Optimal Gain Sensing of Quantum-Limited Phase-Insensitive Amplifiers

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Phase-insensitive optical amplifiers uniformly amplify each quadrature of an input field and are of both fundamental and technological importance. We find the quantum limit on the precision of estimating the gain of a quantum-limited phase-insensitive amplifier using a multimode probe that may also be entangled with an ancilla system. In stark contrast to the sensing of loss parameters, the average photon number N and number of input modes M of the probe are found to be equivalent and interchangeable resources for optimal gain sensing. All pure-state probes whose reduced state on the input modes to the amplifier is diagonal in the multimode number basis are proven to be quantum optimal under the same gain-independent measurement. We compare the best precision achievable using classical probes to the performance of an explicit photon-counting-based estimator on quantum probes and show that an advantage exists even for single-photon probes and inefficient photodetection. A closed-form expression for the energy-constrained Bures distance between two product amplifier channels is also derived.

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Phase-insensitive amplifiers coherently and uniformly amplify every quadrature amplitude of an input electromagnetic field. The prototypical example of such an amplifier is a laser gain medium with population inversion between the active levels. Besides being a key component of lasers, phase-insensitive amplifiers (e.g., erbium-doped fiber amplifiers) are widely deployed in today's optical communication networks for restoring signal amplitudes and to offset detection noise [1]. Many physical mechanisms leading to phase-insensitive amplification are known in diverse platforms (see, e.g., Refs. [2–5]), but they are all constrained by the unitarity of quantum dynamics to add a gain-dependent excess noise [2,6,7] that is minimized when the effective population in the active levels of the gain medium is completely inverted [3,8].

Such minimum-noise phase-insensitive amplifiers—hereafter called quantum-limited amplifiers (QLAs)—are also of fundamental importance in continuous-variable quantum information. This is because the quantum channels defined by QLAs, together with pure-loss channels, are building blocks for constructing all other phase-covariant Gaussian channels by concatenation [9,10]. Because of the ubiquity of loss channels in nature, there is a vast literature on their sensing (see, e.g., Refs. [11–14], and references therein). In contrast, previous work on sensing gain of a QLA is limited to the context of detecting Unruh-Hawking radiation using single-mode probes [15] or assumes access to the internal degrees of freedom of the amplifier [16].

In this Letter, we fill this gap by optimizing the gain sensing precision over all multimode ancilla-entangled probes and all joint quantum measurements, constraining only the energy and number of input modes of the probe. We also propose concrete probes, measurements, and estimators enabling laboratory demonstration of a quantum advantage using present-day technology limited by nonunity-efficiency photodetection. Beyond gain sensing itself, owing to the above-mentioned concatenation theorem, our results combined with those for pure-loss channels [11] are expected to yield fundamental performance limits for a vast suite of detection and estimation problems involving Gaussian channels with excess noise—see, e.g., Refs. [17–32].

Quantum-limited amplifiers.—A canonical realization of a QLA involves an optical parametric amplifier (or *paramp*) effecting a two-mode squeezing interaction between the amplified or *signal* (S) mode (annihilation operator $\hat{a}$) and an *environment* mode (E) (annihilation operator $\hat{e}$), after which the E mode is discarded [3,8,10] (Fig. 1, dashed box). In the interaction picture, the paramp Hamiltonian $\hat{H}_I = i\hbar\kappa(\hat{a}\,\hat{e} - \hat{a}^\dagger\hat{e}^\dagger)$, where κ is an effective coupling strength. Quantum-limited operation obtains when the environment is initially in the vacuum state. Evolution for a time t results in the Bogoliubov transformations $\hat{a}_{\text{out}} = \sqrt{G}\hat{a}_{\text{in}} - \sqrt{G-1}\hat{e}^\dagger_{\text{in}}$; $\hat{e}_{\text{out}} = \sqrt{G}\hat{e}_{\text{in}} - \sqrt{G-1}\hat{a}^\dagger_{\text{in}}$, where $\hat{a}_{\text{in}} = \hat{a}(0)$, $\hat{e}_{\text{in}} = \hat{e}(0)$ are the input (time zero) and $\hat{a}_{\text{out}} = \hat{a}(t)$, $\hat{e}_{\text{out}} = \hat{e}(t)$ are the output (time t) annihilation operators and $\sqrt{G} = \cosh\kappa t \equiv \cosh\tau$. The average output energy $\langle\hat{a}^\dagger_{\text{out}}\hat{a}_{\text{out}}\rangle = G\langle\hat{a}^\dagger_{\text{in}}\hat{a}_{\text{in}}\rangle + (G-1)$, where the last term represents the added noise of a QLA of gain

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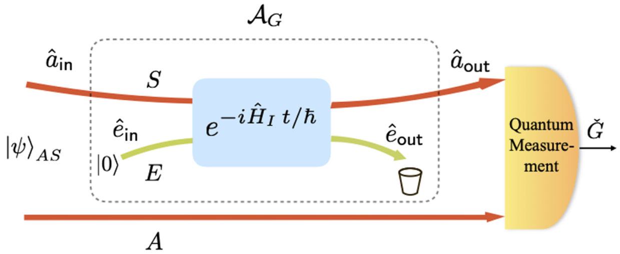

FIG. 1. A general ancilla-assisted parallel strategy for sensing the gain G of a QLA $\mathcal{A}_G$ (dashed box). Each of M signal (S) modes (one of which is shown) of a probe $|\psi\rangle_{AS}$ possibly entangled with an ancilla system A is subject to a two-mode squeezing interaction $\hat{H}_I = i\hbar\kappa(\hat{a}\,\hat{e} - \hat{a}^\dagger\hat{e}^\dagger)$ between the S mode (annihilation operator $\hat{a}$) and an environment (E) mode (annihilation operator $\hat{e}$) initially in the vacuum state. An estimate $\check{G}$ of $G = \cosh^2\kappa t$ is obtained using the optimal joint measurement on the output of AS.

$G \geq 1$ [7]. The state transformation (quantum channel) on the signal mode corresponding to a QLA of gain G is denoted $\mathcal{A}_G$.

Gain sensing setup and background.—Figure 1 also shows a general ancilla-assisted parallel estimation strategy for gain sensing. A pure state $|\psi\rangle_{AS}$ (called the *probe*) of M signal modes entangled with an arbitrary ancilla system A is prepared. Each of the signal modes passes through the QLA, following which the joint AS system is measured using an optimal (possibly probe-dependent) measurement for estimating G. The probe has the general form

$$|\psi\rangle_{AS} = \sum_{\mathbf{n}\geq\mathbf{0}} \sqrt{p_{\mathbf{n}}}|\chi_{\mathbf{n}}\rangle_A|\mathbf{n}\rangle_S, \tag{1}$$

where $|\mathbf{n}\rangle_S = |n_1\rangle_{S_1}|n_2\rangle_{S_2}\cdots|n_M\rangle_{S_M}$ is an M-mode number state of S, $\{|\chi_{\mathbf{n}}\rangle_A\}$ are normalized (not necessarily orthogonal) states of A, and $\{p_{\mathbf{n}} \geq 0\}$ is the probability distribution of $\mathbf{n}$. The number M of available signal modes depends on operational constraints such as measurement time and bandwidth, and will turn out to be fundamental in determining the sensing precision. Additionally, we impose the standard constraint on the average photon number in the signal modes: $\langle\psi|\hat{I}_A \otimes (\sum_{m=1}^M \hat{N}_m)|\psi\rangle = N$, where $\hat{N}_m = \hat{a}_m^\dagger\hat{a}_m$ is the number operator of the mth signal mode and $\hat{I}_A$ is the identity on the ancilla system. This constraint can be simplified as $\sum_{n=0}^\infty np_n = N$, where $p_n = \sum_{\mathbf{n}:n_1+\cdots+n_M=n} p_{\mathbf{n}}$ is the probability mass function of the *total* photon number in the signal modes. A mixed-state probe can be purified using an additional ancilla with the resulting purification being again of the form (1) with the same N and M. Thus, optimization over probes of the form of Eq. (1) suffices.

We are interested in comparing the performance of the optimal quantum probes of the form of Eq. (1) to the best performance achievable using *classical* probes under the same resource constraints, i.e., probes that consist of mixtures of M-mode coherent states, possibly correlated with an arbitrary number M' of ancilla modes. Such probes can be prepared using laser sources, and have the form

$$\rho_{AS} = \iint d^{2M'}\boldsymbol{\alpha} d^{2M}\boldsymbol{\beta} P(\boldsymbol{\alpha},\boldsymbol{\beta})|\boldsymbol{\alpha}\rangle\langle\boldsymbol{\alpha}|_A \otimes |\boldsymbol{\beta}\rangle\langle\boldsymbol{\beta}|_S, \tag{2}$$

where $\boldsymbol{\alpha} = (\alpha^{(1)},\ldots,\alpha^{(M')}) \in \mathbb{C}^{M'}$ and $\boldsymbol{\beta} = (\beta^{(1)},\ldots,\beta^{(M)}) \in \mathbb{C}^M$ index M'- and M-mode coherent states of A and S respectively, and $P(\boldsymbol{\alpha},\boldsymbol{\beta})\geq 0$ is a probability distribution. The signal energy constraint takes the form $\int_{\mathbb{C}^{M'}} d^{2M'}\boldsymbol{\alpha} \int_{\mathbb{C}^M} d^{2M}\boldsymbol{\beta} P(\boldsymbol{\alpha},\boldsymbol{\beta})(\sum_{m=1}^M |\boldsymbol{\beta}^{(m)}|^2) = N$.

Given a probe $|\psi\rangle_{AS}$, we have the output state $\rho_G \coloneqq \mathrm{id}_A \otimes \mathcal{A}_G^{\otimes M}(|\psi\rangle\langle\psi|_{AS})$ [$\rho_G = \mathrm{id}_A \otimes \mathcal{A}_G^{\otimes M}(\rho_{AS})$ for a classical probe (2)], where id_A is the identity channel on A. Estimating G from the state family $\{\rho_G\}$ is subject to the quantum Cramér-Rao bound (QCRB) [12,14,33], a brief description of which follows. A measurement on the AS system is described by a collection of positive operators $\{\hat{\Pi}_y\}_{\mathcal{Y}}$ indexed by the measurement result $y \in \mathcal{Y}$ and summing to the identity. The probability distribution of the result $P(y;G) = \mathrm{Tr}\rho_G\hat{\Pi}_y$, and an estimator $\check{G}(y)$ based on this measurement is called *unbiased* for G if $\int_{\mathcal{Y}} dy\check{G}(y)P(y;G) = G$ for all G in the interval of interest. The (classical) Cramér-Rao bound (CRB) bounds the mean squared error (MSE) $\mathbb{E}[\check{G} - G]^2$ of any unbiased estimator as $\mathbb{E}[\check{G} - G]^2 \geq 1/\mathcal{J}_G[Y]$, where $\mathcal{J}_G[Y] \coloneqq \mathbb{E}[\partial_G \ln P(Y;G)]^2 = -\mathbb{E}[\partial_G^2 \ln P(Y;G)]$, is the (classical) Fisher information (FI) on G of the measurement Y [34]. Different measurements $\{\hat{\Pi}_y\}_{\mathcal{Y}}$ result in different CRBs.

On the other hand, there exists a Hermitian operator $\hat{L}_G$ called the symmetric logarithmic derivative (SLD) satisfying $\partial_G\rho_G \equiv \partial\rho_G/\partial G = (\rho_G\hat{L}_G + \hat{L}_G\rho_G)/2$. The quantum Fisher information (QFI) is defined as $\mathcal{K}_G = \mathrm{Tr}\rho_G\hat{L}_G^2$, and the QCRB $\mathbb{E}[\check{G} - G]^2 \geq \mathcal{K}_G^{-1}$ minimizes the right-hand side of the CRB over all unbiased measurements and defines the quantum-optimal sensing performance. The QFI $\mathcal{K}_\theta$ on θ of an arbitrary state family $\{\rho_\theta\}$ is related to the fidelity $F(\rho_\theta,\rho_{\theta'}) = \mathrm{Tr}\sqrt{\sqrt{\rho_\theta}\rho_{\theta'}\sqrt{\rho_\theta}}$ between the states of the family via [35]

$$\mathcal{K}_\theta = -4\partial_{\theta'}^2 F(\rho_\theta,\rho_{\theta'})|_{\theta'=\theta}. \tag{3}$$

It is expedient for us to work with the QFI on the parameter $\tau = \kappa t = \mathrm{arccosh}\sqrt{G}$. Since the SLDs with respect to τ and G satisfy $\hat{L}_G = (\partial\tau/\partial G)\hat{L}_\tau$, $\mathcal{K}_G = (\partial\tau/\partial G)^2\mathcal{K}_\tau$ so that maximizing either QFI suffices.

Optimal gain sensing.—We first obtain an upper bound $\tilde{\mathcal{K}}_\tau \geq \mathcal{K}_\tau$ on the QFI in the hypothetical situation where the output of ASE is available for measurement. Given a probe $|\psi\rangle_{AS}$, we then hold the state family $\{\Psi_\tau = |\psi_\tau\rangle\langle\psi_\tau|\}$

defined by $|\psi_\tau\rangle_{ASE} = \hat{I}_A \otimes [\otimes_{m=1}^M \hat{U}_m(\tau)]|\psi\rangle_{AS}|\mathbf{0}\rangle_E$, where $\hat{U}_m(\tau) = \exp(-i\hat{H}_I t/\hbar)$ is the paramp unitary acting on the mth signal and environment mode pair parametrized by $\tau = \kappa t$. The QFI $\tilde{\mathcal{K}}_\tau$ of $\{\Psi_\tau\}$ upper bounds $\mathcal{K}_\tau$ due to the monotonicity of the QFI with respect to partial trace over E [35].

We can show (see Ref. [36], Sec. I) that the paramp takes the input $|n\rangle_S|0\rangle_E$ to $\hat{U}(\tau)|n\rangle_S|0\rangle_E = \mathrm{sech}^{(n+1)}\tau \sum_{a=0}^\infty \sqrt{\binom{n+a}{a}}\tanh^a \tau|n+a\rangle_S|a\rangle_E$. Thus, the paramp coherently adds a random number a of photons to both S and E according to the negative binomial distribution $\mathrm{NB}(n+1, \mathrm{sech}^2\tau)$ [40]. For any probe (1), we have

$$|\psi_\tau\rangle_{ASE} = \sum_{\mathbf{a}\geq\mathbf{0}} |\psi_{\mathbf{a};\tau}\rangle\!\rangle_{AS}|\mathbf{a}\rangle_E, \tag{4}$$

where $|\psi_{\mathbf{a};\tau}\rangle\!\rangle_{AS} = \sum_{\mathbf{n}\geq\mathbf{0}} \sqrt{p_\mathbf{n} A_\tau(\mathbf{n},\mathbf{a})}|\chi_\mathbf{n}\rangle_A|\mathbf{n}+\mathbf{a}\rangle_S$ are non-normalized states of AS and $A_\tau(\mathbf{n},\mathbf{a}) = \prod_{m=1}^M \binom{n_m+a_m}{a_m}\mathrm{sech}^{2(n_m+1)}\tau \tanh^{2a_m}\tau$ is a product of negative binomial probabilities. For $|\psi_{\tau'}\rangle_{ASE}$ the output state on ASE obtained by passing $|\psi\rangle_{AS}$ through a QLA of gain $G' = \cosh^2\tau'$, the fidelity between the outputs can be shown after some computation ([36], Sec. II) to be

$$F(\Psi_\tau, \Psi_{\tau'}) = \langle\psi_\tau|\psi_{\tau'}\rangle = \sum_{n=0}^\infty p_n \nu^{n+M}, \tag{5}$$

where $\nu = \mathrm{sech}(\tau'-\tau) = [\sqrt{GG'} - \sqrt{(G-1)(G'-1)}]^{-1} \in (0,1]$. Using Eq. (3), we obtain the sought upper bounds $\tilde{\mathcal{K}}_\tau = 4(N+M)$ and $\tilde{\mathcal{K}}_G = [(N+M)/G(G-1)]$ on the true QFI with respect to τ and G.

Returning to the original problem in which only $\rho_\tau = \mathrm{Tr}_E\Psi_\tau$ is accessible, we have from Eq. (4) that $\rho_\tau = \sum_{\mathbf{a}\geq\mathbf{0}} |\psi_{\mathbf{a};\tau}\rangle\!\rangle\langle\!\langle\psi_{\mathbf{a};\tau}|_{AS}$. For given $\{p_\mathbf{n}\}$ in Eq. (1), consider probes for which $\{|\chi_\mathbf{n}\rangle_A\}$ is an orthonormal set. Such probes, called number-diagonal signal (NDS) probes, are known to be optimal probes for diverse sensing problems [11,24,41]. Orthonormality of the $\{|\chi_\mathbf{n}\rangle_A\}$ implies that $\langle\!\langle\psi_{\mathbf{a};\tau}|\psi_{\mathbf{a}';\tau'}\rangle\!\rangle = \langle\!\langle\psi_{\mathbf{a};\tau}|\psi_{\mathbf{a};\tau'}\rangle\!\rangle\delta_{\mathbf{a},\mathbf{a}'}$, so the output fidelity $F(\rho_\tau,\rho_{\tau'}) = \sum_{\mathbf{a}\geq\mathbf{0}} \langle\!\langle\psi_{\mathbf{a};\tau}|\psi_{\mathbf{a};\tau'}\rangle\!\rangle = F(\Psi_\tau,\Psi_{\tau'})$ of Eq. (5). Thus, the QFIs on τ and G

$$\mathcal{K}_\tau = 4(N+M); \qquad \mathcal{K}_G = \frac{N+M}{G(G-1)} \tag{6}$$

of NDS probes saturate the upper bounds calculated above.

This result exhibits several remarkable features. First, any NDS probe with the given N and M is quantum optimal regardless of its exact signal photon number distribution $\{p_\mathbf{n}\}$. This generalizes the single-mode Fock-state optimality result [15] not just to multimode Fock states but to the infinite class of ancilla-entangled multimode NDS probes including the workhorse of optical quantum information—the two-mode squeezed vacuum (TMSV) state. Second, gain sensing performance explicitly depends on the number M of signal modes. This contrasts sharply with loss sensing, for which the optimal QFI is M independent [11]. Physically, this difference stems from the gain-dependent quantum noise introduced by a QLA that makes the output states of two QLAs with distinct gains distinguishable even for a vacuum input. Increasing the number of signal modes further improves their distinguishability. In contrast, vacuum probes of any M are invariant states of loss channels and are therefore useless for sensing them. Finally, the roles of N and M in Eq. (6) are seen to be equivalent so that one resource can be exchanged for the other, providing additional flexibility in the choice of optimal probes.

For an M-mode signal-only coherent-state probe $|\sqrt{N_1}\rangle_{S_1}\cdots|\sqrt{N_M}\rangle_{S_M}$ with $\sum_{m=1}^M N_m = N$, the output state ρ_G is a product of single-mode Gaussian states. The QFI on G follows from the results of [42] after some algebra:

$$\mathcal{K}_G^{\mathrm{coh}} = \frac{N}{G(2G-1)} + \frac{M}{G(G-1)}. \tag{7}$$

The convexity of QFI in the state [43] and the linear dependence on N of the first term in the above expression imply that no classical probe [Eq. (2)] with M signal modes can beat the QFI of Eq. (7). Both Eqs. (6) and (7) contain a term proportional to N (the *photon* contribution) and another proportional to M (the *modal* contribution). The modal contribution in the optimal quantum and classical QFI is identical, but the quantum-optimal photon contribution is at least twice the classical photon contribution and far exceeds it in the $G \sim 1$ regime (see Fig. 2).

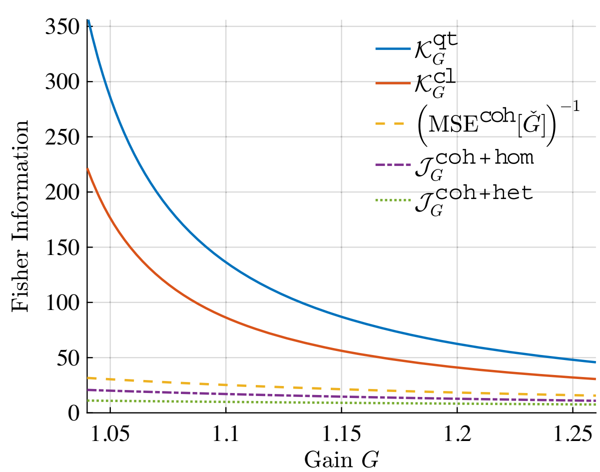

FIG. 2. The optimal quantum (blue) [Eq. (6)] and classical QFI (red) [Eq. (7)] for $N = 6$ and $M = 9$. Also shown are the FI of homodyne (purple dash-dotted), heterodyne detection (green dotted), and the inverse MSE of the photodetection-based estimator [Eq. (8)] (yellow dashed) for a coherent-state probe.

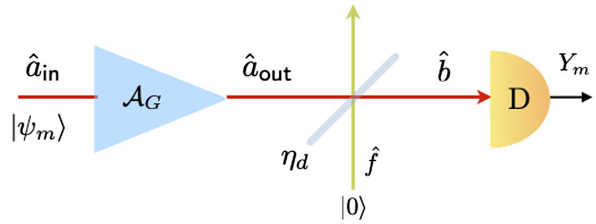

FIG. 3. Gain estimation under inefficient detection. Each mode of a product signal-only probe $\otimes_{m=1}^{M}|\psi_m\rangle$ passes through a QLA $\mathcal{A}_G$. Detection with quantum efficiency η_d is modeled by a beam splitter with mode $\hat{f}$ in vacuum and output mode $\hat{b}$ that is measured using an ideal photodetector D, resulting in photon count Y_m.

Performance of standard measurements.—Suppose that an arbitrary NDS probe (1) is input to a QLA of unknown gain and that we measure the basis $\{|\chi_{\mathbf{n}}\rangle_A\}$ and also the photon number in each of the M output signal modes. Denote the measurement result $(\mathbf{X},\mathbf{Y})$, where $\mathbf{X}=(X_1,\ldots,X_M)$ if $|\chi_{\mathbf{X}}\rangle_A$ is the measurement result on A and $\mathbf{Y}=(Y_1,\ldots,Y_M)$ if Y_m photons are observed in the mth output signal mode. We can then show ([36], Sec. III) that the FI $\mathcal{J}_\tau[\mathbf{X},\mathbf{Y}]=4(N+M)$ for any NDS probe, so that this measurement achieves the quantum-optimal QFI (6).

While this implies that the maximum likelihood estimator based on $(\mathbf{X},\mathbf{Y})$ achieves the quantum limit for a large number of copies [34,40], a quantum-optimal estimator may not exist for a finite sample [34]. For a multimode number-state probe $\otimes_{m=1}^{M}|n_m\rangle_{S_m}$ with $\sum_{m=1}^{M}n_m=N$, consider the estimator

$$\check{G}\coloneqq(Y+M)/(N+M), \tag{8}$$

where $Y=\sum_{m=1}^{M}Y_m$ is the total photon number measured in the signal modes. Using the fact that $Y-N\sim\mathrm{NB}(N+M,\mathrm{sech}^2\,\tau)$, we can show that $\check{G}$ is unbiased and that $\mathrm{Var}[\check{G}]=[G(G-1)/(N+M)]$ so the QCRB (6) is achieved even on a finite sample for any multimode number-state probe.

On the other hand, a G-independent measurement that achieves the coherent-state QFI (7) is unknown. The estimator $\check{G}$ above remains unbiased but has the suboptimal variance $\mathrm{MSE}^{\rm coh}[\check{G}]=[G(G-1)/(N+M)]+[G^2N/(N+M)^2]$ ([36], Sec. IV.C). Homodyne and heterodyne detection in each output mode have the respective (suboptimal) FIs $\mathcal{J}_G^{\rm coh+hom}=[N/G(2G-1)]+[2M/(2G-1)^2]$ and $\mathcal{J}_G^{\rm coh+het}=[(N/2+M)/G^2]$. These Fisher information quantities are compared in Fig. 2.

Practical quantum advantage.—To examine whether a quantum advantage can be demonstrated in the laboratory, we study the estimation of G using single-photon probes and photodetectors of efficiency $\eta_d<1$ (see Fig. 3). For any multimode number-state probe $\otimes_{m=1}^{M}|n_m\rangle$, photon counting in each output mode remains the QFI-achieving measurement and the QFI can be obtained numerically ([36], Sec. IV.B). We also calculate the QFI of a coherent-state probe $\otimes_{m=1}^{M}|\sqrt{N_m}\rangle$ of the same N and M ([36], Sec. IV.A), and also the MSE of the unbiased estimator

$$\check{G}=(\eta_d^{-1}Y+M)/(N+M) \tag{9}$$

generalizing that of Eq. (8) ([36], Sec. IV.C).

Since single-photon states are more readily prepared than multiphoton Fock states [44], we compare their performance relative to coherent states in Fig. 4. The MSE $\mathrm{MSE}^{\text{1-photon}}[\check{G}]$ of $\check{G}$ for single-photon probes (for which $M=N$) is always less than that for coherent states (See Ref. [36], Sec. IV.C, and Fig. 4, left and center). Moreover, for each value of η_d, there is a threshold value of the gain (which is independent of M) beyond which $\mathrm{MSE}^{\text{1-photon}}[\check{G}]$ falls below the QCRB for coherent states (Fig. 4, right), so that a quantum advantage is guaranteed for sensing gain values known to lie beyond the threshold.

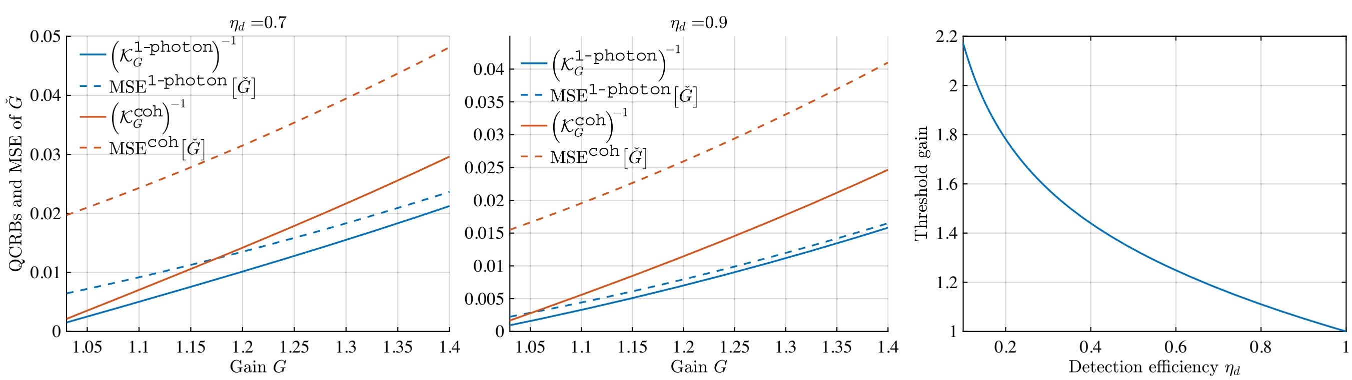

FIG. 4. Performance of single-photon probes with inefficient detection. Left and center: QCRBs of multimode single-photon (blue solid) and coherent-state (red solid) probes along with the MSE of $\check{G}$ of Eq. (9) for single-photon (blue dashed) and coherent-state (red dashed) probes for $\eta_d=0.7$ (left) and $\eta_d=0.9$ (center) with $M=N=20$. Right: the threshold gain beyond which single-photon probes and photon counting beat the coherent-state QCRB.

Energy-constrained Bures distance.—As our final result, we derive the energy-constrained Bures distance [45] between the amplifier channels $\mathcal{A}_G^{\otimes M}$ and $\mathcal{A}_{G'}^{\otimes M}$. This distance is one of several energy-constrained channel divergence measures between bosonic channels, with many applications in quantum information and sensing [24,25,45–53]. Its calculation is equivalent to minimizing the output fidelity $F(\rho_G, \rho_{G'})$ over all M-signal-mode probes (1) with average signal energy N. We show ([36], Sec. V) that this minimum equals $F_{\min}(\rho_G, \rho_{G'}) = \nu^M[(1-\{N\})\nu^{\lfloor N\rfloor} + \{N\}\nu^{\lfloor N\rfloor+1}]$, where $\lfloor N\rfloor$ and $\{N\}$ are, respectively, the integer and fractional parts of N. This results adds QLAs to the short list of channels for which exact values of energy-constrained channel divergences are known and also gives bounds on other divergences between QLAs [54].

Discussion.—We have delineated the optimal precision of sensing the gain of QLAs regardless of their implementation platform and explicit physical realization. Our problem formulation constrained the average signal energy to equal N but since the optimal QFI increases with N, NDS states of average energy N are optimal over all probes with an average energy less than or equal to N.

For multimode number-state probes, we identified a concrete quantum-optimal estimator and showed the in-principle feasibility of quantum-enhanced gain sensing using standard single-photon sources [44] and photon counting even under inefficient detection. Additional loss in the signal path upstream of the QLA can also be accounted for by our calculation techniques. The use of brighter TMSV sources is expected to harness the photon contribution to the QFI of Eq. (6) even better, and finding good measurements and estimators for TMSV probes with imperfect detection is of great interest for future work. Our study can be generalized to the estimation of multiple [55] and distributed [56] gain parameters. The implications of our results for relativistic metrology problems [15,57] also remain to be explored.

Noisy attenuator channels (relevant to quantum illumination, noisy imaging, and quantum reading [17–22] among other applications), noisy amplifier channels (which model laser amplifiers with incomplete inversion [8,10]), and additive noise channels (relevant to noisy continuous-variable teleportation [58]) are compositions of pure-loss channels with QLA channels. Our Letter here, together with complementary results in loss sensing [11], is expected to be basic to a general theory of fundamental limits for sensing such noisy phase-covariant Gaussian channels, while highlighting the role of M as an important resource therein.

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