Gap Statistics for Confined Particles with Power-Law Interactions

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We consider the N particle classical Riesz gas confined in a one-dimensional external harmonic potential with power-law interaction of the form $1/r^k$, where r is the separation between particles. As special limits it contains several systems such as Dyson's log-gas ($k \to 0^+$), the Calogero-Moser model ($k = 2$), the 1D one-component plasma ($k = -1$), and the hard-rod gas ($k \to \infty$). Despite its growing importance, only large-N field theory and average density profile are known for general k. In this Letter, we study the fluctuations in the system by looking at the statistics of the gap between successive particles. This quantity is analogous to the well-known level-spacing statistics which is ubiquitous in several branches of physics. We show that the variance goes as N^{-b_k} and we find the k dependence of b_k via direct Monte Carlo simulations. We provide supporting arguments based on microscopic Hessian calculation and a quadratic field theory approach. We compute the gap distribution and study its system size scaling. Except in the range $-1 < k < 0$, we find scaling for all $k > -2$ with both Gaussian and non-Gaussian scaling forms.

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Introduction.—The Riesz gas, consisting of N particles with long-range interactions confined in a harmonic trap, is one of the classic examples of a strongly interacting many-body system. The model is characterized by power-law interaction potentials of the form $V(r) \sim J r^{-k}$, where r is the distance between two particles, $J > 0$ is the interaction strength, and $k > -2$ (to ensure stability). Special values of k lead us to some important models such as the log-gas ($k \to 0^+$) [1,2], the one-dimensional one-component plasma (1DOCP, $k = -1$) [3–6] and the Calogero-Moser (CM) model ($k = 2$) [7–10]. Experimental realizations of this model in cold atom systems have now become possible [11–13] and hence it is essential to have a complete characterization of its equilibrium and dynamical properties. The long-range nature of the interactions makes this difficult but some progress has recently been made [5,6,14–24]. In Ref. [14] the exact density profile was computed using a field theoretic approach, thereby obtaining a generalization of the Wigner semicircle law for the log-gas [25]. The form of the average density profile and the scaling of its support with increasing N was found to be nontrivial. For the 1D one-component plasma for which the density profile is flat, the distribution of the position of the rightmost particle was computed exactly [6] and found to be different from the Tracy-Widom form [26,27]. Surprisingly, the density profile in the CM gas is identical to that in the log-gas but the edge particle distribution takes a different (non Tracy-Widom) form [28]. Recently, the average density profile, in the presence of a hard wall, has also been computed exactly for all $k > -2$ [29].

One of the interesting observations of Ref. [14] was on the system-size scaling of the mean separation $\langle \Delta \rangle$ between neighboring particles. This has the form $\langle \Delta \rangle \sim N^{-a_k}$, where a_k has a nonmonotonic dependence on k and can have both positive and negative signs. For a complete characterization it is necessary to go beyond the mean and study the fluctuations of this quantity as well as its full distribution. The interplay between the long-range interactions and the confining potential makes this a fascinating and difficult question and this is the main focus of this Letter.

The gap statistics is analogous to level spacing statistics which has been studied in great detail in different areas such as random matrix theory (RMT) [30,31] and quantum chaos [32–35]. In the context of RMT we recall that the equilibrium distribution of particle positions in the log-gas ($k \to 0^+$, $J \to \infty$, with $Jk \to J_0$) at inverse temperature β corresponds to the distribution of eigenvalues of random matrices for the Gaussian orthogonal (GOE), unitary (GUE), and symplectic (GSE) ensembles, corresponding to Dyson indices 1, 2, and 4 respectively. From this correspondence it is known that the distribution of particle spacing, normalized by the mean spacing, is given quite accurately by the Wigner surmise (WS) [25,30,31]. A variant of the WS has also been applied to the CM model ($k = 2$) [36] but to the best of our knowledge, there are no results for other values of k and this Letter provides a complete characterization. Needless to mention, fluctuations at the microscopic level is an avenue that is essentially unexplored in systems with long-range interactions. Probing such fluctuations has now become experimentally accessible given the recent breakthroughs in the technology of quantum gas microscopy [37–42]. Gap fluctuations give us a novel way to probe aspects of the underlying interacting systems that are otherwise completely elusive to diagnostics such as density profiles.

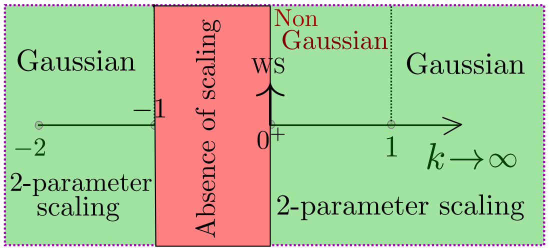

FIG. 1. Schematic phase diagram of the behavior of the gap distribution. We find four regimes in $k \in (-2, \infty)$, where the gap distribution has different scaling properties. In the region $(-2,-1) \cup (0,\infty)$ the scaling limit is achieved by using mean and variance of gap only. The scaling function for $k \in (-2,-1) \cup (1,\infty)$ is Gaussian whereas it is non-Gaussian in $k \in (0, 1)$. In the regime $k \in (-1, 0)$ we are unable to obtain a scaling limit.

Our main results are the following: (i) From direct Monte Carlo (MC) simulations, we find that the system size scaling of the variance of the bulk gap is characterized by a nontrivial exponent b_k that fits the form in Eq. (5). (ii) This proposed form is further validated from our results based on a microscopic Hessian (MH) calculation and a quadratic field theory (FT). (iii) We study the scaling properties of the gap distributions for different k and observe that there exists four regimes as shown in Fig. 1.

Model and definitions.—The harmonically confined Riesz gas consists of N classical particles, confined in a harmonic potential on a line and interacting with each other via pairwise repulsion. We denote the positions of the particles on the line by $x_j (j = 1, 2, \ldots, N)$. The pairwise repulsive interaction is taken as a power law of the distance between the particles, and the total potential energy is given by ($\forall\ k > -2$) [43]

$$E(\{x_j\}) = \sum_{i=1}^{N} \frac{x_i^2}{2} + \frac{J\mathrm{sgn}(k)}{2} \sum_{i \neq j} \frac{1}{|x_i - x_j|^k}, \tag{1}$$

where $J > 0$ and $\mathrm{sgn}(k)$ ensures a repulsive interaction. We consider a thermal distribution of the N particles given by $P_G(x_1, x_2, \ldots, x_N) = e^{-\beta E}/Z$, where Z is the partition function and henceforth we set the inverse temperature $\beta = 1$. Without loss of generality, we assume that the particles are ordered, i.e., $x_1 \leq x_2 \leq x_3 \ldots \leq x_N$. The mean thermal density of particles is defined as $\rho_N^{(\mathrm{eq})}(x) = (1/N)\sum_{i=1}^{N} \langle \delta(x - x_i) \rangle$, where $\langle \ldots \rangle$ denotes a thermal average over the distribution $P_G(\{x_i\})$. For large N the average density has been computed exactly for all k values [14] and has a finite support in the range $[-l_k N^{\alpha_k}/2, l_k N^{\alpha_k}/2]$ (for $k \neq 1$ [44]) where the exponent $\alpha_k = k/(k+2)$ for $k > 1$ and $1/(k+2)$ for $-2 < k < 1$, with l_k known explicitly [14,45]. The average density $\rho_N^{(\mathrm{eq})}(x)$ for large N and temperature $T < N^{2\alpha_k}$ is given by the scaling form $\rho_N^{(\mathrm{eq})}(x) = (l_k N^{\alpha_k})^{-1} F_k[x/(l_k N^{\alpha_k})]$, where the scaling function $F_k(y)$ is known exactly [14].

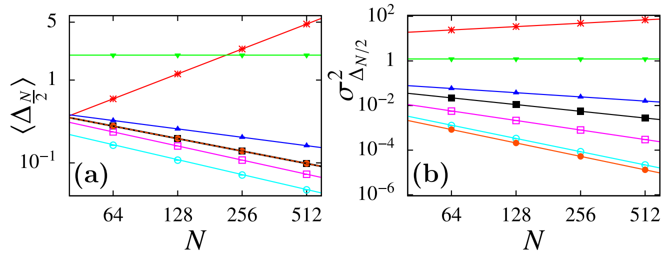

FIG. 2. Mean (left) and variance (right) of midgap as a function of system size for $k = 2$(orange filled circle), 1.5(blue circle), 0.5(red square), 0^+(black filled square), -0.5(blue upward triangle), -1(green downward triangle), and -1.5(red asterisk). Solid lines correspond to their corresponding power-law fitting [Eq. (5)]. The slopes in (a) are $a_k = 0.5, 0.57, 0.6, 0.5, 0.33, 0, -1$ and in (b) are $b_k = 2, 1.97, 1.42, 1, 0.63, 0, -1$, for decreasing k. These are consistent with Eqs. (3) and (5) as elucidated in Fig. 3. The error bars are negligible [45]. In (a) the data for $k = -1.5$ is scaled by a factor 500. $\sim 10^8$ MC samples are used for the computations.

The main quantity of interest here is the interparticle separation $\Delta_i = x_{i+1} - x_i$ and the normalized separation $s_i = \Delta_i / \langle \Delta_i \rangle$. The distribution of s is defined as

$$P_N(s) = \frac{1}{N-1} \sum_{i=1}^{N-1} p_N^{(i)}(s), \tag{2}$$

where $p_N^{(i)}(s) = \langle \delta(s - s_i) \rangle$ is the distribution of the ith normalized gap. We expect that for typical fluctuations, $P_N(s)$ will be dominated by the bulk gaps, but edge gap contributions could be important for atypical s.

Results for mean and variance of the bulk gap.—We expect that for bulk particles $1 \ll i \ll N-1$, the average bulk gap should scale as $\langle \Delta_i \rangle \sim N^{\alpha_k}/N = N^{-a_k}$, where $a_k = 1 - \alpha_k$, i.e.,

$$a_k = \begin{cases} \frac{2}{k+2} & \text{for } k > 1 \\ \frac{k+1}{k+2} & \text{for } -2 < k < 1. \end{cases} \tag{3}$$

We also expect a power-law dependence on the system size of the gap fluctuations $\sigma^2_{\Delta_i} = \langle \Delta_i^2 \rangle - \langle \Delta_i \rangle^2$. In particular, for the mid-gap corresponding to $i = N/2$, we provide theoretical arguments based on MH and FT (see later) for the following conjecture:

$$\sigma^2_{\Delta_{N/2}} \sim N^{-b_k}, \quad \text{where ,} \tag{4}$$

$$b_k = \begin{cases} 2 & \text{for } k > 1 \\ 1 + k & \text{for } 0 < k < 1 \\ 2(k+1)/(k+2) & \text{for } -1 < k < 0 \\ 1 + k & \text{for } -2 < k < -1. \end{cases} \tag{5}$$

We present numerical evidence for the above conjecture in Figs. 2 and 3 where we observe reasonable agreement

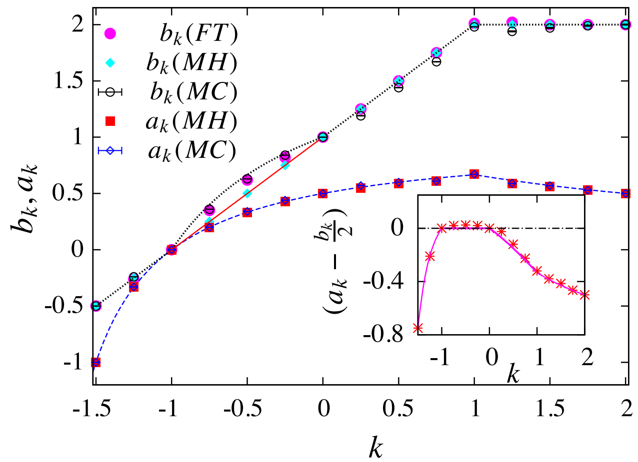

FIG. 3. Comparison of the exponents a_k and b_k (symbols) obtained from simulations (MC), from MH and FT calculations, with Eqs. (3) (dashed line) and (5) (dotted line). In the inset, we plot $[a_k - (b_k/2)]$, which quantifies the relative fluctuations $\sigma_\Delta/\langle\Delta\rangle$ in the MC data. To extract b_k the largest system sizes used were $N = 2048$ for MC, 16384 for both MH and FT. All the error bars are smaller than the point symbols.

between the numerically obtained exponent (MC) and the conjectured values. We believe that the slight deviations from the predictions for few values of k are due to finite size effects, since the error bars are small (see Ref. [45] for discussion of error bars). We verified that the above scaling in Eq. (5) also holds for other gaps deep in the bulk. Interestingly we find that for $-1 \leq k \leq 0$, the ratio $\sigma_{\Delta_i}/\langle\Delta_i\rangle$ as well as $P_N^{(i)}(s)$ are weakly dependent on i (for large N and i in the bulk; see Sec. III of Ref. [45]).

Results for distribution of gap.—The distribution of the normalized gap s in Eq. (2) is a well-studied object in RMT [46–50] where one of the important results is on the universal form of $P_N(s)$ given by the WS. For the distribution of eigenvalues of the random matrices belonging to the three Gaussian ensembles, with Dyson indices 1,2,4 (which for our log-gas corresponds to $\beta J_0 = 1, 2, 4$), it is known that $P_{N\to\infty}(s)$ is in fact accurately described by $P_2(s) \equiv P_{N=2}(s)$ (which is basically the WS) and is given by Refs. [25,30] $P_2(s) = A_0 s^{\beta J_0} e^{-B_0 s^2}$, (for log-gas), where A_0 and B_0 are constants. From our simulations we in fact find that the WS for the log-gas is quite accurate for all $\beta J_0 > 1$. We now examine the distribution $P_N(s)$ for other values of k. Interestingly, we find that for $k = -1$ (as also for log-gas) the distribution converges very fast as can be seen in Figs. 4(b) and 4(d). On the other hand, for other values of k there is no convergence. In particular for the CM model ($k = 2$), our findings [45] are thus in disagreement with the generalized version of WS proposed in Ref. [36]. For generic values of k, as seen in Fig. 4, the distributions $P_N(s)$ do not show convergence with N. Hence, we look at the distribution of the following natural scaling variable

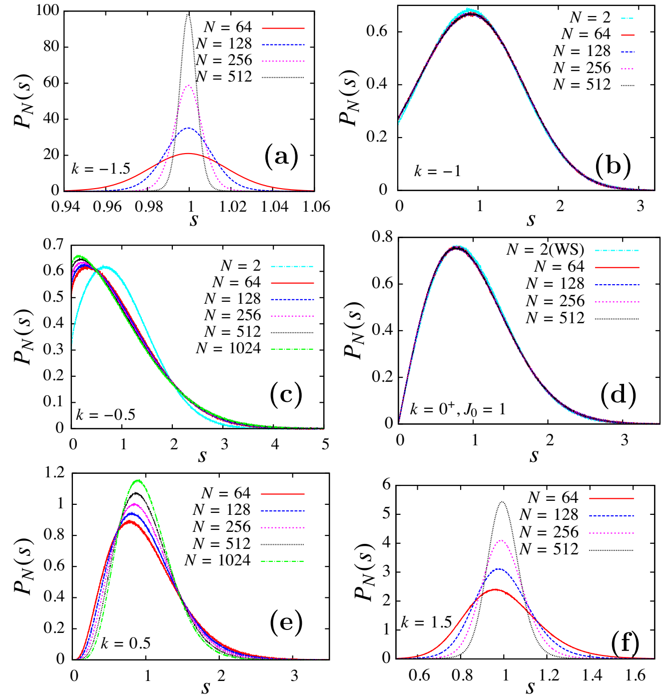

FIG. 4. Plot of distributions $P_N(s)$ for different values of k and N. Except for the 1DOCP (b) and log-gas (d), for other values of k, we do not see convergence in N which naturally implies that it is different from $P_2(s)$ and hence there is no generalization of WS.

$$\tilde{s}_j = \frac{\Delta_j - \langle\Delta_j\rangle}{\sigma_{\Delta_j}}. \tag{6}$$

The distribution of this quantity defined as $\tilde{P}_N(\tilde{s}) = [1/(N-1)]\sum_{i=1}^{N-1}\langle\delta(\tilde{s} - \tilde{s}_i)\rangle$, is computed numerically for different values of k and N. In Fig. 5 we plot $\tilde{P}_N(\tilde{s})$ for $k = -1.5, -0.5, 0.5$, and $k = 1.5$. We find that $\tilde{P}_N(\tilde{s})$ tends to a Gaussian form with zero mean and unit variance in the limit $N \to \infty$, except in the range $-1 \leq k \leq 1$. Interestingly, in the range $-1 < k < 0$, we do not see convergence with N [Fig. 5(b)]. In the range $0 < k < 1$ relative fluctuations die out with N in which case one might expect a Gaussian scaling form. Surprisingly, even though the MH nicely predicts the correct scaling exponent b_k the scaling form of the distribution is non-Gaussian [Fig. 5(c)]. We now present the theoretical arguments which support the conjecture in Eq. (5)—based on MH and FT calculations.

Microscopic Hessian (MH).—Computing analytically the variance of the gap for generic values of k is hard (except for $k = -1$ and $k \to 0$). Here we use the microscopic Hessian method [51,52] to estimate the variance for large N for all values of k. At zero temperature, the system will be in the ground state characterized by the configuration of positions y_i and corresponding gaps $\Delta_i^{\mathrm{GS}} = y_{i+1} - y_i$. Since the system is at low temperature,

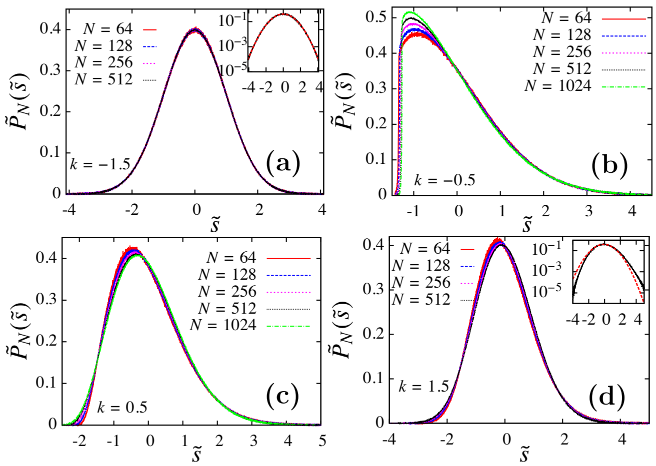

FIG. 5. Plot of $\tilde{P}_N(\tilde{s})$ for different values of k and N. The distributions for $k = \pm 1.5$ are fitted with a Gaussian over 2 standard deviations (insets). This is generally observed for $k \notin [-1, 1]$. The distributions for $k = \pm 0.5$ are very different from Gaussian and this is generally the case for $k \in [-1, 1]$.

we expect that the Hessian of the microscopic Hamiltonian Eq. (1) about the ground state would approximately capture the behavior of the fluctuations of the gap. The joint distribution of fluctuation of gaps $\delta\Delta_i = \Delta_i - \Delta_i^{\mathrm{GS}}$ will be of the form

$$\mathcal{P}_{\mathrm{MH}}(\{\Delta_i\}) \sim e^{-\frac{\beta}{2}\sum_{i,j=1}^{N} H_{ij}\delta\Delta_i\delta\Delta_j}, \tag{7}$$

where the Hessian of the system about the ground state is $H_{ij} = [\partial^2 E/\partial\Delta_i\partial\Delta_j]_{\mathrm{GS}}$ [45]. The variance of the gap $\sigma^2_{\Delta_i} = (\beta H)^{-1}_{ii}$ can thus be obtained by inverting the matrix H numerically. As seen in Fig. 3, the exponent b_k calculated using MH theory matches with the MC result [Eq. (5)] except in the regime $-1 < k < 0$. This is perhaps not surprising since our conjecture suggests that in this regime, the relative fluctuation of the gap, $\sigma_{\Delta_i}/\langle\Delta_i\rangle \sim N^{(a_k - b_k/2)}$ does not decrease with system size—in fact over the range of N considered we see them increasing (see inset of Fig. 3). Next, we discuss the FT calculation.

Field theory (FT).—As discussed in Ref. [14] the Reisz gas for large N can be described by a free-energy functional $\Sigma[\rho_N] = \mathcal{E}[\rho_N] - \beta^{-1}S[\rho_N]$ corresponding to a macroscopic density profile $\rho_N(x)$, where $\mathcal{E}[\rho_N]$ is the energy and $S[\rho_N] = -N\int dx \rho_N \log(\rho_N)$ is the entropy functional. The form of the energy functional depends on k, being local for $k \geq 1$ and nonlocal for $-2 < k < 1$ [14,45]. We use this action to compute the fluctuations of the bulk gap. The probability of a density profile ρ_N is [53]

$$\mathbb{P}[\rho_N] \sim e^{-\beta\delta\Sigma}, \quad \text{with} \quad \delta\Sigma = \Sigma[\rho_N] - \Sigma[\rho_N^{(\mathrm{eq})}], \tag{8}$$

where $\rho_N^{(\mathrm{eq})}$ is mean thermal density. For a given macroscopic density profile $\rho_N(x)$, the gap between two consecutive particles at position x is $\bar{\Delta} = [N\rho_N(x)]^{-1}$. Note that this definition of the gap is different from the gap Δ defined earlier [above Eq. (2)] from the microscopic position configuration. The gap $\bar{\Delta}$ is a coarse-grained version of Δ averaged over many microscopic configurations consistent with the macroscopic density $\rho_N(x)$. As the density profile $\rho_N(x)$ fluctuates, the separation $\bar{\Delta}$ also fluctuates. We expect that for large N, the fluctuation of $\bar{\Delta}$ and Δ would have the same scaling with respect to N.

We first find the distribution of the fluctuation $\delta\rho_N(x)$ around the equilibrium profile $\rho_N^{(\mathrm{eq})}(x)$. Writing $\rho_N(x) = \rho_N^{(\mathrm{eq})}(x) + \delta\rho(x)$ in the expression of the action $\delta\Sigma[\rho_N]$ in Eq. (8) and expanding to quadratic order in $\delta\rho(x)$ we get the distribution of the fluctuation profile $\delta\rho(x)$ (see Ref. [45] for details). Note the action $\delta\Sigma$ now becomes an explicit functional of $\delta\rho(x)$ and $\rho_N^{(\mathrm{eq})}(x)$. The probability distribution of the fluctuation $\delta\bar{\Delta}$ of the gap, defined as $\bar{\Delta} = \langle\bar{\Delta}\rangle + \delta\bar{\Delta}$, is obtained by using the relation

$$\delta\bar{\Delta} \approx -\frac{\delta\rho(x)}{N(\rho_N^{(\mathrm{eq})}(x))^2}, \tag{9}$$

which can be obtained from $\bar{\Delta} = [N\rho_N(x)]^{-1}$.

For N (large but finite) particles there are $(N-1)$ number of gap variables. In order to find the joint distribution of these (discrete) gap variables from the field theory description, we need to discretize $\delta\Sigma[\delta\rho, \rho_N^{(\mathrm{eq})}]$. To do so, we discretize the integral in the action $\delta\Sigma$ along the equilibrium positions $\{y_i\}$ [45]. Recall that the microscopic Hessian was computed about this position configuration in Eq. (7) earlier. Note that $\{y_i\}$, also the minimum energy configuration, leads to the equilibrium macroscopic density $\rho_N^{(\mathrm{eq})}(x)$ that corresponds to mean gaps $\langle\bar{\Delta}_i\rangle = 1/N\rho_N^{(\mathrm{eq})}(y_i)$. Also note that for large-N, $\langle\bar{\Delta}_i\rangle \approx \Delta_i^{\mathrm{GS}}$. We emphasize that this discretization of the density profile is different from the original microscopic position description of the system.

We replace the integrals in the expression of $\delta\Sigma$ as $\int_{-l_N}^{l_N} dx \to \sum_i [1/N\rho_N^{(\mathrm{eq})}(y_i)]$ and evaluate the integrand at points $\{y_i\}$. After some simplifications we get the following joint distribution of the gap variables $\{\delta\bar{\Delta}_i\}$ to leading order in N (see Ref. [45] for details):

$$\mathcal{P}_{\mathrm{FT}}(\{\delta\bar{\Delta}_i\}) \sim e^{-\frac{\beta}{2}\sum_{i,j=1}^{N} M_{ij}\delta\bar{\Delta}_i\delta\bar{\Delta}_j}, \quad \text{where} \tag{10}$$

$$M_{ii} = \begin{cases} J\zeta(k)k(k+1)N^{k+2}[\rho_N^{(\mathrm{eq})}(y_i)]^{k+2} & \text{for } k > 1 \\ 2JN^{k+2}[\rho_N^{(\mathrm{eq})}(y_i)]^{k+2} & \text{for } 0 < k < 1 \\ N^2\beta^{-1}[\rho_N^{(\mathrm{eq})}(y_i)]^2 & \text{for } -2 < k < 0, \end{cases}$$

$$M_{ij} = \begin{cases} 0 & \text{for } k > 1, \\ JN^2 \mathrm{sgn}(k)\frac{\rho_N^{(\mathrm{eq})}(y_i)\rho_N^{(\mathrm{eq})}(y_j)}{|y_i - y_j|^k} & \text{for } -2 < k < 1. \end{cases} \tag{11}$$

For the diagonal term, it is interesting to note [45] that, for $-2 < k < 0$, the contribution from entropy is dominant whereas, for $k > 0$, the contribution from energy is dominant. The variance of $\bar{\Delta}$ is given by $\langle \delta\bar{\Delta}_i^2 \rangle = (\beta M)^{-1}_{ii}$. Assuming that the inverse of the dominant term of the matrix M [see Eqs. (28) and (29) in Ref. [45]] dictates the scaling of the variance we arrive at the conjecture in Eq. (5). We also compute the variance from a direct numerical inversion of the matrix M and as seen in Fig. 3 we find very good agreement with the conjecture in Eq. (5) for all k values. The deviation from the MC results are possibly due to statistical errors, slow equilibration and finite-size effects.

Conclusions.—In this Letter, we have studied the nearest neighbor gap statistics for a harmonically confined Riesz gas, in particular the variance and the distribution. The variance of the bulk gap is characterized by the exponent b_k for which we conjecture a form, Eq. (5), for the k dependence. We provided support for this through direct MC simulations, and numerics based on small fluctuations theories such as microscopic Hessian and quadratic field theory. We studied the normalized gap distribution, $P_N(s)$ and find a convergence, with N, for $k = 0^+, -1$. For other values of k, $P_N(s)$ does not converge with increasing N. This leads us to study $\tilde{s}_i$ [gap normalized by fluctuations, see Eq. (6)]. As summarized in Fig. 1, for $-2 < k < -1$ and $k > 1$ we found that the scaling form of $\tilde{P}_N(\tilde{s})$ is Gaussian while for all other k values, we find strong non-Gaussian behavior. In fact, for $-1 < k < 0$, we found that there is no convergence with N. Moreover in this regime, the fluctuations are of the same order as the mean, leading to the failure of the Hessian theory. Remarkably, the quadratic field theory approach is able to predict the expected scaling exponent even in this regime. It is worth reemphasizing that the analytical microscopic treatment of fluctuations is extremely difficult. We have proposed two different analytical approaches which are able to successfully capture the main features seen by direct simulations: (i) mapping between the microscopic variables and the coarse-grained macroscopic density field. This provides an enormous simplification for the otherwise intractable and highly nonlocal microscopic model. (ii) Hessian approximation which results in an all-to-all connected Harmonic network and provides a powerful tool for tackling long-ranged systems. Some interesting outstanding problems include understanding of the non-Gaussian behavior, including large deviations, of the gap distribution and its analytical derivation for special cases such as the 1DOCP ($k = -1$), CM ($k = 2$) and hard rods ($k \to \infty$).

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