Fractional Integrable Nonlinear Soliton Equations

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Nonlinear integrable equations serve as a foundation for nonlinear dynamics, and fractional equations are well known in anomalous diffusion. We connect these two fields by presenting the discovery of a new class of integrable fractional nonlinear evolution equations describing dispersive transport in fractional media. These equations can be constructed from nonlinear integrable equations using a widely generalizable mathematical process utilizing completeness relations, dispersion relations, and inverse scattering transform techniques. As examples, this general method is used to characterize fractional extensions to two physically relevant, pervasive integrable nonlinear equations: the Korteweg–deVries and nonlinear Schrödinger equations. These equations are shown to predict superdispersive transport of nondissipative solitons in fractional media.

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Fractional calculus is an effective tool when describing physical systems with power law behavior such as in anomalous diffusion, where the mean squared displacement is proportional to t^{α}, $\alpha > 0$ [1–4]. This form of transport has been observed extensively in biology [5–8], amorphous materials [9–11], porous media [12–14], and climate science, [15] among others. Equations in multiscale media can express fractional derivatives in any governing term [16,17], including dispersion, such as that found in the 1D nonlinear Schrödinger equation (NLS) in optics [18–24] and the Korteweg–deVries equation (KdV) in water waves [25]. In the case of integer derivatives, NLS and KdV are famously integrable equations, leading to solitonic solutions and an infinite set of conservation laws [26]. Integrable equations are key signposts in nonlinear dynamics as they provide exactly solvable cases and, moreover, are an essential element of Kolmogorov-Arnold-Moser theory underlying our understanding of chaos. The fundamental solution of 1D dispersive integrable equations is the soliton, a robust nondispersive localized wave. While in the space of possible nonlinear evolution equations integrable cases are extremely rare, they arise frequently in application.

In this Letter, we present a new class of integrable *fractional* nonlinear evolution equations which predict superdispersive transport in fractional media. Fractional media is "rough" or multiscale media that is neither regular nor random; it includes fractals but is more general as it need not be self-similar. We use the fractional NLS (fNLS) and fractional KdV (fKdV) equations as case studies. We show their integrability, demonstrate exact fractional soliton solutions, and make physical predictions about the speed of these localized waves. To date, to our knowledge, no nonlinear fractional evolution equation has been known to be integrable.

The building blocks of our demonstration are three mathematical ingredients. Two are familiar to physicists as they are well-known concepts in physics. They are completeness and the dispersion relations. However, in our case the dispersion relation will use fractional, rather than integer, derivatives. The third building block is the fundamental ingredient of integrability, namely, the inverse scattering transform (IST) well known to researchers in nonlinear dynamics.

Different versions of the fNLS equation have been studied in, e.g., [20,27–29], and soliton type solutions have been found, but unlike the fNLS and fKdV equations that we introduce, none of these are integrable. The fractional operators in the fNLS and fKdV equations are nonlinear generalizations of the Riesz fractional derivative. In fact, the linear limit of the fNLS equation is the well-known fractional Schrödinger equation derived using a Feynman path integral over Lévy flights [30,31]. Fractional equations defined using the Riesz fractional derivative (alternately termed the Riesz transform [32] or fractional Laplacian [33]) are effective tools when describing behavior in complex systems because the Riesz fractional derivative is closely related to non-Gaussian statistics [34]. It has found physical applications in describing movement of water in porous media [35], transport of temperature in fluid dynamics [36], and power law attenuation in materials [37] among many others [38–40].

The KdV and NLS equations arise in many physical problems. The KdV equation is applicable in shallow water waves, internal waves, fluid dynamics, plasma physics, and lattice dynamics among others [25]. Furthermore, KdV is a universally important equation whenever weak dispersion balances weak quadratic nonlinearity; cf. Refs. [18,19]. Similarly, the NLS equation arises in the quasimonochromatic approximation with dispersion balancing weak nonlinearity and occurs widely in physical applications, e.g., water waves, nonlinear optics, spin waves in ferromagnetic films, plasma physics, Bose-Einstein condensates, etc., [18,19,41,42]. The KdV equation was shown to be solvable using the IST and to admit soliton solutions when associated with the linear time-independent Schrödinger equation in Ref. [43]. Then, the NLS equation with decaying data was solved and shown to possess solitons via the IST in Ref. [44]. Soon after, the method was extended to the modified KdV and sine-Gordon equations as well as general classes of equations written in terms of a linearized dispersion relation [19,45]. IST is now a large field; cf. Refs. [18,26,46–48].

Here we show how to extend this formulation to encompass fractional integrable nonlinear evolution equations. As examples of this technique, we show that fKdV and fNLS are solvable by the IST. These are two examples of many possible fractional integrable equations that can be characterized by this method.

The IST and anomalous dispersion relations.—It is well known that linear evolution equations for $q = q(x,t)$ of the form

$$q_t + \gamma(\partial_x) q_x = 0 \tag{1}$$

can be solved by Fourier transforms when $\gamma(\partial_x)$ is a rational function of ∂_x; cf. Ref. [19]. We can do this because the completeness of plane waves gives an integral representation of $\gamma(\partial_x)$. The solution to Eq. (1) is explicitly

$$q(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk \hat{q}(k,0) e^{ikx - ik\gamma(ik)t}, \tag{2}$$

where $\hat{q}(k,0)$ is the Fourier transform of $q(x,t)$ with respect to x evaluated at $t = 0$. However, as Riesz showed [32], the solution (2) makes sense for much more general γ. Specifically, Fourier transforms can be used to solve linear fractional evolution equations, e.g., $\gamma(\partial_x) = |-\partial_x^2|^\epsilon$, with 2ϵ the order of the fractional derivative; we take $0 < \epsilon < 1$ throughout this letter.

Here we show that similar analysis applies to nonlinear evolution equations using the IST. We do this by associating a class of integrable nonlinear equations with a linear scattering problem (ingredient 1, IST) characterizing the fractional equation with an anomalous dispersion relation (ingredient 2, dispersion), and defining the fractional operator associated with this dispersion relation using the completeness of squared eigenfunctions of the scattering equation (ingredient 3, completeness).

We will apply ingredients 1 and 2 to find the fKdV and fNLS equations, and use ingredient 3 to define the fractional operators in these equations. Associated with the nondimensionalized time-independent Schrödinger equation for $v(x,t)$ with potential $q(x,t)$,

$$v_{xx} + [k^2 + q(x,t)]v = 0, \qquad |x| < \infty \tag{3}$$

is the following class of integrable nonlinear equations for $q(x,t)$ [45]:

$$q_t + \gamma(L^A) q_x = 0, \qquad L^A \equiv -\frac{1}{4}\partial_x^2 - q + \frac{1}{2} q_x \int_x^\infty dy, \tag{4}$$

where $\int_x^\infty dy$ operates on the function to which L^A is applied by integrating it. Hence, Eq. (4) can be solved by the IST using Eq. (3). We obtain the fKdV equation by choosing $\gamma(L^A) = -4L^A|4L^A|^\epsilon$; this will be justified shortly.

Similarly, associated with the following 2×2 scattering problem—termed the Ablowitz-Kaup-Newell-Segur (AKNS) system—for the vector-valued function $\mathbf{v}(x,t) = [v_1(x,t), v_2(x,t)]^T$ (T represents transpose)

$$v_x^{(1)} = -ikv^{(1)} + q(x,t)v^{(2)}, \tag{5}$$

$$v_x^{(2)} = ikv^{(2)} + r(x,t)v^{(1)} \tag{6}$$

is the set of integrable nonlinear equations [45]

$$\sigma_3 \partial_t \mathbf{u} + 2A_0(\mathbf{L}^A)\mathbf{u} = 0, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{7}$$

where $\mathbf{u} = (r,q)^T$ and the operator

$$\mathbf{L}^A \equiv \frac{1}{2i}\begin{pmatrix} \partial_x - 2rI\_q & 2rI\_r \\ -2qI\_q & -\partial_x + 2qI\_r \end{pmatrix} \tag{8}$$

with $I_- = \int_{-\infty}^x dy$. Note that I_- operates both on the function immediately to its right and the functions to which $\mathbf{L}^A$ is applied. Taking $r = \mp q^*$, $*$ the complex conjugate, and $A_0(\mathbf{L}^A) = 2i(\mathbf{L}^A)^2|2\mathbf{L}^A|^\epsilon$, we find fNLS to be the second component of Eq. (7).

These definitions are justified when we note that $\gamma(L^A)$ and $A_0(\mathbf{L}^A)$ can be related to the dispersion relation of the linearization of Eqs. (4) and (7). Specifically, if we put $q = e^{i(kx - w(k)t)}$ into the linearizations of Eqs. (4) and (7), we have

$$\gamma(k^2) = \frac{w_K(2k)}{2k}, \qquad A_0(k) = -\frac{i}{2} w_S(-2k), \tag{9}$$

where w_K is the dispersion relation for the linear fKdV equation, and w_S is the same for the linear fractional Schrödinger equation. Therefore, $\gamma(L^A)$ and $A_0(\mathbf{L}^A)$ for fKdV and fNLS are generated from the dispersion relations for linear fKdV and the linear fractional Schrödinger equation. These equations are, naturally,

$$q_t + |-\partial_x^2|^\epsilon q_{xxx} = 0, \qquad iq_t = |-\partial_x^2|^{\epsilon/2} q_{xx}, \tag{10}$$

where $|-\partial_x^2|^\epsilon$ is the Riesz fractional derivative. So, the corresponding dispersion relations are $w_K(k) = -k^3|k|^{2\epsilon}$ and $w_S(k) = -k^2|k|^\epsilon$, which lead to the aforementioned definitions of $\gamma(L^A)$ and $A_0(\mathbf{L}^A)$.

Spectral definitions of fKdV and fNLS by completeness.—To define the fKdV and fNLS equations, we need to determine what operating on a function with $\gamma(L^A)$ or $A_0(\mathbf{L}^A)$ means. We do this using ingredient 3, completeness of the associated linear scattering system.

In Ref. [45] it was shown that the eigenfunctions of L^A are any of the three functions $\{\partial_x\varphi^2, \partial_x\psi^2, \partial_x(\varphi\psi)\}$ which we represent generically as Ψ^A, each with eigenvalue $\lambda = k^2$. Here, ψ and φ solve the time-independent Schrödinger equation (3) subject to appropriate asymptotic boundary conditions at $x = \pm\infty$. Furthermore, the eigenfunctions of $\mathbf{L}^A$ are $\mathbf{\Psi}^A$ and $\bar{\mathbf{\Psi}}^A$ each with eigenvalue $\lambda = k$. These may be written in terms of solutions to Eqs. (5) and (6) (see Supplemental Material [49]).

Starting from $\gamma(L^A)$ and $A_0(\mathbf{L}^A)$ operating on Ψ^A and $\mathbf{\Psi}^A$, we can write

$$\gamma(L^A)\Psi^A = \gamma(k^2)\Psi^A, \tag{11}$$

$$A_0(\mathbf{L}^A)\mathbf{\Psi}^A = A_0(k)\mathbf{\Psi}^A. \tag{12}$$

To extend this to $\gamma(L^A)$ and $A_0(\mathbf{L}^A)$ operating on any function, we need to be able to express any function in terms of Ψ^A and $\mathbf{\Psi}^A$; i.e., we need a completeness relation for each set of eigenfunctions.

In Ref. [50] it was shown that the eigenfunctions Ψ^A are complete. Assuming $q(x,t)$ is sufficiently decaying and smooth in x, an arbitrary, and sufficiently regular, function $h(x)$ may be expanded in terms of the eigenfunctions Ψ^A as

$$h(x) = \int_{\Gamma_\infty} dk \frac{\tau^2(k)}{4\pi i k} \int_{-\infty}^{\infty} dy\, G(x,y,k)h(y), \tag{13}$$

where time is suppressed and $\Gamma_\infty = \lim_{R\to\infty}\Gamma_R$ with Γ_R the semicircular contour in the upper half plane evaluated from $k = -R$ to $k = R$. τ is the transmission coefficient defined by the relation $\varphi(x,k)\tau(k) = \psi(x,-k) + \rho(k)\psi(x,k)$, ρ is the reflection coefficient, and

$$G(x,y,k) = \partial_x(\psi^2(x,k)\varphi^2(y,k) - \varphi^2(x,k)\psi^2(y,k)). \tag{14}$$

This completeness relation reduces to Fourier completeness in the linear limit. From Eqs. (11) and (13), the operation of $\gamma(L^A)$ on a sufficiently smooth and decaying function h follows as

$$\gamma(L^A)h(x) = \int_{\Gamma_\infty} dk\gamma(k^2)\frac{\tau^2(k)}{4\pi i k}\int_{-\infty}^{\infty} dy G(x,y,k)h(y). \tag{15}$$

Hence, Eqs. (13)–(15) provide an explicit representation of fKdV, i.e., Eq. (4) with $\gamma(L^A) = -4L^A|4L^A|^\epsilon$, which may be written as

$$q_t + \int_{\Gamma_\infty} dk|4k^2|^\epsilon \frac{\tau^2(k)}{4\pi i k}\int_{-\infty}^{\infty} dy G(x,y,k)(6qq_y + q_{yyy}) = 0. \tag{16}$$

Notice that Eq. (16) is in nondimensional coordinates x and t. In the linear limit $q \to 0$, we have $\gamma(L^A) \to \gamma(-\partial_x^2/4)$. So, for fKdV, $\gamma(L^A) \to -\partial_x^2|-\partial_x^2|^\epsilon$, which is the Riesz fractional derivative. If we then set $\epsilon = 0$, we recover the KdV equation:

$$q_t + 6qq_x + q_{xxx} = 0. \tag{17}$$

We note that $\tau(k,t)$ has a finite number of simple poles along the imaginary axis denoted $k_j = i\kappa_j$ for $j = 1, 2, \ldots, J$, so the above representation can be evaluated by contour integration (see Supplemental Material [49]).

Similarly, the eigenfunctions $\mathbf{\Psi}^A$ are also complete [51]. Thus, we can write the operation of $A_0(\mathbf{L}^A)$ on a sufficiently smooth and decaying vector-valued function $\mathbf{h}(x) = [h_1(x), h_2(x)]^T$ as

$$A_0(\mathbf{L}^A)\mathbf{h}(x) = \sum_{n=1}^{2}\int_{\Gamma_\infty^{(n)}} dk A_0(k) f_n(k)\int_{-\infty}^{\infty} dy \mathbf{G}_n(x,y,k)\mathbf{h}(y),$$
$$\mathbf{G}_1(x,y,k) = \mathbf{\Psi}^A(x,k)\mathbf{\Psi}(y,k)^T, \quad f_1(k) = -\tau^2(k)/\pi,$$
$$\mathbf{G}_2(x,y,k) = \bar{\mathbf{\Psi}}^A(x,k)\bar{\mathbf{\Psi}}(y,k)^T, \quad f_2(k) = \bar{\tau}^2(k)/\pi, \tag{18}$$

where $\Gamma_R^{(1)}$ ($\Gamma_R^{(2)}$) is the semicircular contour in the upper (lower) half plane evaluated from $-R$ to $+R$; $\mathbf{\Psi}(x,k)$, $\bar{\mathbf{\Psi}}(x,k)$ are eigenfunctions of $\mathbf{L}$; $\mathbf{\Psi}^A(x,k)$, $\bar{\mathbf{\Psi}}^A(x,k)$ are eigenfunctions of $\mathbf{L}^A$; and $\tau(k)$, $\bar{\tau}(k)$ are transmission coefficients defined similarly to fKdV. Notice that $\mathbf{G}_n$ are 2×2 matrices (see Supplemental Material [49]).

Thus, Eq. (18) gives a representation for the fNLS equation (7) with $A_0(\mathbf{L}^A) = 2i(\mathbf{L}^A)^2|2\mathbf{L}^A|^\epsilon$ and $r = \mp q^*$; see the Supplemental Material [49]. In the linear limit, fNLS is represented in terms of the Riesz fractional derivative, and for $\epsilon = 0$ we recover NLS:

$$iq_t = q_{xx} \pm 2q^2q^*. \tag{19}$$

With explicit expressions for $\gamma(L^A)$ and $A_0(\mathbf{L}^A)$ in Eqs. (15) and (18), the fKdV and fNLS equations are characterized. Further, because these equations are inside of the time-independent Schrödinger and AKNS classes of integrable nonlinear equations in Eqs. (4) and (7), fKdV and fNLS are solvable by the IST.

Soliton solutions of fKdV and fNLS.—Given an initial state $q(x,0)$ with sufficient smoothness and decay, we can solve fKdV and fNLS, i.e., obtain $q(x,t)$, using the IST. To do this, we first map the initial state into scattering space, evolve the resulting scattering data in time, and reconstruct the solution in physical space from these data. It turns out that solving fKdV and fNLS is remarkably similar to solving KdV and NLS.

We note that, given the explicit representation of fKdV in Eq. (16), and fNLS in the Supplemental Material [49], these equations can also be solved numerically in discrete time by finding the kernels $G/\mathbf{G}_j$ and evaluating the integrals with respect to y and k at each time step.

The fractional soliton solutions of fKdV and fNLS are given in Eqs. (20) and (21). These correspond to bound states of the Schrödinger and AKNS scattering problems with one complex eigenvalue $k_K = i\kappa$ and $k_S = \xi + i\eta$, respectively,

$$q_K(x,t) = 2\kappa^2 \mathrm{sech}^2(\kappa[(x-x_1) - (4\kappa^2)^{1+\epsilon}t]) \quad (20)$$

$$q_S(x,t) = 2\eta e^{-2i\xi x + 4i(\xi^2-\eta^2)|2k_S|^\epsilon t} \mathrm{sech}(z_\epsilon(x,t)), \quad (21)$$

where $z_\epsilon(x,t) = 2\eta(x - x_0 - 4\xi|2k_S|^\epsilon t)$ and x_0, x_1 can be characterized in terms of scattering data.

It can also be shown that the fractional solitons solve their respective equations by evaluating $\gamma(L^A)\partial_x q_K$ and $A_0(\mathbf{L}^A)\partial_x q_S$ using contour integration methods (this computation for the fKdV equation is given in the Supplemental Material [49].) Further, higher order solitons can be calculated and their interactions are elastic.

Physical predictions.—The fKdV and the fNLS equations describe the transport of fluid and photons in multiscale fluid channels and laser fiberoptic systems, respectively. The multiscale characteristic of these materials represents a certain "roughness" which is averaged over in fKdV and fNLS. The solitonic solutions of these equations describe how localized waves of fluid or probability are transported in such systems. Both fKdV and fNLS predict solitons with anomalous motion, that is, superdispersive transport where speeds are larger than expected from regular or ordered systems (note that subdispersive transport can also be realized by modifying the dispersion relation). Specifically, the group velocity of fKdV and fNLS and the phase velocity of fNLS are given by

$$v_K(\epsilon,\kappa) = (4\kappa^2)^{1+\epsilon}, \quad (22)$$

$$v_S(\xi,\eta) = 2^{2+\epsilon}\xi(\xi^2+\eta^2)^{\epsilon/2}, \quad (23)$$

$$v_\theta(\xi,\eta) = 2^{1+\epsilon}(\xi^2-\eta^2)(\xi^2+\eta^2)^{\epsilon/2}/\xi. \quad (24)$$

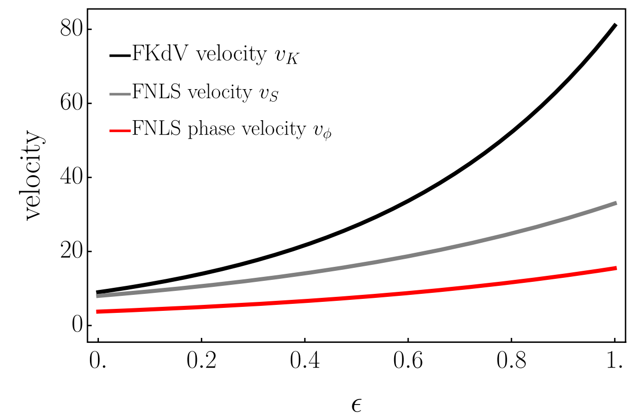

FIG. 1. Localized waves predicted by the fKdV and fNLS equations (22)–(24) show superdispersive transport as their velocity increases as ϵ increases from 0 to 1. Like anomalous diffusion where the mean squared displacement is proportional to t^α, the velocity in anomalous dispersion is proportional to A^ϵ, where A is the amplitude of the wave. The parameter values used are $\kappa = 3/2$, $\xi = 2$, and $\eta = 1/2$.

In a wave tank of height 5 cm we expect solitons with amplitude and KdV speed around 2/3 cm and 0.3 cm/s, respectively. One can similarly associate physical values to solitons in fiberoptics [52], spin waves in ferromagnetic films [53], Bose-Einstein condensates [54], or any of the many other contexts in which NLS is applicable.

Figure 1 shows the velocities in Eqs. (22)–(24) as they interpolate between KdV (NLS) for $\epsilon = 0$ and $\epsilon = 1$. Notice that fKdV and fNLS predict a power law relationship between the amplitude of the wave κ^2 and η, respectively, and the speed of the wave characterized by ϵ. Experimentally verifying these relations relies on comparing the amplitude of water waves and the amplitude and phase of laser pulses in optical fibers to their speed in multiscale media.

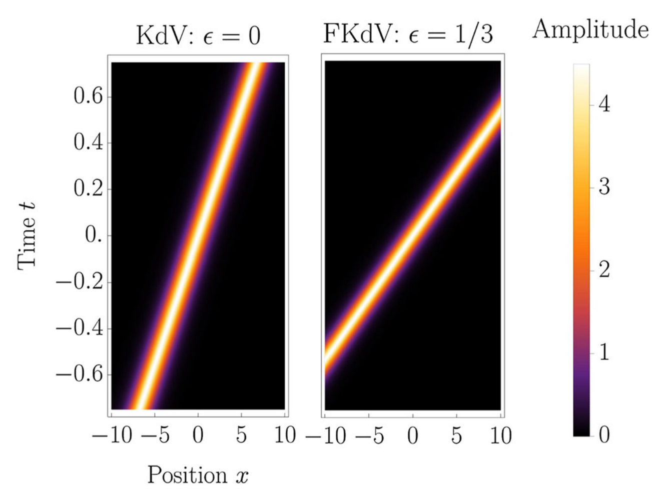

FIG. 2. Note that soliton solutions to the fKdV equation propagate without dissipating or spreading out. The parameter values used are $\kappa = 3/2$ and $x_0 = 0$.

Importantly, the physical properties of fractional solitons, besides the change in velocity described by Eqs. (22)–(24), are identical to regular ones. From Fig. 2, fractional solitons propagate without dissipating or spreading out. An open question is to compare the solitons predicted by fKdV and fNLS to solitary waves predicted by other, nonintegrable versions of these equations. This could be done by studying how the velocity of each equation varies with the fractional parameter ϵ and whether soliton-soliton interactions are elastic or inelastic and what the predicted phase shifts are.

Conclusion.—We demonstrated a new class of integrable equations, namely, 1D fractional integrable nonlinear evolution equations derivable from a general method. As ubiquitous examples of this class, we presented integrability and solitonic solutions of the fractional nonlinear Schrödinger and Korteweg–deVries equations. We demonstrated the three basic mathematical ingredients of our procedure: completeness, dispersion relations, and inverse scattering transform techniques. We also gave fractional soliton solutions to these equations and demonstrated superdispersive transport as a physical implication of the equations. Such fractional equations model multiscale materials and open new directions in integrable nonlinear dynamics for such systems, both artificial and naturally occurring. Our method provides a context for the discovery and understanding of 1D fractional nonlinear evolution equations generally, with integrability acting as a key signpost for fractional nonlinear dynamics.

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[1] M. F. Shlesinger, B. J. West, and J. Klafter, Lévy Dynamics of Enhanced Diffusion: Application to Turbulence, Phys. Rev. Lett. **58**, 1100 (1987).

[2] R. Metzler and J. Klafter, The random walk's guide to anomalous diffusion: A fractional dynamics approach, Phys. Rep. **339**, 1 (2000).

[3] B. J. West, P. Grigolini, R. Metzler, and T. F. Nonnenmacher, Fractional diffusion and Lévy stable processes, Phys. Rev. E **55**, 99 (1997).

[4] W. Wang, A. G. Cherstvy, A. V. Chechkin, S. Thapa, F. Seno, X. Liu, and R. Metzler, Fractional Brownian motion with random diffusivity: Emerging residual nonergodicity below the correlation time, J. Phys. A **53**, 474001 (2020).

[5] M. J. Saxton, A biological interpretation of transient anomalous subdiffusion. I. Qualitative model, Biophys. J. **92**, 1178 (2007).

[6] I. Bronstein, Y. Israel, E. Kepten, S. Mai, Y. Shav-Tal, E. Barkai, and Y. Garini, Transient Anomalous Diffusion of Telomeres in the Nucleus of Mammalian Cells, Phys. Rev. Lett. **103**, 018102 (2009).

[7] A. V. Weigel, B. Simon, M. M. Tamkun, and D. Krapf, Ergodic and nonergodic processes coexist in the plasma membrane as observed by single-molecule tracking, Proc. Natl. Acad. Sci. U.S.A. **108**, 6438 (2011).

[8] B. M. Regner, D. Vučinić, C. Domnisoru, T. M. Bartol, M. W. Hetzer, D. M. Tartakovsky, and T. J. Sejnowski, Anomalous diffusion of single particles in cytoplasm, Biophys. J. **104**, 1652 (2013).

[9] H. Scher and E. W. Montroll, Anomalous transit-time dispersion in amorphous solids, Phys. Rev. B **12**, 2455 (1975).

[10] G. Pfister and H. Scher, Time-dependent electrical transport in amorphous solids: As_2Se_3, Phys. Rev. B **15**, 2062 (1977).

[11] Q. Gu, E. A. Schiff, S. Grebner, F. Wang, and R. Schwarz, Non-Gaussian Transport Measurements and the Einstein Relation in Amorphous Silicon, Phys. Rev. Lett. **76**, 3196 (1996).

[12] D. A. Benson, S. W. Wheatcraft, and M. M. Meerschaert, Application of a fractional advection-dispersion equation, Water Resour. Res. **36**, 1403 (2000).

[13] D. A. Benson, R. Schumer, M. M. Meerschaert, and S. W. Wheatcraft, Fractional dispersion, Lévy motion, and the made tracer tests, Transp. Porous Media **42**, 211 (2001).

[14] M. M. Meerschaert, Y. Zhang, and B. Baeumer, Tempered anomalous diffusion in heterogeneous systems, Geophys. Res. Lett. **35**, L17403 (2008).

[15] E. Koscielny-Bunde, A. Bunde, S. Havlin, H. E. Roman, Y. Goldreich, and H.-J. Schellnhuber, Indication of a Universal Persistence Law Governing Atmospheric Variability, Phys. Rev. Lett. **81**, 729 (1998).

[16] B. J. West, Colloquium: Fractional calculus view of complexity: A tutorial, Rev. Mod. Phys. **86**, 1169 (2014).

[17] W. P. Zhong, M. R. Belić, B. A. Malomed, Y. Zhang, and T. Huang, Spatiotemporal accessible solitons in fractional dimensions, Phys. Rev. E **94**, 012216 (2016).

[18] M. J. Ablowitz and H. Segur, *Solitons and the Inverse Scattering Transform* (SIAM, Philadelphia, 1981).

[19] M. J. Ablowitz, *Nonlinear Dispersive Waves: Asymptotic Analysis and Solitons* (Cambridge University Press, Cambridge, England, 2011), Vol. 47.

[20] B. A. Malomed, Optical solitons and vortices in fractional media: A mini-review of recent results, in *Photonics* (Multidisciplinary Digital Publishing Institute, Basel, 2021), Vol. 8, p. 353.

[21] S. Longhi, Fractional Schrödinger equation in optics, Opt. Lett. **40**, 1117 (2015).

[22] P. Li, B. A. Malomed, and D. Mihalache, Metastable soliton necklaces supported by fractional diffraction and competing nonlinearities, Opt. Express **28**, 34472 (2020).

[23] L. Zeng, B. A. Malomed, D. Mihalache, Y. Cai, X. Lu, Q. Zhu, and J. Li, Bubbles and *W*-shaped solitons in Kerr media with fractional diffraction, Nonlinear Dyn. **104**, 4253 (2021).

[24] S. He, B. A. Malomed, D. Mihalache, X. Peng, X. Yu, Y. He, and D. Deng, Propagation dynamics of abruptly autofocusing circular airy Gaussian vortex beams in the fractional Schrödinger equation, Chaos Solitons Fractals **142**, 110470 (2021).

[25] D. J. Korteweg and G. De Vries, Xli. On the change of form of long waves advancing in a rectangular canal, and on a

new type of long stationary waves, Philos. Mag. **240**, 422 (1895).

[26] M. Ablowitz, P. Clarkson, and P. A. Clarkson, *Solitons, Nonlinear Evolution Equations and Inverse Scattering* (Cambridge University Press, Cambridge, England, 1991), Vol. 149.

[27] Y. Qiu, B. A. Malomed, D. Mihalache, X. Zhu, X. Peng, and Y. He, Stabilization of single-and multi-peak solitons in the fractional nonlinear Schrödinger equation with a trapping potential, Chaos Solitons Fractals **140**, 110222 (2020).

[28] P. Li, B. A. Malomed, and D. Mihalache, Symmetry-breaking bifurcations and ghost states in the fractional nonlinear Schrödinger equation with a PT-symmetric potential, arXiv:2106.05446.

[29] U. Al Khawaja, M. Al-Refai, G. Shchedrin, and L. D. Carr, High-accuracy power series solutions with arbitrarily large radius of convergence for the fractional nonlinear Schrödinger-type equations, J. Phys. **51**, 235201 (2018).

[30] N. Laskin, Fractional quantum mechanics and Lévy path integrals, Phys. Lett. A **268**, 298 (2000).

[31] N. Laskin, *Fractional Quantum Mechanics* (World Scientific, Singapore, 2018).

[32] M. Riesz, L'intégrale de riemann-liouville et le probléme de cauchy, Acta Math. **81**, 1 (1949).

[33] A. Lischke, G. Pang, M. Gulian, F. Song, C. Glusa, X. Zheng, Z. Mao, W. Cai, M. M. Meerschaert, M. Ainsworth, and G. E. Karniadakis, What is the fractional Laplacian? A comparative review with new results, J. Comput. Phys. **404**, 109009 (2020).

[34] M. M. Meerschaert and A. Sikorskii, *Stochastic Models for Fractional Calculus* (De Gruyter, Berlin, 2011).

[35] A. de Pablo, F. Quirós, A. Rodríguez, and J. L. Vázquez, A fractional porous medium equation, Adv. Math. **226**, 1378 (2011).

[36] P. Constantin and J. Wu, Behavior of solutions of 2D quasi-geostrophic equations, SIAM J. Math. Anal. **30**, 937 (1999).

[37] S. Holm, *Waves with Power-Law Attenuation* (Springer, New York, 2019).

[38] C. Pozrikidis, *The Fractional Laplacian* (CRC Press, Boca Raton, 2018).

[39] C. Bucur and E. Valdinoci, *Nonlocal Diffusion and Applications*, Lecture Notes of the Unione Matematica Italiana (Springer, New York, 2016).

[40] J. A. Carrillo, M. D. P. Manresa, A. Figalli, G. Mingione, and J. L. Vázquez, *Nonlocal and Nonlinear Diffusions and Interactions: New Methods and Directions* (Springer, New York, 2017).

[41] J. C. Bronski, L. D. Carr, B. Deconinck, and J. N. Kutz, Bose-Einstein Condensates in Standing Waves: The Cubic Nonlinear Schrödinger Equation with a Periodic Potential, Phys. Rev. Lett. **86**, 1402 (2001).

[42] A. Boardman, Q. Wang, S. Nikitov, J. Shen, W. Chen, D. Mills, and J. Bao, Nonlinear magnetostatic surface waves in ferromagnetic films, IEEE Trans. Magn. **30**, 14 (1994).

[43] C. S. Gardner, J. M. Greene, M. D. Kruskal, and R. M. Miura, Method for Solving the Korteweg-devries Equation, Phys. Rev. Lett. **19**, 1095 (1967).

[44] A. Shabat and V. Zakharov, Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media, Sov. Phys. JETP **34**, 62 (1972).

[45] M. J. Ablowitz, D. J. Kaup, A. C. Newell, and H. Segur, The inverse scattering transform-Fourier analysis for nonlinear problems, Stud. Appl. Math. **53**, 249 (1974).

[46] F. Calogero and A. Degasperis, *Spectral Transform and Solitons* (Elsevier, New York, 2011).

[47] S. Novikov, S. V. Manakov, L. Pitaevskii, and V. E. Zakharov, *Theory of Solitons: The Inverse Scattering Method* (Springer Science & Business Media, New York, 1984).

[48] M. Remoissenet, *Waves Called Solitons: Concepts and Experiments* (Springer Science & Business Media, New York, 2013).

[49] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.128.184101 for detailed description of scattering theory for fKdV and fNLS, an alternate representation of the fKdV operator, an explicit form of fNLS, and verification of the fKdV soliton solution.

[50] R. L. Sachs, Completeness of derivatives of squared Schrödinger eigenfunctions and explicit solutions of the linearized KdV equation, SIAM J. Math. Anal. **14**, 674 (1983).

[51] D. Kaup, Closure of the squared Zakharov-Shabat eigenstates, J. Math. Anal. Appl. **54**, 849 (1976).

[52] L. F. Mollenauer, R. H. Stolen, and J. P. Gordon, Experimental Observation of Picosecond Pulse Narrowing and Solitons in Optical Fibers, Phys. Rev. Lett. **45**, 1095 (1980).

[53] A. B. Ustinov, B. A. Kalinikos, V. E. Demidov, and S. O. Demokritov, Formation of gap solitons in ferromagnetic films with a periodic metal grating, Phys. Rev. B **81**, 180406(R) (2010).

[54] K. E. Strecker, G. B. Partridge, A. G. Truscott, and R. G. Hulet, Bright matter wave solitons in Bose-Einstein condensates, New J. Phys. **5**, 73 (2003).