeeded in creating wave-turbulence states where measurements can be explained using the WWT theory in the fully dispersive limit [16,17]. On the other hand, much interest has been devoted to studying the dynamics of vortices experimentally in 2D BECs, as such states are close to hydrodynamic 2D classical turbulence [18,19]. Unfortunately, experiments of acoustic 2D BECs are still lacking and no theoretical predictions are available.

In this Letter, we develop the theory of weak wave turbulence of weakly dispersive 2D sound and obtain a modified wave kinetic equation. We derive a new stationary power-law flux spectrum of Kolmogorov-Zakharov type, which corresponds to a cascade of energy from large to small spatial scales. We then determine the flux spectrum exponent and the value of the prefactor constant analytically. This prediction cannot be naively guessed by dimensional arguments, unlike the existing 3D results. Our theory is then tested and validated with numerical simulations of weakly nonlinear sound in the 2D forced-dissipated Gross-Pitaevskii (GP) equation (17) which is a dynamical model for BEC.

Let us consider the classical wave-kinetic equation describing the evolution of the wave action spectrum $n_{\boldsymbol{k}} = n(\boldsymbol{k}, t)$ (with $\boldsymbol{k}$ and t being the wave number and time) driven by three-wave resonant interactions [1,2],

$$\frac{\partial n_{\boldsymbol{k}}}{\partial t} = \mathrm{St}_{\boldsymbol{k}} \tag{2}$$

with the wave-collision integral

$$\mathrm{St}_{\boldsymbol{k}} = 2\pi \int (\mathcal{R}^{\boldsymbol{k}}_{\boldsymbol{12}} - \mathcal{R}^{\boldsymbol{1}}_{\boldsymbol{k2}} - \mathcal{R}^{\boldsymbol{2}}_{\boldsymbol{k1}}) d\boldsymbol{k}_1 d\boldsymbol{k}_2,$$
$$\mathcal{R}^{\boldsymbol{k}}_{\boldsymbol{12}} = \mathcal{N}^{\boldsymbol{k}}_{\boldsymbol{12}} \delta^{\boldsymbol{k}}_{\boldsymbol{12}} \delta(\omega^{\boldsymbol{k}}_{\boldsymbol{12}}), \tag{3a}$$

$$\mathcal{N}^{\boldsymbol{k}}_{\boldsymbol{12}} = |V^{\boldsymbol{k}}_{\boldsymbol{12}}|^2 [n_{\boldsymbol{k}_1} n_{\boldsymbol{k}_2} - n_{\boldsymbol{k}} n_{\boldsymbol{k}_1} - n_{\boldsymbol{k}} n_{\boldsymbol{k}_2}],$$
$$\delta^{\boldsymbol{k}}_{\boldsymbol{12}} = \delta(\boldsymbol{k} - \boldsymbol{k}_1 - \boldsymbol{k}_2), \qquad \delta(\omega^{\boldsymbol{k}}_{\boldsymbol{12}}) = \delta(\omega_{\boldsymbol{k}} - \omega_{\boldsymbol{k}_1} - \omega_{\boldsymbol{k}_2}), \tag{3b}$$

$\omega_{\boldsymbol{k}} = \omega(\boldsymbol{k})$ and $V^{\boldsymbol{k}}_{\boldsymbol{12}} \equiv V(\boldsymbol{k}, \boldsymbol{k_1}, \boldsymbol{k_2})$ is a three-wave interaction amplitude, that depends on the particular type of waves.

The central object in turbulence is the 1D energy spectrum defined as the distribution of energy in $k = |\boldsymbol{k}|$, so that the energy (per unit of mass) is defined as $E = \int E(k) dk$. It is related to the wave action spectrum as $E(k) = 4\pi \omega_k k^2 n_k$ in 3D and $E(k) = 2\pi \omega_k k n_k$ in 2D.

In this Letter, we consider turbulence of weakly dispersive acoustic waves for which

$$\omega_k = c_s k [1 + (ak)^2]. \tag{4}$$

Here, $a = \text{const}$ is a dispersion length such that $ak \ll 1$. The interaction amplitude V^k_{12} in the case of acoustic hydrodynamics [2,11,12,20] is given by

$$V^k_{12} = V_0 \sqrt{k k_1 k_2}, \qquad V_0 = \text{const}. \tag{5}$$

To find the energy spectra, note that Eq. (3) conserves the total energy of the system and, therefore, it can be rewritten as the following continuity equation for $E(k)$,

$$\frac{\partial E(k)}{\partial t} + \frac{\partial \varepsilon_k}{\partial k} = 0, \quad \text{where } \varepsilon_k = -\int_{k_1 < k} \mathrm{St}_{\boldsymbol{k}_1} \omega_{\boldsymbol{k}_1} d\boldsymbol{k}_1. \tag{6}$$

Dimensionally, the energy flux $\varepsilon(k) \propto St_k \propto n_k^2$, so that n_k and $E(k) \propto \sqrt{\varepsilon}$. Assuming full self-similarity (i.e., no other dimensional parameters enter into the game), and that ε_k is independent of k in an inertial range of scales, one can reconstruct $E(k)$ from the dimensional reasoning up to a dimensionless constant C,

$$E(k) = C \sqrt{\varepsilon \omega_k} / k^2. \tag{7a}$$

For example, for nondispersive 3D acoustic waves with $\omega_k = c_s k$ we recover the Zakharov-Sagdeev spectrum [12],

$$E(k) \propto k^{-3/2}, \tag{7b}$$

which is a flux spectrum describing an energy cascade from large to small scales. It can also be obtained as an exact solution of Eq. (3) [12].

Recall that nondispersive acoustic waves admit triad wave number and frequency resonances for colinear wave vectors only, i.e., $\boldsymbol{k} \| \boldsymbol{k}_1 \| \boldsymbol{k}_2$. In this case, the arguments of $\delta^{\boldsymbol{k}}_{\boldsymbol{12}}$ and $\delta(\omega^k_{12})$ in Eq. (3) for $\mathrm{St}_{\boldsymbol{k}}$ coincide which creates a singularity. Fortunately, in 3D, this singularity is integrable and $\mathrm{St}_{\boldsymbol{k}}$ remains finite. Therefore, spectrum Eq. (7b) in 3D is a valid solution (see Refs. [14,15,20] for further discussions). In 2D, the singularity is NOT integrable and Eq. (3) for the nondispersive case is therefore invalid. Furthermore, the energy spectrum (7b) obtained dimensionally is wrong in 2D because the additional dimensional parameter a becomes essential (unlike in 3D).

To solve the problem of acoustic turbulence in 2D, we take into account the dispersion correction in the frequency (4). We choose a reference system with $\boldsymbol{k} \| \hat{\boldsymbol{x}}$. Integrating in Eq. (3) $\mathcal{R}^{\boldsymbol{k}}_{\boldsymbol{12}}$ over $\boldsymbol{k}_2$ with the help of $\boldsymbol{\delta}^{\boldsymbol{k}}_{\boldsymbol{12}}$ and over k_{1y} using $\delta(\omega^k_{12})$. We get

$$\Delta^{\boldsymbol{k}}_{\boldsymbol{12}} \stackrel{\text{def}}{=} \int_{\boldsymbol{k}_2, k_{1y}} \delta(\omega^k_{12}) \delta^{\boldsymbol{k}}_{\boldsymbol{12}} d\boldsymbol{k}_1 d\boldsymbol{k}_2$$
$$= |\partial_{k_{1y}} (\omega_{\boldsymbol{k}} - \omega_{\boldsymbol{k_1}} - \omega_{\boldsymbol{k}-\boldsymbol{k_1}})|^{-1} dk_{1x} \approx \frac{k_1 k_2 dk_{1x}}{c_s k |k_{1y}|}, \tag{8a}$$

where we retained the leading order in $ak \ll 1$ only. Analyzing resonance conditions (1) with the dispersion law (4), for small ak we find $k_{1y} \approx \sqrt{6} a k_1 k_2$ (see Supplemental Material [21]). Substituting this into Eq. (8a), we have

$$\Delta^{\boldsymbol{k}}_{\boldsymbol{12}} \approx dk_{1x} / (\sqrt{6} c_s a k). \tag{8b}$$

We can use this expression to modify the dimensional analysis, which now involves an additional dimensional quantity a. For this, it is only important to observe that $\Delta^{\boldsymbol{k}}_{\boldsymbol{12}} \propto 1/a$. Therefore, $St_{\boldsymbol{k}} \propto n_k^2 / a$ which, together with $\mathrm{St}_{\boldsymbol{k}} \propto \varepsilon$ [arising from Eq. (6)], gives $n_k \propto \sqrt{\varepsilon a}$. This leads to the following energy spectrum of weakly dispersive 2D acoustic turbulence,

$$E(k) = C_1\sqrt{\varepsilon a \omega_k}/k^{3/2} = C_1\sqrt{\varepsilon a c_s}/k, \qquad (9)$$

where C_1 is a dimensionless constant. Note that the spectrum $E(k) \propto 1/k$ was suggested in [11] without proving that it is a solution of the kinetic equation. Below, we will prove that (9) is the unique stationary power-law flux solution of Eq. (3) and find C_1 analytically. To do this, we compute $\Delta^{\mathbf{1}}_{\mathbf{k2}}$ and $\Delta^{\mathbf{2}}_{\mathbf{k1}}$, similarly to Eqs. (8a) [see the Eqs. (27) in Supplemental Material [21]] and present the collision term in (3) in simpler form,

$$\mathrm{St}_{\mathbf{k}} = \frac{2\pi}{\sqrt{6}c_s a k}\int dk_1 dk_2 [\mathcal{N}^{\mathbf{k}}_{\mathbf{12}}\delta(k-k_1-k_2) - \mathcal{N}^{\mathbf{1}}_{\mathbf{k2}}\delta(k_1-k-k_2) - \mathcal{N}^{\mathbf{2}}_{\mathbf{k1}}\delta(k_2-k-k_1)]. \qquad (10)$$

Let us seek a stationary solution of Eq. (3) in the form

$$n_k = Ak^{-x}, \qquad x = \text{const}. \qquad (11)$$

There is, of course, a trivial solution with $x = 1$ corresponding to the thermodynamic energy equipartition, but we will be interested in nonzero flux states only. Assuming that the integrals in Eq. (10) converge (which is to be checked *a posteriori*), let us apply the Kraichnan-Zakharov transform

$$k_1 = \frac{k^2}{\tilde{k}_1}, \quad k_2 = \frac{k\tilde{k}_2}{\tilde{k}_1} \quad \text{and} \quad k_2 = \frac{k^2}{\tilde{k}_1}, \quad k_1 = \frac{k\tilde{k}_2}{\tilde{k}_1} \qquad (12)$$

to the second and the third terms in Eq. (10), respectively. This gives (after dropping tildes on the integration variables for uniformity of notations)

$$\mathrm{St}_{\mathbf{k}} = \frac{2\pi}{\sqrt{6}c_s a k}\int dk_1 dk_2 \mathcal{N}^{\mathbf{k}}_{\mathbf{12}} \times \left[1 - \left(\frac{k_1}{k}\right)^{-y} - \left(\frac{k_2}{k}\right)^{-y}\right]\delta(k-k_1-k_2), \qquad (13)$$

with $y = 5 - 2x$. Thus, $\mathrm{St}_{\mathbf{k}} = 0$ if $y = -1$, so that $x = 3$ and thus

$$n_k \propto \frac{1}{k^3}, \text{so} \quad E(k) = 2\pi k \omega_k n_k \approx 2\pi k^2 c_s n_k \propto \frac{1}{k}. \qquad (14)$$

This is an exact stationary solution of the Eqs. (3) and (10) coinciding with the dimensional prediction (9).

To demonstrate the existence and uniqueness of solution (14), let us nondimensionalize the collision integral for arbitrary x:

$$\mathrm{St}_k = \frac{2\pi A^2 V_0^2}{\sqrt{6}ac_s} I(x) k^{3-2x}, \qquad (15a)$$

$$I(x) = \int_0^1 q(1-q)[q^{-x}(1-q)^{-x} - q^{-x} - (1-q)^{-x}] \times [1 - q^{-y} - (1-q)^{-y}]dq. \qquad (15b)$$

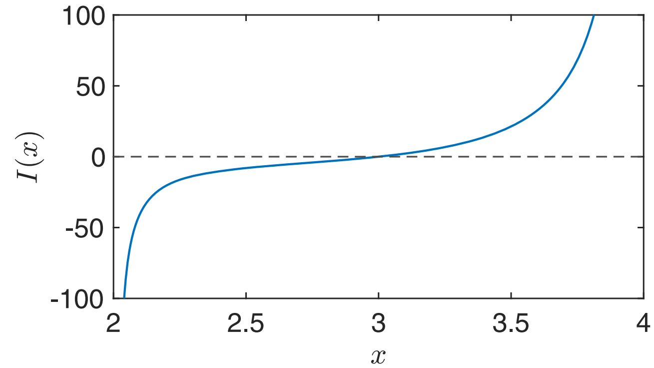

FIG. 1. Integral (13) in the window of convergence.

This integral converges if and only if $2 < x < 4$ or $x = 1$, as we prove in the Supplemental Material [21]. In Fig. 1 we show the plot $I(x)$ obtained numerically for the convergence range $2 < x < 4$. As we see, $x = 3$ is the only point at which $I(x) = 0$, which proves the uniqueness of the stationary power-law flux solution (9). The existence of the solution amounts to the finiteness of the constant C_1. We compute C_1 in Sec. III. C of Supplemental Material [21] by substituting (13) into the definition of the flux (6). The result is as follows:

$$\varepsilon_k = \frac{4\pi^4 A^2 V_0^2}{\sqrt{6}a}, \qquad C_1 = \frac{6^{1/4}\sqrt{c_s}}{\pi V_0}. \qquad (16)$$

Notice that Eq. (3) is valid for sufficiently small nonlinearities and stochasticity of the phases [1,2]. Let us define the interaction frequency γ_k as a frequency with which the wave packets are destroyed by the nonlinear interactions. It will also correspond to the nonlinear frequency broadening: a characteristic width of the time-Fourier spectrum at a fixed $\boldsymbol{k}$. Applicability of Eq. (3) requires frequency γ_k to be smaller than the characteristic frequency of interacting waves $\delta\omega_k$. For the 3D waves, we roughly (and definitely not rigorously) may take $\delta\omega_k = \omega_k$. However, due to the nonintegrable singularity of the nondispersive 2D system, in 2D we should take only the dispersive part of the frequency, $\delta\omega_k = c_s a^2 k^3$, ignoring its linear part $c_s k$, disappearing in the reference system, comoving with velocity c_s in the $\boldsymbol{k}$ direction. On the other hand, for the wave turbulence to be considered acoustic, the dispersion must remain a small correction, i.e., $ak \ll 1$.

To test our theoretical predictions, we perform direct numerical simulations of the 2D GP equation for the complex wave function ψ. Written in terms of the healing length ξ, the speed of sound c_s and the bulk density ρ_0, this equation reads

$$i\frac{\partial\psi}{\partial t} = \frac{c_s}{\sqrt{2}\xi}\left[-\xi^2\nabla^2 + \frac{1}{\rho_0}|\psi|^2 - 1\right]\psi + iD_c\nabla^6\psi + F(\boldsymbol{r}, t), \qquad (17)$$

where we have also included a large-scale forcing F and a hyperviscous dissipation term. The healing length and the speed of sound both depend on the physical properties of the superfluid and on ρ_0, and they can be chosen arbitrarily in the dimensionless GP equation.

The GP equation is a well established model for BEC and it can be mapped to an effective compressible irrotational fluid flow via the Madelung transformation, $\psi(\boldsymbol{r},t) = \sqrt{\rho(\boldsymbol{r},t)}\exp[i\phi(\boldsymbol{r},t)/\sqrt{2}c_s\xi]$, with $\rho(\boldsymbol{r},t)$ and $\phi(\boldsymbol{r},t)$ being the fluid density and the velocity potential, respectively. Perturbations about a still fluid with uniform density $\rho(\boldsymbol{r},t) \equiv \rho_0$ behave as a dispersive sound with frequency given by the Bogoliubov dispersion relation, $\omega_k = c_s k\sqrt{1+\xi^2k^2/2}$. In the weakly dispersive limit $\xi k \ll 1$, it becomes $\omega_k = c_s k(1+\xi^2k^2/4)$, i.e., the dispersion relation (4) with $a = \xi/2$. In this limit, the three-wave interaction coefficient of Eq. (17) is of the form (5), although some ambiguities and discrepancies in the value of the coefficient V_0 can be found in the previous works [2,11,12,20]. In Sec. I. B of the Supplemental Material [21], we provide the corrected derivation which leads to $V_0 = 3\sqrt{c_s}/4\sqrt{2}$. Substituting this into Eq. (16) we have the following prediction for the prefactor constant,

$$C_1 = \frac{2^{11/4}}{3^{3/4}\pi} \approx 0.94. \tag{18}$$

We simulate Eq. (17) using the standard pseudospectral code FROST [22] in a periodic domain of size L using $N_c^2 = 1024^2$ and $N_c^2 = 512^2$ collocation points, denoted by Run 1 and Run 2, respectively. The nonlinear term is dealiased twice with the 2/3 rule following the scheme introduced in [23] in order to conserve momentum (in addition to the energy and the number of particles) in the ideal case (with $F = D_c = 0$). The Fourier transform of the forcing F obeys the Ornstein-Uhlenbeck process $dF_{\mathbf{k}} = -\alpha F_{\mathbf{k}}dt + f_0 dW_{\mathbf{k}}$, where $W_{\mathbf{k}}$ is the Wiener process. The forcing acts only on wave numbers such that $2\pi \leq kL \leq 3\times 2\pi$. In addition, the condensate amplitude is kept constant during the evolution. We set the initial data with uniform condensate with $|\psi_0|^2 = \rho_0$, the forcing then adds the acoustic disturbances, and we evolve the system until it reaches a steady state. We then perform averages over time. In numerics, we have set $c_s = 1$, $\rho_0 = 1$, and $\xi = 2L/N_c$. For forcing and dissipation we set $\alpha = 1$, $f_0 = 1.25\times 10^{-4}, 5\times 10^{-4}$, and $D_c = 4.1\times 10^{-15}, 2.1\times 10^{-11}$ for Runs 1 and 2, respectively.

In the absence of forcing and dissipation, Eq. (17) conserves the total energy (Hamiltonian) of the system. The energy per unit of mass, written in terms of the hydrodynamic variables, consists of the kinetic, internal, and quantum energies [24]:

$$E = \frac{1}{L^2\rho_0}\int\left[\frac{\rho}{2}(\nabla\phi)^2 + \frac{c_s^2}{2\rho_0}(\rho-\rho_0)^2 + c_s^2\xi^2(\nabla\rho)^2\right]d^2\boldsymbol{r}. \tag{19}$$

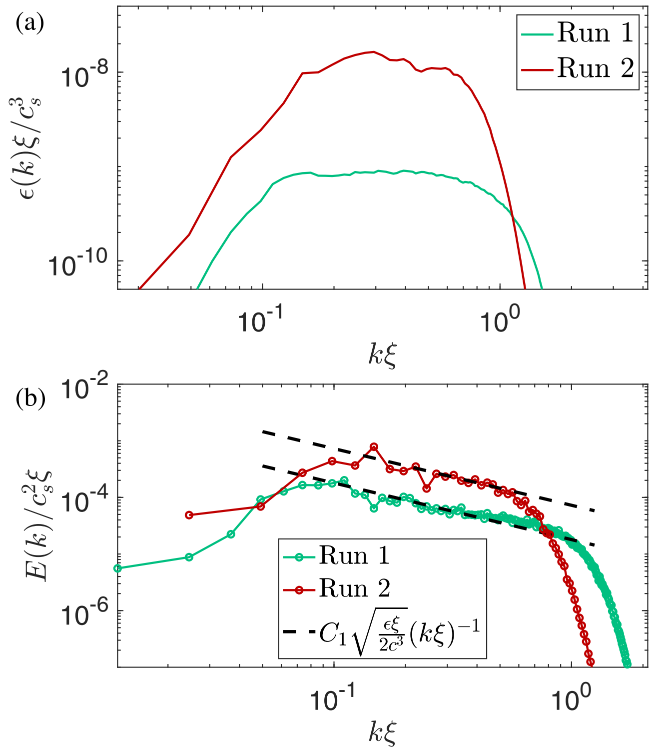

FIG. 2. (a) Energy flux. (b) Energy spectra. Dashed lines correspond to theoretical predictions (9) and (18) using the corresponding flux values in the inertial range.

In the dispersiveless limit ($\xi \to 0$), we retrieve the standard energy for a for a compressible, isentropic, irrotational fluid [25]. The total energy spectrum is computed writing the energy as usual in quantum turbulence [24]. We also calculate the k-space energy flux $\epsilon(k)$ directly using Eq. (17) [see the Eq. (31) of Supplemental Material [21] for exact definitions].

In Fig. 2 we show the fluxes and spectra for Run 1 and Run 2. We see that the fluxes have a pronounced plateau which indicates the presence of an inertial range (free of forcing and dissipation effects). Both runs display a stationary power-law spectrum. Remarkably, both, the power-law exponent and the prefactor C_1 (calculated based on the averaged flux in the inertial range), closely agree with the theoretical predictions (9) and (16).

In Fig. 3 we show the spatiotemporal spectrum for Run 1. We see that this spectrum follows closely the Bogoliubov dispersion law, which indicates that the nonlinearity is sufficiently weak. The ω width of the spectrum at each fixed k represents the nonlinear frequency broadening; we define it as $\gamma_k = [\int_0^\infty(\omega-\omega_k)^2|\hat{\psi}(\omega,k)|^2d\omega/\int_0^\infty|\hat{\psi}(\omega,k)|^2d\omega]^{1/2}$. In Fig. 4 we show the ratios $\gamma_k/\delta\omega_k$, where $\delta\omega = c_s a^2 k^3$ is the dispersive correction. Recall that the WT theory is applicable when $\gamma_k/\delta\omega_k \ll 1$. We see that this quantity is indeed small in the scaling range in Run 1, and only marginally small in a rather narrow range in Run 2. This indicates that the WT theory has a

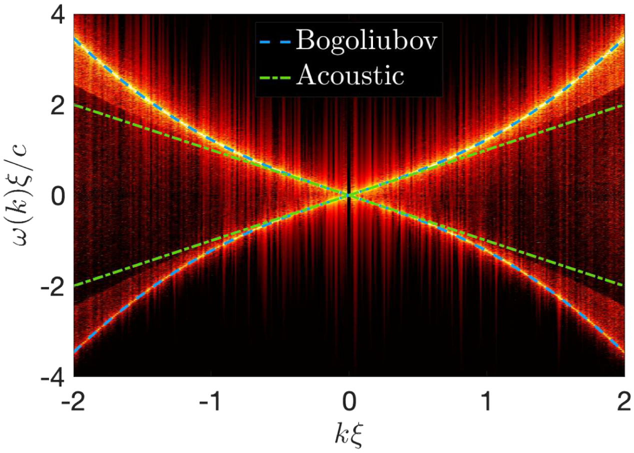

FIG. 3. Spatiotemporal spectrum $S(\omega, k) = |\hat{\psi}(\omega, k)|^2$ normalized by the time-averaged spectra $|\hat{\psi}(t, k)|^2$ of Run 1 (1024^2). The dashed and dot-dashed lines show the Bogoliubov and the pure acoustic dispersion relation.

good predictive power even when the formal applicability condition is on the borderline of validity.

The main result of our Letter is the 1D energy spectrum of 2D weakly dispersive acoustic waves, Eq. (9), found as the unique stationary constant-flux solution of the kinetic equation (3) with convergent collision integral. From the physical view point such a convergence means that the main contribution to the energy balance of waves with wave number k comes from their energy exchange with the "neighboring" waves with wave numbers k' of the order of k. In the language of hydrodynamic turbulence we are dealing here with the step-by-step cascade energy transfer, local in the wave number space. We found the energy flux to be positive, meaning that the energy is transferred from small to large k, i.e., it is a direct energy cascade.

We tested our analytical predictions by numerical simulations of the forced-dissipated GP equation (17) in the presence of a strong condensate. The analytically found spectrum (9) was confirmed by the numerics, including both the power-law exponent and the prefactor C_1 without any adjustable parameter. Such a double validation is a rare success in the theory of wave turbulence, where numerical tests were attempted by numerical simulations for various types of waves but, in most cases, only the spectrum exponent was confirmed. Wave turbulence is therefore a valid and productive approach for describing 2D superfluid BEC turbulence where interacting sound waves represent the principal mechanism of energy dissipation. Since measurements of the spectrum are experimentally accessible in BEC [16,17], our results present verifiable predictions which could guide future experiments. The focus of the present Letter was on the weak turbulence of 2D acoustic waves, and the strong turbulence regimes would be an interesting subject for future studies.

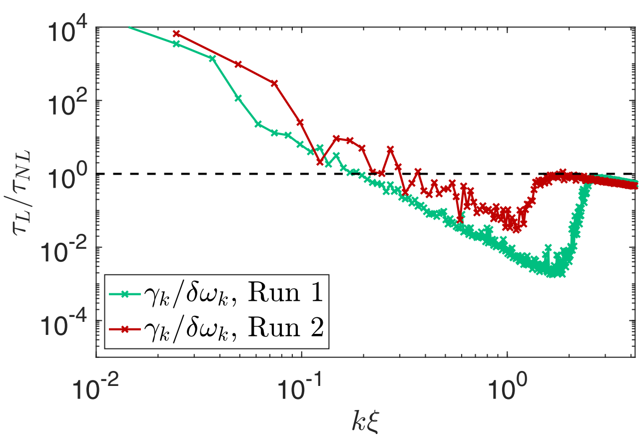

FIG. 4. Linear to nonlinear time ratios for Run 1 (1024^2) and Run 2 (512^3) computed from the spatiotemporal in Fig. 3.

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*krstulovic@oca.eu

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