Exact Anomalous Current Fluctuations in a Deterministic Interacting Model

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We analytically compute the full counting statistics of charge transfer in a classical automaton of interacting charged particles. Deriving a closed-form expression for the moment generating function with respect to a stationary equilibrium state, we employ asymptotic analysis to infer the structure of charge current fluctuations for a continuous range of timescales. The solution exhibits several unorthodox features. Most prominently, on the timescale of typical fluctuations the probability distribution of the integrated charge current in a stationary ensemble without bias is distinctly non-Gaussian despite diffusive behavior of dynamical charge susceptibility. While inducing a charge imbalance is enough to recover Gaussian fluctuations, we find that higher cumulants grow indefinitely in time with different exponents, implying singular scaled cumulants. We associate this phenomenon with the lack of a regularity condition on moment generating functions and the onset of a dynamical critical point. In effect, the scaled cumulant generating function does not, irrespectively of charge bias, represent a faithful generating function of the scaled cumulants, yet the associated Legendre dual yields the correct large-deviation rate function. Our findings hint at novel types of dynamical universality classes in deterministic many-body systems.

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Introduction.—The central limit theorem (CLT) is one of the bedrock accomplishments of probability theory. In the standard formulation, the CLT asserts that sums of random, independent, identically distributed variables converge towards the normal distribution when the sample size becomes large. Validity of the CLT however transcends uncorrelated processes, as it applies for macroscopic fluctuating observables in a wide array of dynamical processes in nature, including classical or quantum deterministic dynamical systems which typically exhibit highly nontrivial temporal correlations. It appears as though the CLT only ceases to hold away from equilibrium, i.e., upon breaking reversibility at the microscopic level.

Another hallmark result of statistical analysis is the large deviation principle (LDP) [1–3], stipulating that atypically large (rare) fluctuations are exponentially unlikely. In this regard, the main object of interest is a dynamical partition sum, the moment generating function (MGF) of the process also known as the full counting statistics (FCS). The rate function describing large deviations can be inferred from the logarithm of MGF. In spite of many important cases where MGF can be computed explicitly [4–14], there are virtually no explicit results available when it comes to genuinely *interacting* many-particle systems governed by deterministic and reversible microscopic evolution laws, whether in or out of equilibrium.

A recent numerical study [15] has found robust signature of anomalous dynamical fluctuations in the *integrable* Landau-Lifshitz ferromagnet, hinting that lack of ergodicity can play a pivotal role and may lead to inapplicability of the CLT. The precise microscopic mechanism leading to such an unconventional behavior has not been identified however. In this Letter, we report major progress on this question. We compute the exact FCS for a simple model of *interacting* charged degrees of freedom governed by a reversible deterministic equation of motion in a stationary equilibrium state. By deducing the late-time behavior of cumulants in a closed analytic form, we encounter two novel regimes of dynamical behavior characterized by divergent scaled cumulants of transferred charge.

Current fluctuations on typical and large scale.—We consider an infinitely extended deterministic dynamical many-body system with charge conservation. The time-integrated current density, $J(t) = \int_0^t d\tau j(\tau)$, where $j(\tau)$ is the charge-current density (at the origin) propagated by time τ, can be viewed as a dynamical fluctuating observable, measuring the net transferred charge between two halves of the system in the time interval t for each particular initial configuration. Obtaining the FCS of $J(t)$ amounts to computing the MGF [3,16]

$$G(\lambda|t) \equiv \langle e^{\lambda J(t)} \rangle \equiv \int dJ \mathcal{P}(J|t) e^{\lambda J}, \qquad (1)$$

corresponding to a Laplace transformation of the normalized (time-dependent) current distribution $\mathcal{P}(J|t)$ of $J(t)$, computed with respect to a stationary equilibrium measure. The formal variable $\lambda \in \mathbb{C}$ is commonly known as fugacity (or counting field). Owing to detailed balance we have the symmetry $\mathcal{P}(J|t) = \mathcal{P}(-J|t)$, implying $G(\lambda|t) = G(-\lambda|t)$. The associated cumulant generating function (CGF), $\log G(\lambda|t) = \sum_{n=0}^{\infty} c_n(t)\lambda^n/n!$, encodes an entire hierarchy of connected n-point dynamical correlation functions of time-integrated current densities

$$c_n(t) = \int_0^t \prod_{k=1}^{n} d\tau_k \langle j(\tau_n) j(\tau_{n-1}) \cdots j(\tau_1)\rangle^c. \qquad (2)$$

Assuming that, for asymptotically large times, the MGF grows as $G(\lambda|t) \asymp \exp[t^{\alpha} F(\lambda)]$, we introduce the *scaled* CGF

$$F(\lambda|t) \equiv t^{-\alpha} \log G(\lambda|t), \qquad F(\lambda) \equiv \lim_{t\to\infty} F(\lambda|t), \quad (3)$$

which may be viewed as a "dynamical free energy" of the process. We stress that exponent α is intrinsic to the system. In nonequilibrium processes, such as current-carrying steady-states arising in boundary-driven systems, or in stochastic systems with an intrinsic asymmetric drift, one finds ballistic scaling with $\alpha = 1$. For ergodic diffusive systems in equilibrium, such as the simple symmetric exclusion processes [6], one instead has $\alpha = 1/2$.

In this study, we focus exclusively on *equilibrium* states where, by detailed balance, odd moments vanish, $\langle [J(t)]^{2n-1}\rangle = 0$, while variance $\langle [J(t)]^2\rangle^c \sim t^{1/z}$ sets the scale of *typical* fluctuations $J(t) \sim t^{1/2z}$ governed by the dynamical exponent $z \geq 1$. Probabilities of atypically large fluctuations on a timescale t^{ζ}, in the range $1/2z < \zeta \leq 1$, can be quantified in terms of the rescaled current $\mathcal{J}(t) = t^{-\zeta} J(t)$. Assuming that the probabilities of measuring atypical values of the current $\mathcal{J}$ are exponentially suppressed, we anticipate that

$$\mathbb{P}[\mathcal{J}(t) = j] \asymp \exp[-t^{v(\zeta)} I_{\zeta}(j)], \qquad (4)$$

where $I_{\zeta}(j)$, with $j \equiv \lim_{t\to\infty} t^{-\zeta} J(t)$, is the LD *rate function*, and $v(\zeta)$ the associated "speed" that depends, in general, on the adjustable scale parameter ζ. In the context of LD theory, one is typically interested in the *largest* deviations corresponding to a ballistic scaling exponent $\zeta = 1$, with the associated speed $v(\zeta) = \zeta$. Fluctuations within the range of scales $1/2z < \zeta < 1$ are commonly referred to as "moderate deviations." Scaled CGF $F(\lambda)$ is not just a formal object: for $\lambda \in \mathbb{R}$ it represents a convex function that takes a distinguished role in LD theory. Provided $F(\lambda)$ is everywhere differentiable on its domain, the Gärtner-Ellis theorem [3] states that its Legendre transform, $I(j) = \max_{\lambda}[\lambda j - F(\lambda)]$ provides the rate function $I(j) \equiv I_{\zeta=1}(j)$, quantifying probabilities of exponentially rare events.

Now we touch a delicate but pivotal point. In the literature devoted to applications of LD theory, it is commonly understood that $F(\lambda)$ provides the generating series for the *scaled* cumulants $s_n = \lim_{t\to\infty} t^{-\alpha} c_n(t)$ when expanded around $\lambda = 0$, $s_n = (d/d\lambda)^n F(\lambda)|_{\lambda=0}$. At the technical level, however, faithfulness of $F(\lambda)$ hinges on interchangeabilty of the limit $t \to \infty$ and the series expansion of $F(\lambda|t)$. Indeed, there is no reason to *a priori* assume (i) that the sequence of analytic function $F(\lambda|t)$ converges necessarily to an analytic limiting function $F(\lambda)$ nor (ii) that, assuming $F(\lambda)$ is analytic, its expansion coefficients yield scaled cumulants s_n. As we argue next, not only are both of these scenarios viable, but they lead to certain profound physical consequences.

Evading the central limit theorem.—Before considering our model, we explain how robustness of the CLT is in fact deeply rooted in analytic properties of $G(\lambda|t)$. Whenever there exists a disc $D(r)$ of radius $r > 0$ centered at $\lambda = 0$, such that $F(\lambda|t)$ are uniformly bounded on $D(r)$ for all times t, and $F(\lambda)$ exists, then—proven in a theorem by Bryc [17] assuming $\alpha = 1$ (see also Ref. [18])—the central limit theorem holds as a consequence of *finite* scaled cumulants s_n (ensured by Vitali's convergence theorem). To put it simply, an irregular (i.e., nongeneric) behavior can only be achieved when Bryc's analyticity conditions are not fulfilled.

With this in mind, we now imagine a system of interacting ballistically propagating quasiparticles, where it is expected (e.g., based on generalized hydrodynamics [13,19]) that MGF, cf. Eq. (1), grows asymptotically with dynamical exponent $\alpha = 1$. If the model admits a $\mathbb{Z}_2$ parity symmetry (e.g. charge conjugation) under which the charge current flips sign then, by analogy with quantum spin chains [20–22], the charge Drude weight (in a parity-invariant equilibrium state) identically vanishes, and charge transport is governed by a *subballsitic* dynamical exponent $z > 1$. Recalling that $c_2(t) \sim t^{1/z}$, in this scenario scaled cumulants s_n *cannot* correspond to coefficients of $F(\lambda)$ (in fact, s_n need not even exist). Interestingly, however, as we are about to demonstrate next, even ballsitic charge transport (i.e., $\alpha = z = 1$) does not by itself guarantee faithfulness of $F(\lambda)$. To substantiate our claims, we explicitly compute the FCS for the charge-current fluctuations in a classical deterministic cellular automaton of hard-core interacting charged particles introduced and studied earlier in Refs. [23–25]. We first give a closed-from solution for the FCS and subsequently discuss its most salient features.

Cellular automaton with solvable FCS.—We consider a reversible cellular automaton introduced in Ref. [23], realized as a space-time circuit composed of elementary two-body maps. Each lattice site $(\ell, t) \in \mathbb{Z} \times \mathbb{Z}$ is occupied with one of three "species" of particles taking values in $\mathcal{Q} = \{\emptyset, +, -\}$, representing a particle of positive $(+)$ or negative charge $(-)$, and a charge-neutral vacancy $(\emptyset)$.

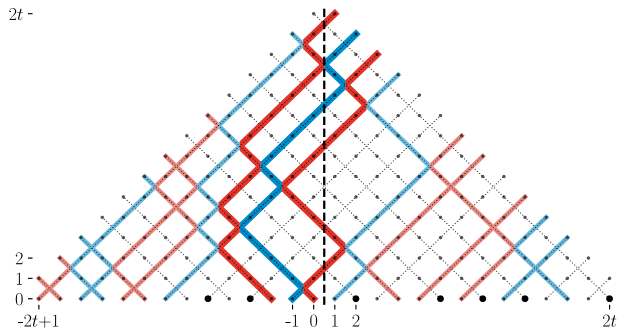

FIG. 1. Coordinate frame (time vertical, space horizontal) of a deterministic charged hard-core lattice gas (red: + particles, blue: − particles, while thin black lines indicate vacancies). Example of a light cone (pyramid) section of a typical trajectory, for which initial data on a saw of $4t$ subsequent links uniquely determine the transport through the midpoint (dashed line) for all times times from 0 to $2t$.

The local two-particle propagator $\Phi: \mathcal{Q}\times\mathcal{Q}\to\mathcal{Q}\times\mathcal{Q}$ acts simply as a permutation (i.e., no interaction) whenever the sum of charges is nonzero, $(\emptyset, q)\leftrightarrow(q,\emptyset)$ for $q\in\{\emptyset,+,-\}$, while oppositely charged particles repel each other and thus retain their initial positions, $(q,q')\leftrightarrow(q,q')$ for $q,q'\in\{\pm\}$. The dynamics consists of brickwork application of Φ, as depicted in Fig. 1.

Dynamical properties of such an automaton are distinctly nonergodic; a phase-space trajectory cannot explore the entire phase space: besides conservation of total charge, it possesses an exponentially large number of conserved quantities (related trivially to the fact that vacancies are inert). The exact charge Drude weight and the diffusion constant have been computed in Ref. [24].

Our central result is an explicit computation of the MGF which is outlined below. The ensemble average in $G(\lambda|t)$ can be computed in a two-stage "nested" way: for every "frozen" sublattice $\Sigma\subset\mathbb{Z}$ of occupied sites at $t=0$ we perform an average over all possible charge configurations $\{q_\ell\}$, and subsequently average over all sublattices Σ. Let $\Lambda_\pm\subset\Sigma$ denote the sublattices of charged particles at $t=0$ that move from the right to the left half (respectively, vice versa) during time interval $[0,2t]$. The integrated current through the origin corresponds to the total transferred charge, $J(t)=\sum_{\ell\in\Lambda_-}q_\ell-\sum_{\ell\in\Lambda_+}q_\ell$. We make the following observations: (a) particle worldlines cannot cross each other, implying in particular that at most one of the subsets $\Lambda_\pm$ can be nonempty, (b) the signed number of worldlines crossing the origin is given by the difference between the number of vacancies passing through the origin from the left or right up to time $2t$. Introducing a separable invariant probability measure $\mathbb{P}(\{q_\ell\})=\prod_\ell p(q_\ell)$, with $p(\pm)=\frac{1}{2}\rho(1\pm b)$, $p(\emptyset)=\bar{\rho}=1-\rho$ corresponding to densities of particles and vacancies, we derived (see Ref. [26] for details) the following exact double-sum representation for the MGF:

$$G(\lambda|t)=\sum_{l,r=0}^{t}\binom{t}{l}\binom{t}{r}\bar{\rho}^{l+r}\rho^{2t-l-r}\prod_{\varepsilon=\pm}[\mu_\varepsilon(\lambda)]^{d_\varepsilon(l,r)}, \quad (5)$$

where $\mu_\pm(\lambda)\equiv\cosh(\lambda)\mp b\sinh(\lambda)$, $d_\varepsilon(l,r)\equiv[|l-r|+\varepsilon(l-r)]/2$, and $b\in[0,1)$ is the "charge bias." In the following, we systematically carry out an asymptotic analysis of $G(\lambda|t)$.

Dynamical free energy and cumulants.—Performing asymptotic analysis on $F(\lambda|t)$, see Eq. (3), for $\alpha=1$, $\lambda\in\mathbb{R}$, we inferred the following scaled CGF

$$F(\lambda)=\log[1+\Delta^2(\mu_b+\mu_b^{-1}-2)], \quad (6)$$

with $\mu_b\equiv\cosh\lambda+|b|\sinh|\lambda|$ and $\Delta^2\equiv\rho(1-\rho)\in[0,1/4]$. We stress, importantly, that $F(\lambda)$ does not provide (irrespectively of b) the generating series for scaled cumulants s_n. As already announced, we find (regardless of b) that all scaled cumulants $s_{n>2}$ are *singular*. We deduced the following asymptotic behavior [30]

$$c^{[b]}_{2n>2}(t)\sim t^{n-1/2}, \qquad c^{[0]}_{2n}(t)\sim t^{n/2}, \quad (7)$$

and succeeded in obtaining a compact generator of cumulant asymptotics, $c_n^{[b\geq 0]}(t)\asymp(d/d\lambda)^n\mathcal{F}^{[b\geq 0]}(\lambda)|_{\lambda=0}$ with $\exp[\mathcal{F}^{[b\geq 0]}(\lambda)]=\sum_{\varepsilon=\pm}\exp(a_\varepsilon^2)[1+\mathrm{erf}(a_\varepsilon)]$, where $a_\pm\equiv t^{1/2}\Delta[\frac{1}{2}(1-b^2)\lambda^2\pm b\lambda]$ for $b\in[0,1)$.

Typical fluctuations and CLT.—We first examine fluctuations of $J(t)$ on the "typical timescale," associated with scaling exponent $\zeta_{\rm typ}=1/2z$. To this end, we explicitly compute cumulants $\kappa_n(t)$ characterizing the time-dependent distribution $\mathcal{P}_{1/2z}(\mathcal{J}|t)$. Recalling a theorem by Marcinkiewicz [31], stating that the Gaussian distribution is the unique distribution with finitely many nonzero cumulants, the CLT applies if and only if $\lim_{t\to\infty}\kappa_n(t)=0$ for all $n>2$. From the scaling relation $\kappa_n(t)=t^{-n/2z}c_n(t)$ we readily deduce the scalings $\kappa^{[b]}_{n>2}(t)\sim t^{-1/2}$ and $\kappa^{[0]}_{n>2}(t)\sim t^0$. For a finite bias $b>0$, all the higher cumulants $\kappa^{[b]}_{n>2}(t)$ of the distribution $\mathcal{P}^{[b]}_{1/2z}(\mathcal{J}|t)$ decay with time, yielding a Gaussian asymptotic profile, $\mathcal{P}_{\rm typ}(j)\equiv\lim_{t\to\infty}\mathcal{P}_{\zeta_{\rm typ}}(\mathcal{J}=j|t)$, where $\zeta_{\rm typ}=1/2$

$$\mathcal{P}^{[b]}_{\rm typ}(j)=\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left[-\frac{j^2}{2\sigma^2}\right], \qquad \sigma^2=2(b\Delta)^2. \quad (8)$$

Upon switching off the bias, $b=0$, typical fluctuations occur on a scale $\zeta_{\rm typ}=1/4$, where we inferred a *non-Gaussian* profile characterized by finite cumulants $\kappa_n^{[0]}=\lim_{t\to\infty}\kappa_n^{[0]}(t)<\infty$, with the following integral representation [26] (see Fig. 2)

$$\mathcal{P}^{[0]}_{\rm typ}(j)=\frac{1}{\sqrt{2\pi}\Delta}\int_{\mathbb{R}}du\exp\left[-\left(\frac{u^2}{2\Delta}\right)^2-\frac{j^2}{2u^2}\right]. \quad (9)$$

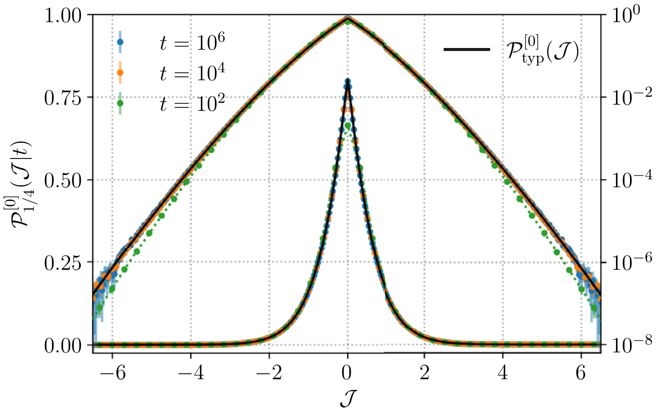

FIG. 2. Rescaled current distribution for unbiased $b = 0$, half-filled $\rho = 0.5$ charged hard-core lattice gas in normal or log scale (lower or upper data). Colored dashed lines show convergence of exact distributions $\mathcal{P}^{[0]}_{1/4}(\mathcal{J}|t)$ to the asymptotic form (9) (solid black line). Estimated current distribution (dots), agrees with exact solution within statistical errors ($N_{\rm sample} = 10^9$).

Explicitly, $\mathcal{P}^{[0]}_{\rm typ}(j) = (2\Delta)^{-1/2} M_{1/4}(\sqrt{2/\Delta}|j|)$, where $M_\nu(x) \equiv \sum_{k=0}^{\infty}(-x)^k/\{k!\Gamma[(1-\nu)-\nu k]\}$ is the M-Wright function [32]. The associated MGF reads explicitly $\mathcal{G}^{[0]}(\eta) = e^{w^4}(1+\mathrm{erf}\, w^2) = E_{1/2}(w^2)$, where $w^2 = \eta^2\Delta/2$ and $E_\nu(x) = \sum_{k=0}^{\infty} x^k/\Gamma(1+\nu k)$ is the Mittag-Leffler function [note that $\mathcal{G}^{[0]}(\eta) = \exp[\mathcal{F}^{[0]}(\eta t^{-1/4})]$].

Large deviation principle.—What remains is to characterize fluctuations on the largest scale $\zeta = 1$. Since $F(\lambda)$ is strictly convex and differentiable in its entire domain $\lambda \in \mathbb{R}$ for all values of b, the Gärtner-Ellis theorem ensures that the Legendre transform is involutive [3], and that $I(j)$ corresponds to a unique strictly convex and differentiable LD rate function $I(j)$ [whose Legendre transform yields back $F(\lambda)$].

In the general case with finite bias, the leading-order behavior near $\lambda = 0$ reads $F^{[b]}(\lambda) = (b\Delta)\lambda^2 + \mathcal{O}(|\lambda|^3)$, implying a quadratic rate function for perturbatively small j, $I^{[b]}(j) = (j/2b\Delta)^2 + \mathcal{O}(j^4)$. At $b = 0$, the behavior is markedly different; owing to the absence of the leading order terms in SCGF, $F^{[0]}(\lambda) = (\Delta/2)^2\lambda^4 + \mathcal{O}(\lambda^6)$, we find $I^{[0]}(j) = (3/4)(j^2/\Delta)^{2/3} + \mathcal{O}(j^2)$. Unlike $I^{[b]}(j)$, $I^{[0]}(j)$ is not twice differentiable at $j = 0$. On the other hand, at large $|\lambda|$ we have $F(\lambda) \sim |\lambda|$, implying that j is confined within the compact interval $[-1, 1]$, cf. Refs. [8,33]. This is a direct manifestation of causality: owing to the fact that charges propagate with unit velocity and interact locally, the maximal transferred charge in a time interval t is upper bounded by t. The near-horizon behavior can be found analytically [26].

We have also computed a family of rate functions associated with the "moderate deviation principle" for a continuous range of timescales $(1/2z < \zeta < 1)$ [26].

Singular scaled cumulants and criticality.—Lack of analyticity of scaled CGF $F(\lambda)$ is often found in Markovian stochastic systems driven away from equilibrium by means of boundary reservoirs, where it is attributed to a first-order dynamical phase transition (DPT), see Refs. [33–39]. We are not aware of similar dynamical features taking place in equilibrium. Despite that, we can observe certain conspicuous similarities.

Significance of divergent scaled cumulants is most transparently discussed in the complex fugacity plane in the framework of the Lee-Yang theory [40] of phase transitions [41,42]. Presently, we find that $\mathcal{O}(t)$ Lee-Yang zeros of $G(\lambda|t)$ condense along certain contours in the λ plane. By fourfold symmetry, there are four zeros of $G(\lambda|t)$ closest to the origin $\lambda = 0$, at a distance $r(t)$, corresponding to the convergence radius of a complex Taylor series $\log G(\lambda|t) = \sum_n c_n(t)\lambda^n/n!$. Applying the standard analysis (see Refs. [35,43–45]), and using the known asymptotics of $c_n(t)$, we deduce the scaling $r^{[b]}(t) \sim t^{-1/2}$, $r^{[0]}(t) \sim t^{-1/4}$ (see Ref. [26] for details). The vanishing convergence radius, $r_\infty \equiv \lim_{t\to\infty} r(t) = 0$, signifies that $\lambda_c = 0$ is a *dynamical critical point*. Based on this, one might draw an incorrect conclusion that scaled CGF $F(\lambda)$ develops a nonanalyticity at the critical point. In reality, only $F^{[b]}(\lambda)$ is found to be nonanalytic, owing to the discontinuities in its odd-order derivatives at the origin. Conversely, $F^{[0]}(\lambda)$, which depends on $\mu_0(\lambda) = \cosh\lambda$ and is derived via Eq. (3), represents a real analytic function; while its expansion coefficients are unrelated to cumulants, $F^{[0]}(\lambda)$ is the Legendre dual of the LD rate function $I^{[0]}(j)$.

In contrast to first-order DPTs seen in out-of-equilibrium stochastic processes [where both the scaled CGF $F(\lambda)$ and LD rate function exhibit a cusp], we encounter, in the biased case $b > 0$, a cusp only in the second derivative, $(d/d\lambda)^2 F^{[b]}(\lambda)$. This indicates, at a formal level, a DPT of *third order* at $\lambda = \lambda_c$, with the value at the cusp being the dynamical charge-current susceptibility, $s_2 = \lim_{t\to\infty} t^{-1} c_2(t) = \int_0^t d\tau \langle j(\tau) j(0)\rangle^c$. Note that result (5) can be reinterpreted as a Curie-Weiss like partition sum [46], where b plays the role of a magnetic field with a line of first order phase transitions at $b_c = 0$, ending at $\lambda = \lambda_c$.

The Lee-Yang theory permits us to establish that divergent scaled cumulants, with an extra assumption that $c_n(t)/c_{n+2}(t) \sim t^{-\gamma_n}$ with $\lim_{n\to\infty}\gamma_n > 0$, imply $r_\infty = 0$ (i.e., $\lambda_c = 0$), and vice versa. In this scenario "Bryc's regularity conditions" ensuring applicability of CLT are violated. Indeed, in the present model Lebesgue's criterion of dominated convergence is not satisfied by the time sequence of *real* analytic functions $F(\lambda|t)$, irrespectively, of bias b. One should, however, be cautious, as neither divergent s_n nor nonanalytic $F(\lambda)$ automatically imply a departure from Gaussianity. The fate of $\mathcal{P}_{\rm typ}(j)$ is instead predicated on the asymptotic scaling of the higher cumulants $c_n(t) = (d/d\lambda)^n \log G(\lambda|t)|_{\lambda=0}$: writing $c_{n>2}(t) \sim t^{\nu_n}$, one finds a Gaussian $\mathcal{P}_{\rm typ}(j)$ *if and only if* the exponents ν_n

can be upper bounded by threshold exponents $\nu_n < n/2z$ and is otherwise *violated*.

Fluctuations of particle current.—It is instructive to add that fluctuations of the total transferred particle number behave regularly. By disregarding internal charge degrees of freedom, the model reduces to free ballistically propagating particles with $z = \alpha = 1$. Setting accordingly $b = 1$, the MGF can be easily summed up explicitly, yielding an analytical (and faithful) scaled CGF of the form $F^{[1]}(\lambda) = \log\{1 + 2\Delta^2[\cosh(\lambda) - 1]\}$, expectedly recovering the celebrated Levitov-Lesovik formula [47,48] (here specialized to a single particle channel with perfect transmission at "infinite temperature").

Conclusion.—We have examined the structure of charge current fluctuations in a simple classical deterministic model of interacting charged particles. We derived an exact closed-form expression for the MGF in equilibrium at arbitrary background charge density, encoding the FCS of transferred charge. By performing asymptotic analysis, we deduced a number of remarkable properties: (I) in the presence of charge bias, fluctuations of the integrated current density on the typical timescale are described by a Gaussian distribution; at vanishing bias we instead discover a distinctly non-Gaussian profile, thereby establishing that the CLT can be evaded despite detailed balance; (II) cumulants $c_n(t)$ exhibit, irrespectively of bias, indefinite temporal growth with *distinct* algebraic exponents, implying *divergent* scaled cumulants; (III) the scaled CGF yields, via the Legendre transform, a bona fide large-deviation rate function; and (IV) for finite charge bias, the scaled CGF is a nonanalytic function of fugacity λ, with a discontinuous third derivative at the critical point $\lambda_c = 0$.

Singular behavior of scaled cumulants, $s_n(t) \to \infty$, may be suggestively interpreted as lack of "sufficiently strong" temporal clustering associated with a hierarchy of dynamical multipoint current-density correlations along a temporal seam attached to a fixed point in space, see Eq. (2). Such an anomalous structure is a manifestation of enhanced memory effects that are, like other unconventional transport phenomena such as finite Drude weights [22,49,50] and charge superdiffusion [51–53], likely inherently tied to stable (quasi)particles that propagate ballistically though the system. This expectation is further corroborated by a recent numerical study of the lattice Landau-Lifshitz model [15], widely viewed as an archetypicical completely integrable system [54,55], which similarly displays divergent scaled cumulants and absence of Gaussianity in the presence of particle-hole symmetry. Therefore, despite exhibiting diffusive charge dynamics at the level of the dynamical charge correlations, integrable systems of this sort do not belong to the same universality class as generic (i.e., ergodic) diffusive systems such as, for example, the SSEP (whose current fluctuations have been computed analytically in Ref. [6], complying with the predictions of the MFT [56–58]).

We conclude by pointing out several most pressing issues. Our expectation is that a similar unconventional behavior of current fluctuations arise in many other classical and quantum solitonic systems and related integrable modes (including similar superintegrable automata [59,60]). We specifically have in mind the distinguished conserved charges associated with manifest discrete or continuous symmetries [52], while other local conservation laws presumably behave in a regular way. Second, while nonanalytic scaled CGF $F^{[b]}(\lambda)$ is conventionally understood as a precursor of a dynamical phase transition, we currently lack a more insightful physical interpretation of the encountered third-order critical point. It is also important to investigate whether there is any degree of universality in the non-Gaussian probability function of the charge current at the typical scale. Last but not least, our model offers an opportunity to translate many of these exciting questions into the realm of nonequilibrium physics by either studying evolution of inhomogeneous initial profiles or mesoscopic driven models.

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