Quantum-Critical Resistivity of Strange Metals in a Magnetic Field

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Resistivity in the quantum-critical fluctuation region of several metallic compounds such as the cuprates, the heavy fermions, Fe chalogenides and pnictides, Moiré bilayer graphene, and WSe_2 is linear in temperature T as well as in the magnetic field H_z perpendicular to the planes. Scattering of fermions by the fluctuations of a time-reversal odd polar vector field $\mathbf{\Omega}$ has been shown to give a linear in T resistivity and other marginal Fermi-liquid properties. An extension of this theory to an applied magnetic field is presented. A magnetic field is shown to generate a density of vortices in the field $\mathbf{\Omega}$ proportional to H_z. The elastic scattering of fermions from the vortices gives a resistivity linear in H_z with the coefficient varying as the marginal Fermi-liquid susceptibility $\ln(\omega_c/T)$. Quantitative comparison with experiments is presented for cuprates and Moiré bilayer graphene.

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High temperature cuprates [1,2] have a linear in T resistivity for doping in the region above T_c that is bounded by a phase with a "pseudo-gap" on one side and crossover to a Fermi liquid on the other. This and related anomalies [3] in this region suggested a quantum-critical origin for the anomalies [4,5] and the prediction that the pseudogap phase breaks time-reversal and inversion symmetries. Linear in T resistivity and other anomalies, similar to those in the cuprates, are also found in several Fe-based compounds in the fluctuation region of their antiferromagnetic quantum-critical point [6–9], in heavy fermion compounds [10,11], and more recently in twisted bilayer graphene [12,13] and in the twisted bilayer compound WSe_2 [14] with hitherto undiscovered order parameters. An important recent discovery [14–17] is that in all of them the resistivity is linear also in an applied magnetic field $|H|$. The magnitude of the magnetoresistivity is similar to the zero-field resistivity at temperature T for $\mu_B H$ of $O(k_B T)$. Where investigated [18,19], the linear in $|H|$ resistivity is found only for the component H_z applied perpendicularly to the planes.

Three important general points should be noted. First, a transport scattering rate linear in $|H|$ and (nearly) independent of temperature can only be due to elastic scattering of fermions from time-reversal odd axial objects induced by the magnetic field. Second, the fact that only the component of the field orthogonal to the high conducting plane in all these metals is effective excludes magnetic moment due to spins in favor of magnetic moments due to orbital loop currents. Third, the magnitude mentioned above implies that the theory of the linear in $|H|$ resistivity must be closely related to the theory that gives linear in T resistivity.

A theory that gives linear in T resistivity and other anomalies in cuprates rests on the theory of quantum-critical fluctuations [20,21] that are a prelude to a state of loop-current order. The new experiments invite extension of this theory to the effects of a magnetic field. The occurrence of the linear in T and in H resistivity, as well as the associated $T\ln(\omega_c/\pi T)$ entropy in the quantum-critical regions in at least all the other compounds where results are available [7,10], is to be expected if their quantum criticality is described by a model that maps to the quantum-XY model coupled to fermions (QXY-F). The mapping has been shown [22] for the planar ferromagnetic or antiferromagnetic model or an incommensurate Ising model. Here, I will first present a theory for the magnetic field dependence of the resistivity in the cuprate compounds for which more quantitative information is available than the other compounds and briefly comment on the other cases.

Loop-current order in cuprates can be represented as a time-reversal odd polar vector $\mathbf{\Omega}$ on a lattice, sketched in Fig. 1(a). Using conservation laws alone, Else and Senthil [23] have recently shown that, to get resistivity proportional to T for $T \rightarrow 0$ in the pure limit, the critical fluctuations must be of an order parameter of such a symmetry. Such an order parameter has indeed been found to be consistent with experiments using a variety of different techniques [24–28].

The orbital magnetic susceptibility of the model is obtained from the fluctuations already derived in Refs. [20,29–31]. The model at $H = 0$ is specified by the interaction energy of the angles $\theta_{i,\tau}$ of $\mathbf{\Omega}_i$ at neighboring sites, by the kinetic energy due to their angular momentum $\mathbf{L}_{zi}$, and the coupling of spatial and temporal fluctuations in $\theta_{i,\tau}$ to the fermions. The QXY-F model, just as the classical XY model, does not belong to the universality class of the Ginzburg-Landau-Wilson theories and their quantum extensions. The quantum-critical fluctuations are driven

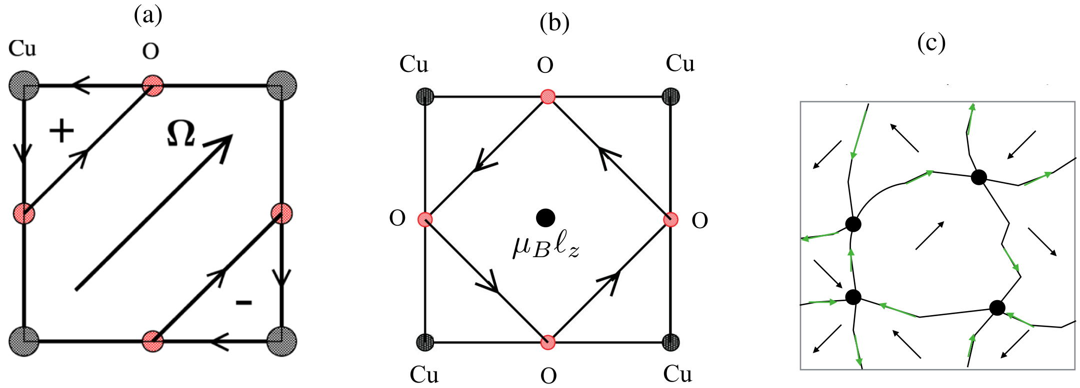

FIG. 1. Representation of the current distribution in the cu-o unit cell for (a) the vector field $\mathbf{\Omega}$, which has one of four possible angles θ_i in the unit cell i; and (b) the angular momentum ℓ_z, which is a generator of rotations of $\mathbf{\Omega}$ and has a magnetization at its core. (c) Represents a fluctuation of $\mathbf{\Omega}$ over regions of many cells. A current represented by the green arrows runs at the boundary between any two orientations of $\mathbf{\Omega}$. At all corners of the variations in $\mathbf{\Omega}$ a vortex or ℓ_z, represented by the black dot, is required to exist. At $H = 0$, the vortices are of equally up and down orientations. But an applied finite H leads to a net orbital angular momentum due to unequal density of vortices of different orientation.

by proliferation of topological defects, 2D spatial vortices, and warps that are spatially local events interacting logarithmically in imaginary time [20]. The critical correlations, $C(\boldsymbol{r},\tau) \equiv \langle e^{-i\theta(\boldsymbol{r},\tau)} e^{i\theta(0,0)} \rangle$, have been obtained by quantum Monte Carlo calculations [29,30] as well as derived by renormalization group [31]. As shown in an appendix in Ref. [32], the orbital magnetic susceptibility $\chi_{LL}(\boldsymbol{r},\tau)$ defined by Eq. (1) is proportional to that of $C(\boldsymbol{r},\tau)$. Near criticality, the (dimensionless) dynamic orbital magnetic susceptibility is

$$\chi_{LL}(\boldsymbol{r},\tau) \equiv \mu_B^2 \langle \boldsymbol{L}_z^+(\boldsymbol{r},\tau) \boldsymbol{L}_z(\boldsymbol{0},0) \rangle = \mu_B^2 \langle L_z^2 \rangle \frac{\tau_c^2}{\tau} e^{-(\tau/\xi_\tau)^{1/2}} \ln\frac{r}{a} e^{-r/\xi_r}. \quad (1)$$

$\mu_B^2 \langle L_z^2 \rangle$ is the expectation value of the square of the magnitude of the orbital magnetic moment per unit-cell volume. We take this to be given by the amplitude of the measured [25] ordered staggered moment per unit cell $(\ell_z \mu_B)^2$. The amplitude $\langle L_z^2 \rangle$ is nearly temperature independent in the region of interest. τ_c is the short time cutoff obtainable from experiments. The spectral function in Eq. (1) is of the form proposed phenomenologically [33] to give the fluctuations of marginal Fermi liquid, rather than the $1/\tau^2$ of the Landau Fermi liquid. In terms of the frequency ω and temperature T,

$$\chi_{LL}(\omega, T) = \frac{\mu_B^2 \langle L_z^2 \rangle}{\omega_c} \left(\ln\left| \frac{\omega_c}{\max(\omega, \pi T)} \right| - i \tanh\frac{\omega}{2T} \right) \quad (2)$$

at criticality. $\omega_c = 1/\tau_c$ is the ultraviolet cutoff. This functional form is also the principal result of theories on interesting models of mathematical interest such as the SYK model [34] and holographic models [35] and of other models [3,36–38]. The magnetic field couples to the angular momentum as $-\mu_B \sum_i \boldsymbol{H} \cdot L_{iz}(\tau)$. In the quantum-critical regime H_z induces a static macroscopic $\langle L_z \rangle$ given by

$$\mu_B \langle L_z \rangle = \chi'_{LL} H_z, \qquad \chi'_{LL}(T) = \frac{\mu_B^2 \ell_z^2}{\omega_c} \log\left(\frac{\omega_c}{\pi T} \right). \quad (3)$$

From the experimental observations [25] that the ordered staggered moment per cell is about $0.1\mu_B$, and $\omega_c \approx 2000$ K [2,32], χ'_{LL} is estimated to be about $10^{-5} \mu_B^2/(\text{Kelvin cell})$. So a magnetic field of 50 Tesla can be estimated to produce a static magnetization $\approx 5 \times 10^{-4} \mu_B$, not including the numerical factor due to the logarithmic temperature dependence in χ_{LL}. An important question in the present context is how such a moment would be distributed. To think of this, it is useful to know the physical description of ℓ_z, the quasiquantized unit of orbital angular momentum in the present problem. A loop current carrying the lattice representation of angular momentum is shown in Fig. 1(b) [39]. It has been shown [32,40,41] to be the generator of rotations of the magnetoelectric vector $\mathbf{\Omega}$ in the plane, from one of its four orientations to the clockwise or anticlockwise orientation:

$$e^{i(\pi/4)\ell_z} |\hat{\mathbf{\Omega}}\rangle = |\hat{\mathbf{\Omega}} + \pi/2\rangle. \quad (4)$$

The pictorial representation in Fig. 1(c) of L_z corresponds to a vortex in the vector field $\mathbf{\Omega}$ with quantized angle but a magnetic moment given by the area and the current carried by the core cell around which the four orientations of $\mathbf{\Omega}$ meet. Over long wavelengths, one may ignore the granularity of the lattice so that L_z is similar to the vortex in more familiar $U(1)$ fields such as superconductors in a magnetic field or superfluids in rotation. Instead of quantization of the magnetic moment in terms of fundamental constants, it is nonuniversal and given by the magnitude of the vectors Ω, which have very weak temperature dependence. From the estimates given above, the density of the moments n_L is about 5×10^{-3}/unit cell for a field of 50 Tesla so that their separation is about 50 unit cells. In an ordered state of $\mathbf{\Omega}$, such moments would crystallize at low enough temperature due to their long-range mutual interactions. But we are considering the region in which they live in a bath of $\mathbf{\Omega}$s quantum fluctuating in time and space. Therefore, such moments would remain disordered at the temperatures of interest and diffuse at a very slow rate because of their enormous effective mass. If the motion of $\langle L_z \rangle$ is very slow compared to the motion of fermions with which they scatter, the scattering should be considered elastic.

The s-wave scattering rate $1/\tau_L$ of such local magnetic field generated by density n_L of the defects can be easily estimated [see Fig. 2(a)]:

$$1/\tau_L = 2\pi n_L (g_0 \mu_B \ell_z)^2 N(0),$$
$$n_L = \frac{\chi'_{LL} H_z}{\mu_B \ell_z} \approx \ell_z \frac{\mu_B H_z}{\omega_c} \ln(\omega_c/\pi T). \tag{5}$$

$N(0)$ is the density of states of fermions at the chemical potential and g_0 is the coupling energy [32] of the fermions to a vortex with orbital moment $\mu_B \ell_z$. This is to be compared with the inelastic scattering of fermions by the fluctuations $\chi''(\omega, T)$ [see Fig. 2(b)]. This is calculated from the analytic continuation of the imaginary part of the self-energy at zero frequency, which has been derived often [33,42,43]:

$$1/\tau(T) = 2UIm\Sigma''(0, T),$$
$$\Sigma(i\omega_n) = g_0^2 \sum_{\omega_m, \boldsymbol{k}} G(\boldsymbol{k}, i\omega_m)\chi(i\omega_n - i\omega_m). \tag{6}$$

U is a dimensionless Umklapp factor, which is necessary for finite resistivity. Recently, in an asymptotically exact theory for resistivity due to fluctuations of the QXY-CF model, it has been shown that U is temperature independent [44]. A way to estimate U is to compare the transport scattering rate with the imaginary part of the self-energy in the direction on the Fermi surface of maximum velocity. This gives U of $O(1)$ [2] for the cuprates where both have been measured. Eq. (6) gives

$$1/\tau(T) \approx \pi U (g_0 \mu_B \ell_z)^2 N(0) \frac{k_B T}{\omega_c}. \tag{7}$$

$1/\tau_L$ and $1/\tau(T)$ are of similar magnitude at $\mu_B H / k_B T$ of $O(1)$ for $\ln(\omega_c/\pi T) \approx 1$. They are similar because the inelastic scattering rate comes from the imaginary part of the same fluctuations whose real part gives n_L to give the elastic scattering rate and the coupling energy to fermions is identical. Specifically, the ratio of the scattering rates is

$$(1/\tau_L) \div [1/\tau(T)] \approx \frac{2\ell_z}{U} \frac{\mu_B H}{k_B T} \ln(\omega_c/\pi T). \tag{8}$$

The result of Eq. (8) is subject to a cutoff at low temperatures if one is not at critical parameters (the critical point is also expected to shift in a magnetic field if the usual magnetic susceptibility of the system is different on the two sides of critical point) and a high temperature cutoff on the scale of the upper cutoff ω_c.

We can compare the result in Eq. (8) quantitatively with experiments. The data for the resistivity in the most extensively investigated case, for a cuprate near criticality, is represented in Ref. [16] by $\rho(T, H) = \alpha k_B T + \beta(T)\mu_B H$. We can write using Eqs. (7) and (8) that $\beta(T) = \alpha(2\ell_z/U)\ln(\omega_c/\pi T)$. $\beta(T)$ from low T to the highest available temperature, 180 K, and a logarithmic fit to it by $0.14 \ln(1500/\pi T)$ are given in Fig. 3. The coefficient 0.14 should be compared with 0.19 that is estimated from parameters above and the value of $\alpha \approx 1.1$ deduced in the experiment [16]. A logarithmic fit appears reasonable for $T \gtrsim 30$ K, below which the data saturates. The parameter ω_c is about 1600 K, which may be compared with the $O(3000)$ K deduced [2] from the fit to the logarithmic

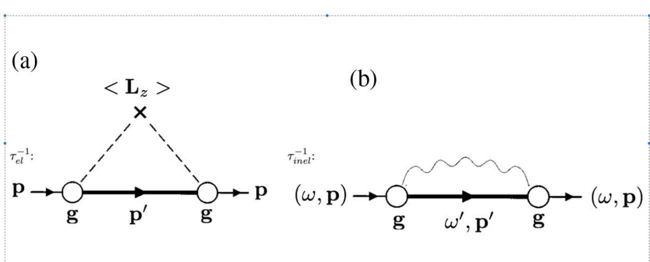

FIG. 2. (a) Elastic scattering of fermions by vortices of angular momentum $\langle L_z \rangle$. (b) Inelastic scattering of fermions by fluctuations $\chi''(\omega, T, \boldsymbol{q})$.

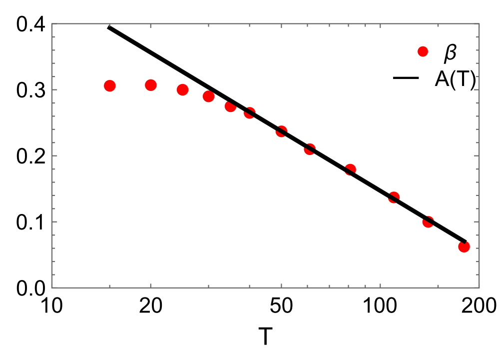

FIG. 3. The temperature dependence of the linear in H resistivity in $La_{2-x}Sr_xCuO_4$ for $x = 0.19$. The data, shown as red dots, are taken from Fig. 1(b) of Ref. [16]. $\beta(T)$ is obtained from the fit to the resistivity (after subtracting a small residual value) $\rho(T, H) - \rho_0 = \alpha T + \beta(T)H$ at $H = 70$ Tesla. $A(T) = 0.14 \ln|\omega_c/\pi T|$, with $\omega_c \approx 1600$ K.

C_v/T measured [45] between 0.3 and 10 K. The peril of deducing a number from a logarithm in a range far above the data should be kept in mind. The data in Fig. 1(a) in [16] shows systematic rounding toward zero below about 30 K even in a field of 70 Tesla. One may be tempted to ascribe it to not being very close to criticality, but a closer look at all the data at various fields suggests a more mundane reason. The data show a large region of rounding from the zero-field transition temperature (≈41 K) toward zero resistivity at low temperatures even in large fields. This is generally the rule in 2D strongly type II superconductors or superconducting films due to an enhanced region of phase fluctuations in a field.

An independent way to test the prediction made here is to see if a direct measurement of magnetization in the range in which the resistivity satisfies Eq. (8) shows the same logarithmic enhancement.

I now briefly discuss the other compounds, beginning with those for which the quantum criticality is that of antiferromagnetism. Significantly, the important critical fluctuations for planar and incommensurate Ising ferromagnets or antiferromagnets (or charge density waves) are of the phase variable given by the XY model [22,46]. It is very interesting to note that the measured spectral functions for the quantum-critical fluctuations for planar antiferromagnetism in $BaFe_{1.85}Co_{0.15}As_2$ [47] or incommensurate antiferromagnetism in the heavy fermion $CeCu_6$ [48,49] are consistent with the product form in momentum and energy [50] as in Eq. (1) for the QXY-SF model.

The data [15] in $BaFe_2(As_{1-x}P_x)_2$ with $T_c \approx 30$ K, which is available only to 60 K with fields up to 59 Tesla, has a severe rounding of resistivity toward zero at low temperatures for fields less than 50 Tesla so that linearity of H above this field is observed only in a narrow range of temperatures. We therefore cannot usefully compare the data in $BaFe_2(As_{1-x}P_x)_2$. The fit of the resistivity data made as $\propto \sqrt{(\mu_B H)^2 + (k_B T)^2}$ earlier [15] is not good under closer examination of the detailed data kindly received from the authors [51]. That fit also does not work for the cuprate or for the twisted bilayer graphene [17], as stated by the authors. However, an H^2 dependence of magnetoresistance at low fields is conventional and well understood and there is no reason why it should be completely absent in the metals under discussion.

The relevant order parameter for twisted bilayer (TB) graphene and TB-WSe_2 is not known yet from experiments, although there are theoretical calculations suggestive of loop-current ordered states [52,53] in TB graphene. TB-WSe_2 is similar except for the large spin-orbit coupling. Their structure has a triangular motif and it is expected that the nearest neighbor repulsion is comparable to the kinetic energy. In this situation, loop-current order is a likely instability [54–56]. It should be ascertained if only the component of the magnetic field perpendicular to the plane is responsible for the resistance linear in the field. If this

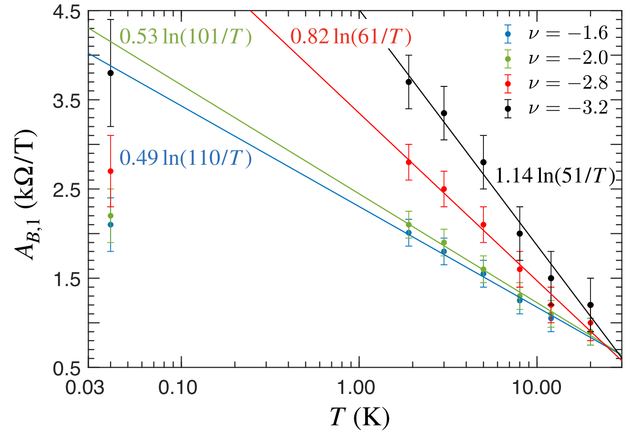

FIG. 4. The temperature dependence of the linear in H resistivity in twisted bilayer graphene. The data is from Ref. [17] and replotted by the authors. $A_{B,1} \equiv \beta(T)$ is obtained just as for the cuprate compound in Fig. 3. The fit to experiments for the coefficient of the resistivity $\propto \mu_B H$ is given for the four band-fillings where the resistivity at zero magnetic field is most nearly linear in T.

holds, more experiments to test the time reversal, inversion, and possible chirality given by loop currents are suggested to decipher their long-range order. The data on WSe_2 is not yet detailed enough to compare with theory for $\beta(T)$, but it is for TB graphene [17]. The data at all band-fillings ν where the resistivity is most nearly linear in T is plotted to obtain $A_{B,1} \equiv \beta(T)$, defined earlier. The coefficient varies from about 0.5 to about 1 for the various band-fillings ν and the cut-off from about 50 K to about 100 K. So we gather that the spontaeously generated magnetic moment per unit-cell in twisted bi-layer graphene is similar to that in the cuprates, but the cut-off ω_c is more than an order of magnitude less. There are no independent numbers from other experiments to compare. But the scale of the fluctuation energies an order of magnitude smaller than the cuprates appears reasonable. The saturation at the lowest point at 40 mK is almost certainly due to rounding of resistivity due to impending superconductivity, while at the highest temperature πT is essentially the upper cutoff and so a saturation is inevitable.

The experiments in a magnetic field test a crucial microscopic aspect underlying the application of the theory of quantum-criticality of the xy model. The success of the results is due to having the kinetic energy operator be a magnetic angular momentum which serves as a generator of rotation in the fluctuations of the order parameter. When the angular momentum is not due to spins as in spin-antiferromagnets, the order parameter must then describe loop-current order. It is already understood that d-wave superconductivity is not possible if the self-energy of the fermions is angle-independent as it is in cuprates without the fermions coupling to the fluctuations of angular momentum [2,32,57]. To conclude, one might also add that the mechanism of superconductivity in all these systems is inevitably related to the fluctuations that give resistivity linear in T and in H.

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