Editors' Suggestion

Bogoliubov-Born-Green-Kirkwood-Yvon Hierarchy and Generalized Hydrodynamics

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We consider fermions defined on a continuous one-dimensional interval and subject to weak repulsive two-body interactions. We show that it is possible to perturbatively construct an extensive number of mutually compatible conserved charges for any interaction potential. However, the contributions to the densities of these charges at second order and higher are generally nonlocal and become spatially localized only if the potential fulfils certain compatibility conditions. We prove that the only solutions to the first of these conditions are the Cheon-Shigehara potential (fermionic dual to the Lieb-Liniger model) and the Calogero-Sutherland potentials. We use our construction to show how generalized hydrodynamics emerges from the Bogoliubov-Born-Green-Kirkwood-Yvon hierarchy, and argue that generalized hydrodynamics in the weak interaction regime is robust under nonintegrable perturbations.

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Finding an efficient description for the nonequilibrium dynamics of quantum many-particle systems has been a key problem in theoretical physics since the birth of quantum mechanics [1]. During the last two decades there has been an upsurge in interest as a result of significant experimental advances [2] and potential technological applications [3]. In spite of remarkable progress in our understanding of out-of-equilibrium quantum matter [4–10], however, an efficient and accurate description of its dynamics remains out of reach.

In essence, what makes this problem so hard is the lack of general methods to tackle it. Exact methods are restricted to a small number of fine-tuned many-body systems [11–15] and there is currently no controlled approximation scheme applicable to generic interacting systems. Numerical methods are typically limited by the accessible system sizes [5,16] or by the class of initial states that can be accommodated [17]. Even in one dimension, where techniques based on matrix product states [18–20] give access to large systems, their applicability is limited to short times by the rapid growth of quantum entanglement. Furthermore, continuum models—which describe many relevant experiments—provide additional obstacles to numerical approaches.

For weakly interacting systems the situation is substantially simpler because, at least in principle, a general description of the dynamics can be attained by using the celebrated Bogoliubov-Born-Green-Kirkwood-Yvon (BBGKY) hierarchy [21–23], which encodes the Heisenberg equations of motion for the reduced density matrices. To illustrate it let us consider for simplicity and concreteness a system of spinless fermions: in this case the reduced density matrices are written as

$$\rho_n(\mathbf{x},\mathbf{y};t) = \mathrm{Tr}[\rho(t)\psi^\dagger_{x_1}\dots\psi^\dagger_{x_n}\psi_{y_n}\dots\psi_{y_1}], \tag{1}$$

where $\psi^\dagger_x$ and ψ_x are fermionic creation and annihilation operators and $\rho(t)$ denotes the density matrix specifying the state of the system at time t. Assuming that the fermions interact via a two-body potential $V(x-y)$ the BBGKY hierarchy takes the form

$$\begin{aligned} &i\partial_t\rho_n(\mathbf{x},\mathbf{y};t) - (H^{(n)}_{\mathbf{x}} - H^{(n)}_{\mathbf{y}})\rho_n(\mathbf{x},\mathbf{y};t) \\ &= \sum_{j=1}^{n}\int dw[V(x_j-w)-V(y_j-w)]\rho_{n+1}(\mathbf{x},w,\mathbf{y},w;t), \end{aligned} \tag{2}$$

where $H^{(n)}_{\mathbf{x}}$ is the first-quantized n-particle Hamiltonian in the position representation. In principle, the system (2) gives a complete account of the nonequilibrium dynamics of many-particle quantum systems. In practice it is necessary to truncate it, which can be justified, e.g., for weak interactions. Retaining only two-particle cumulants gives rise [24] to the ubiquitous quantum Boltzmann equation (QBE) for the Wigner function

$$f_t(x,p) = \int dz e^{-ipz}\mathrm{Tr}[\rho(t)\psi^\dagger_{x+\frac{z}{2}}\psi_{x-\frac{z}{2}}]. \tag{3}$$

The QBE holds in the regime of weak spatial variations and large times, i.e., the Euler scaling limit [24], and under the assumption of local relaxation can be further reduced to a set of hydrodynamic equations, which are obtained from the continuity equations of particle number (or mass), energy, and momentum [22,23].

The situation is very different, and significantly richer, in quantum integrable systems [25,26], which are characterized by having an extensive number of (mutually compatible) conservation laws with densities that are sufficiently local in space. These conservation laws give rise to the existence of stable quasiparticles over macro states at finite energy densities [27]. Integrable models do not thermalize but relax to a much wider class of equilibrium states known as generalized Gibbs ensembles [28,29], which can be fully characterized in terms of their respective quasiparticle density in momentum space $\rho(k)$. In these systems the dynamics of local observables close to local equilibrium is described by GHD [30,31], which can be expressed as an evolution equation for a space and time dependent density $\rho_{x,t}(k)$ of the stable quasiparticles

$$\partial_t \rho_{x,t}(k) + \partial_x [v_{x,t}(k) \rho_{x,t}(k)] = 0. \tag{4}$$

Here $v_{x,t}(k)$ is a (known) quasiparticle velocity that depends on $\rho_{x,t}(\cdot)$. GHD is obtained from the system of continuity equations for the extensive number of conservation laws, postulating local relaxation to an equilibrium state with quasiparticle density $\rho_{x,t}(k)$, and then inferring its evolution equation (4). The GHD equation (4) can be viewed as a kinetic theory governed by a *dissipationless* Boltzmann equation [30] where the velocity is a nonlinear functional of the density itself [32,33]. Hence it suggests the existence of an operator $n(x,k)$ expressed in terms of fermions ψ_x localized near x, whose expectation value is $\rho_{x,t}(k)$, and for which the BBGKY hierarchy would re-organize into a dissipationless QBE. Such an operator, however, is only known in the noninteracting case, where it is given by the Wigner operator [whose expectation value is Eq. (3)].

In this Letter we show that under certain conditions the BBGKY hierarchy can be reformulated as GHD equations. We establish that, surprisingly, one can construct conserved charges for *arbitrary local* but weak interaction potentials by appropriately dressing the noninteracting modes. Crucially, however, these conserved charges are generally nonlocal and acquire good locality properties only for integrable models. This means that even though it is possible to explicitly construct operators fulfilling a dissipationless QBE for generic interacting models, these operators are nonlocal and the equation cannot be used to infer the dynamics of physical observables. In contrast, in the integrable case the evolution equation reduces to the GHD equation for expectation values of local mode occupation numbers. As discussed in the following, this also gives a simple operational criterion to assess the integrability of a given interaction potential.

In the remainder of this Letter we consider a system of interacting fermions on a ring of size L with Hamiltonian $H = H_0 + H_1$

$$H_0 = \sum_p p^2 \psi_p^\dagger \psi_p, \quad H_1 = \frac{1}{L} \sum_{\boldsymbol{p}} V(\boldsymbol{p}) \psi_{p_1}^\dagger \psi_{p_2}^\dagger \psi_{p_3} \psi_{p_4}, \tag{5}$$

where all summations are on "free momenta" $p = 2\pi n/L$ with $n \in \mathbb{Z}$ and where $\psi_p^\dagger$ and ψ_p are fermionic creation and annihilation operators with canonical anticommutation relations $\{\psi_{p_1}^\dagger, \psi_{p_2}^\dagger\} = 0 = \{\psi_{p_1}, \psi_{p_2}\}$ and $\{\psi_{p_1}^\dagger, \psi_{p_2}\} = \delta_{p_1,p_2}$. Here $V(\mathbf{p})$ is a two-particle interaction potential, which by virtue of the anticommutation relations can be cast in the form

$$V(\boldsymbol{p}) = \frac{1}{4} \delta_{p_1+p_2,p_3+p_4} \mathcal{A}_{p_1 p_2} \mathcal{A}_{p_3 p_4} [V(p_1 - p_3)], \tag{6}$$

where $\mathcal{A}_{k_1 \cdots k_n}$ is an operator that acts by antisymmetrizing in $\{k_1, \ldots, k_n\}$. Without loss of generality we take $V(k)$ to be even in k and $V(0) = 0$ [34].

In particular, the choice

$$V(k) = E_a(k) := -\frac{\beta}{a^2} \int dx \frac{\sigma''(x/a)(e^{ikx} - 1)}{x + \beta\sigma(x/a)}, \tag{7}$$

where $\sigma(x)$ is a smooth odd function with $\sigma'(x) \geq 0$, $\sigma'(0) > 0$, $\lim_{x\to\infty} \sigma(x) = 1$, and $\lim_{x\to\infty} x^2 \sigma''(x) = 0$, produces the second-quantized (and regularized) Cheon Shigehara model of Refs. [35,36], i.e., an integrable fermionic dual of the Lieb-Liniger model [37]. We stress that the regularization (7) allows for a perturbative expansion at second order without renormalization.

In order to obtain GHD we follow the logic of Refs. [30,31], but aim to explicitly construct the conserved charges from the BBGKY hierarchy. These are derived by assuming that the potential $V(k)$ can be treated perturbatively. We note that for the potential (7) perturbation theory is believed to have a finite radius of convergence, cf. Refs. [38–40].

Our first step is to find all the conserved charges of Eq. (5). We begin by considering an operator Q and assuming that it has a regular perturbative expansion

$$Q = \sum_{m=0}^{\infty} Q_m, \tag{8}$$

where Q_m is of order m in $V(k)$. Q is conserved if the operators Q_m fulfil

$$[H_0, Q_m] = -[H_1, Q_{m-1}], \tag{9}$$

where we set $Q_{-1} = 0$. In addition, we restrict our discussion to cases where the particle number operator $N = \sum_p \psi_p^\dagger \psi_p$ is part of the tower of conserved charges. This implies that all Q_n are expressed as sums of monomials involving equal numbers of $\psi_p^\dagger$ and ψ_q. It is easy to see that the zeroth order of Eq. (9) is solved by the following number-conserving charges

$$Q_{f;0} = \sum_p f(p)\psi_p^\dagger \psi_p, \tag{10}$$

where $f(p)$ is any function, which we take to be smooth. We note that charges corresponding to different f's commute, but Eq. (10) is clearly not the most general choice for a conserved charge of H_0. Any energy-diagonal operator (i.e., any operator that is diagonal in an eigenbasis of H_0) could be taken as a conserved charge of H_0, with any V-dependent prefactor appearing at higher orders. The densities of such more complicated energy-diagonal operators, however, are generically nonlocal in real space. We therefore restrict our search to zeroth-order charges of the form (10) and leave out energy-diagonal contributions at higher orders. We expect this "minimal ansatz" to be sufficient for the perturbative construction of complete sets of conserved charges in all integrable models of the form (5) featuring a single species of quasiparticles, i.e., cases in which the stable quasiparticles can be thought of as being "adiabatically connected" to the free fermions $\psi_p^\dagger \psi_p$.

Starting from Eq. (10) we can directly use Eq. (9) and our minimal ansatz to recursively generate the higher orders of Q_f. In particular, for $m = 1$ the equation can be solved for any potential giving [41]

$$Q_{f;1} = \frac{1}{L}\sum_{\boldsymbol{k}} g_{f;1}^{(4)}(\boldsymbol{k})\psi_{k_1}^\dagger \psi_{k_2}^\dagger \psi_{k_3}\psi_{k_1+k_2-k_3}, \tag{11}$$

where the function $g_{f;1}^{(4)}(\boldsymbol{k})$ is nonsingular for any choice of $V(k)$, and has the same regularity as $V(k)$: its explicit expression is given in Eq. (SM-14) [41].

In contrast, for $m = 2$ Eq. (9) does not always admit a solution in the framework of our minimal ansatz, i.e., starting from the zeroth order (10) and omitting energy-diagonal contributions at higher orders. This is because $[H_1, Q_{f;1}]$ generically contains an energy-diagonal component and therefore cannot be expressed in the form $[H_0, Q_{f;2}]$. Indeed, the energy-diagonal part of the latter commutator is always zero. In particular, defining $\mathcal{S}_6 \coloneqq \mathcal{M}_6 \backslash \mathcal{N}_6$ with

$$\mathcal{M}_6 \coloneqq \left\{ k_j : \sum_{j=1}^{3} k_j^2 = \sum_{j=4}^{6} k_j^2, \sum_{j=1}^{3} k_j = \sum_{j=4}^{6} k_j \right\}, \tag{12}$$

$$\mathcal{N}_6 \coloneqq \{k_j : \{k_1, k_2, k_3\} = \{k_4, k_5, k_6\}\}, \tag{13}$$

we find the following solvability condition:

Condition 1.—For $Q_{f;1}$ given in Eq. (11), Eq. (9) admits solution for $m = 2$ and all f only if

$$\mathcal{A}_{k_1k_2k_3}\mathcal{A}_{k_4k_5k_6}\left[\frac{V(k_4-k_1)V(k_3-k_5)}{(k_1-k_4)(k_2-k_4)}\right] = 0, \tag{14}$$

for every $\boldsymbol{k} \in \mathcal{S}_6$. This condition is equivalent to the vanishing of the 2nd order contribution to the three-particle S-matrix for non-coinciding sets of incoming and outgoing momenta [44].

If Condition 1 is fulfilled the second order charge can be expressed as [34]

$$Q_{f;2} = \frac{1}{L}\sum_{\boldsymbol{k}} g_{f;2}^{(4)}(\boldsymbol{k})\psi_{k_1}^\dagger \psi_{k_2}^\dagger \psi_{k_3}\psi_{k_1+k_2-k_3} + \frac{1}{L^2}\sum_{\boldsymbol{k}} g_{f;2}^{(6)}(\boldsymbol{k})\psi_{k_1}^\dagger \psi_{k_2}^\dagger \psi_{k_3}^\dagger \psi_{k_4}\psi_{k_5}\psi_{k_1+k_2+k_3-k_4-k_5}, \tag{15}$$

where $\{g_{f;2}^{(n)}(\boldsymbol{k})\}_{n=4,6}$ are regular and their explicit expressions are given in Eqs. (SM-33) and (SM-34) [41].

Proceeding in this way we obtain a set of charges

$$\{Q_f = Q_{f;0} + Q_{f;1} + Q_{f;2},\ \text{any smooth}\, f(k)\}, \tag{16}$$

conserved up to $O(V^3)$. Our construction can be readily extended to higher orders. Crucially the charges (16) are mutually compatible [it is shown in Ref. [34] that they commute up to $O(V^3)$] and for smooth $f(k)$ and $V(k)$ they are quasilocal, i.e., their density is exponentially localized. The latter property can be shown by expressing $Q_{f;m}$ in terms of real space fermions $\psi(x) = \sum_p \psi_p e^{ixp}/\sqrt{L}$. For instance, considering Eq. (11) we have

$$Q_{f;1} = \int d\boldsymbol{x} C_{f;1}(x_1 - x_4, \ldots, x_3 - x_4)\psi_{x_1}^\dagger \ldots \psi_{x_4}, \tag{17}$$

where

$$C_{f;1}(y_1, y_2, y_3) = \frac{1}{L^3}\sum_{\boldsymbol{k}} g_{f;1}^{(4)}(\boldsymbol{k}) e^{i(k_1y_1 + k_2y_2 - k_3y_3)}. \tag{18}$$

In the thermodynamic limit $C_{f;1}(\mathbf{x})$ becomes the Fourier transform of $g_{f;1}^{(4)}(\boldsymbol{k})$, which is smooth for $f(k)$ and $V(k)$ smooth. Therefore, quasilocality is guaranteed by standard Fourier analysis [42]. If $V(k)$ is regular but not smooth, the densities still decay in real space, but generically as power laws.

If Condition 1 is not fulfilled one can construct charges at second order by either: (i) adding appropriate energy-diagonal contributions to $Q_{f,0}$ (or $Q_{f;1}$) to subtract the energy diagonal part of $[H_1, Q_{f,1}]$ [43]; (ii) appropriately deforming the dispersion relation of the free model: $k^2 \mapsto k^2 + \epsilon\eta(k)$, with $\epsilon \ll 1$ and $\eta(\cdot)$ a suitable function. Using this deformed dispersion in Eq. (12) one can ensure that $\mathcal{M}_6$ coincides with $\mathcal{N}_6$ in any finite volume L and hence Condition 1 is always fulfilled for finite L. Both strategies (i) and (ii), however, necessarily produce second-order charges where the coefficient of the six-fermion term [$g_{f;2}^{(6)}(\boldsymbol{k})$ in Eq. (15)] has singularities in momentum space. In real space, these singularities translate into a charge density that does not decay, i.e., the second-order charges are nonlocal.

In summary, restricting to zeroth order charges with a quadratic term, Condition 1 is necessary and sufficient for the system (5) to have a complete set of charges with densities that decay in space at second order. We remark that if one changes the dispersion relation, for instance by considering a system on the lattice, the situation becomes richer and will be discussed in a separate work [44].

Let us now characterize the solutions to Condition 1. We have the following [34]

Property 1.—The only potentials fulfilling Condition 1 and admitting a power-series expansion around 0 are

$$V_{ab}(k) = a(1 - \sqrt{b}k \coth \sqrt{b}k), \qquad a, b \in \mathbb{R}. \quad (19)$$

Restricting to $b > 0$ (we seek potentials that are well defined for all $k \in \mathbb{R}$) we see that Eq. (19) corresponds to the inverse-sinh-squared Calogero-Sutherland potential up to mass and momentum rescaling [26]. In particular we find the two following limiting cases:

$$\lim_{\substack{b\to 0\\ ab=-3\gamma}} V_{ab}(k) = \gamma k^2, \qquad \lim_{\substack{b\to\infty\\ a=-\gamma/\sqrt{b}}} V_{ab}(k) = \gamma|k|. \quad (20)$$

The first is nothing but the Fourier transform of the integrable Cheon-Shigehara potential (7) at order $O(\beta)$ when the regulator is removed, while the second is the inverse-squared Calogero-Sutherland potential [26]. We have conducted a numerical check of Condition 1 for several classes of singular potentials but have failed to find any additional solutions.

We note that to the best of our knowledge, Eqs. (19) and (20) correspond to the only known integrable potentials for Eq. (5) in the thermodynamic limit. Importantly both cases give rise to theories with a single species of quasiparticles (smoothly connected to free fermions for vanishing V). For the k^2 potential one can use a standard result of Fourier analysis [42,45] to construct a complete set of "ultralocal" charges [25] with density supported on a single point. To this end it is sufficient to take the set of charges constructed choosing $f(k) \in \{k^{2n}\}_{n=1}^{\infty}$.

Given the set of conserved charges (16) we can now define the quasiparticle number operator $n(k, x)$. Our starting point are the operatorial densities $q_f(x)$ of the charges (16), which can be chosen with at least a power-law decaying density if Condition 1 is fulfilled. A convenient choice is

$$\begin{aligned} q_f(x) =& \frac{1}{L}\sum_{k_1,k_2} f(k_1) e^{ix(k_2-k_1)} \psi^\dagger_{k_1}\psi_{k_2} \\ &+ \frac{1}{L^2}\sum_{\boldsymbol{k},k_4} [g^{(4)}_{f;1}(\boldsymbol{k}) + g^{(4)}_{f;2}(\boldsymbol{k})] e^{ix(k_4+k_3-k_2-k_1)} \psi^\dagger_{k_1}\cdots\psi_{k_4} \\ &+ \frac{1}{L^3}\sum_{\boldsymbol{k},k_6} g^{(6)}_{f;2}(\boldsymbol{k}) e^{ix(k_6+k_5+k_4-k_3-k_2-k_1)} \psi^\dagger_{k_1}\cdots\psi_{k_6} \quad (21) \end{aligned}$$

corrected by higher orders in V. To construct $n(k, x)$ we consider linear combinations of $\{q_f(x)\}$ with smooth f's and select the contributions of a single noninteracting mode k. The linear combination is then chosen such that we obtain the density associated with $f(p) = \delta_{k,p}$. Since Eq. (21) is linear in f this is always possible if $\{f(k)\}$ is a complete set in $L^2(\mathbb{R})$. Specifically, we propose the following definition

$$n(k, x) := \frac{L}{2\pi}\frac{\partial}{\partial f(k)} q_f(x), \quad (22)$$

where k obeys free quantisation conditions. This operator is characterized by the following properties: (i) The moments of $n(k, x)$ are the densities

$$\frac{2\pi}{L}\sum_k f(k) n(k, x) = q_f(x). \quad (23)$$

As shown below this implies that the thermodynamic limit of the expectation value of $n(k, x)$ on a translationally invariant state gives the density of quasiparticles.

(ii) $n(k, x)$ is conserved, i.e., it fulfils

$$\begin{aligned} &\partial_t n(k, x) + \partial_x j_n(k, x) = 0, \\ &j_n(k, x) := \frac{L}{2\pi}\frac{\partial}{\partial f(k)} j_f(x), \end{aligned} \quad (24)$$

where $j_f(x)$ is the current associated with $q_f(x)$ via the continuity equation [34]

$$\partial_t q_f(x) + \partial_x j_f(x) = 0. \quad (25)$$

(iii) An explicit expression of $n(k, x)$ can be obtained from Eq. (21).

If the moments of $n(k, x)$ are sufficiently local in space [46] then (i) and (ii) allow us to interpret Eq. (24) as an operatorial progenitor of the GHD equation (4). Property (iii) makes the relation between GHD and the BBGKY hierarchy explicit. The expectation value of $n(k, x)$, which fulfils the GHD equation (4), is recovered as a specific sum of cumulants fulfilling the BBGKY hierarchy. The expansion of $\langle n(k, x)\rangle$ up to order $\mathcal{O}(V^n)$ involves cumulants of order $2n + 2$. Therefore, in order to recover the full GHD equation one needs the *entire* BBGKY hierarchy.

Our previous discussion implies that, at second order, $n(k, x)$ is sufficiently local only if $V(\boldsymbol{k})$ is taken to be the Cheon-Shigehara or Calogero-Sutherland potential. Remarkably, however, at first order the moments are quasilocal for *any* smooth potential $V(\boldsymbol{k})$, integrable or not [49]. This suggests that in the weak-interaction regime GHD physics is robust against nonintegrable perturbations. Namely, having a weak nonintegrable potential in Eq. (5) does not preclude the formulation of the BBGKY hierarchy in terms of a GHD equation at first order in perturbation theory, but only at second order. This means that GHD will be applicable on a longer timescale than naively expected and perhaps partially explains the relevance of GHD in modeling actual experiments [51–53].

In order to fully make contact with GHD, the thermodynamic limit of the expectation values of $n(k,x)$ and $j_n(k,x)$ in an energy eigenstate should match the known formulas for thermodynamic Bethe ansatz (TBA) integrable models. Namely, at order m in perturbation theory one should have

$$\langle n(k,x)\rangle|_m = \rho(k) + O(V^{m+1}), \tag{26}$$

$$\langle j_n(k,x)\rangle|_m = v(k)\rho(k) + O(V^{m+1}), \tag{27}$$

where $\rho(k)$ is the quasiparticle density in momentum space and $v(k)$ is the group velocity of stable particle and hole excitations around the energy eigenstate [54]. For a given state both these quantities depend nontrivially on the two body interactions characterizing the integrable model [30,31,55,56]. More precisely, they are determined by two integral equations involving the two-particle "scattering phase shifts" [57]. Upon assuming local equilibration [58], the relations (26) and (27) allow one to go from the operatorial continuity equation (24) to the GHD equation (4).

One can verify that for $m = 1$ Eqs. (26) and (27) are fulfilled for any potential. Namely, using $\langle\psi_p^\dagger\psi_q\rangle \equiv \delta_{p,q}\vartheta(q)$ and $\langle\psi_p^\dagger\psi_q^\dagger\rangle = 0$, we find at first order [43]

$$\langle n(k,x)\rangle = \frac{\vartheta(k)}{2\pi}\left[1 + \int dq K(k-q)\vartheta(q)\right] + O(V^2), \tag{28}$$

$$\langle j_n(k,x)\rangle = \frac{\vartheta(k)}{\pi}\left[k + \int dq K(k-q)q\vartheta(q)\right] + O(V^2), \tag{29}$$

where $K(k) = \partial_k[V(k)/k]$. These agree with the first order expansion of $\rho(k)$ and $v(k)$ in an integrable model with a single species of quasiparticles and a scattering phase shift $V(k)/k$ (perturbatively small). The latter is then interpreted as the scattering phase shift of the effective integrable model describing (5) at first order. In particular, using the potentials (19) and (20) we recover the first order expansion of the known TBA expressions of $\rho(k)$ and $v(k)$ in the Lieb-Liniger and Calogero-Sutherland models [34]. The extension of these results to higher orders in the integrable case will be reported elsewhere [44].

We note that at first order in V the above programme can be generalized to the bosonic case, i.e., when the operators ψ entering the Hamiltonian (5) satisfy canonical commutation relations. We are again able to construct charges for any potential, but now the expectation value of these charges in an eigenstate is in general divergent at first order in V, as reported in Sec. VI of the Supplemental Material [41].

Discussion.—In this Letter we showed how to systematically derive a dissipationless Boltzmann equation for certain "dressed" quasiparticles from the BBGKY hierarchy for weakly interacting many-particle systems and derived an explicit expression for the density of the corresponding mode occupation operator. In order to enable a GHD description this density must have good locality properties, which we find to be the case precisely for the known integrable potentials. This suggests that our "integrability condition" provides an exact characterization of all integrable systems (even those not solvable by Bethe ansatz) with a single species of quasiparticles. Our construction further suggests that in weakly interacting models the GHD description is robust against nonintegrable perturbations, which is relevant for applications of GHD to cold-atom experiments. Our work can be extended in a number of directions. First, the case of integrable models with several species of stable quasiparticles can be analyzed in a similar way, but a number of interesting complications occur [44]. Second, from our "operatorial" GHD equation it should be possible to derive corrections to GHD [58–62] in a systematic fashion. Finally, our approach can be used to study the effects of general weak integrability breaking interactions [63–70]. An important goal is to investigate how, and over what timescales, they render perturbed GHD descriptions invalid.

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