Editors' Suggestion

Self-Healing of Non-Hermitian Topological Skin Modes

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A unique feature of non-Hermitian (NH) systems is the NH skin effect, i.e., the edge localization of an extensive number of bulk-band eigenstates in a lattice with open or semi-infinite boundaries. Unlike extended Bloch waves in Hermitian systems, the skin modes are normalizable eigenstates of the Hamiltonian that originate from the intrinsic non-Hermitian point-gap topology of the Bloch band energy spectra. Here, we unravel a fascinating property of NH skin modes, namely self-healing, i.e., the ability to self-reconstruct their shape after being scattered off by a space-time potential.

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Introduction.—Self-healing is the fantastic property of certain classical or quantum (matter) waves to reconstruct their original shape after being scattered off by a potential (an obstacle) [1–3]. Such a special property is rather generally shared by diffraction-free and, thus, non-normalizable (delocalized) states of the underlying wave equation. Important examples include Bessel waves of the Helmholtz equation [1,2,4] and self-accelerating (Airy) waves of the Schrödinger equation [3,5,6]. Self-healing has been demonstrated for optical [1,3,7–10], acoustic [11–14], and matter waves [15,16], with a variety of applications in different areas of science such as in microscopy and biomedical imaging [17–19], material processing [20], particle manipulation [21,22], sensing [8–10], and quantum communications [23]. However, in a norm-preserving (Hermitian) system, any normalizable (bound) wave function cannot be strictly self-healing. An interesting and open question is whether infinitely many self-healing normalizable waves can exist in non-Hermitian (NH) systems [24]. An important class of such systems is provided by NH lattices, where the role of topology and its far-reaching physical consequences are attracting an enormous interest [25–100] (for a recent review, see [79]). A unique feature of NH lattices is the skin effect [29–31,33,55], i.e., the localization of an extensive number of bulk eigenstates at the edges under open (OBC) or semi-infinite (SIBC) boundary conditions. The localized skin modes replace the extended Bloch waves of Hermitian lattices and their origin can be traced back to the nontrivial point-gap topology of the bulk energy spectra under periodic boundary conditions (PBC), thus, establishing a bulk-edge correspondence for skin modes [27,55].

In this Letter, we unveil that topological skin edge modes share the fascinating property of being self-healing waves. Like non-normalizable diffraction-free waves in Hermitian systems, in one-dimensional (1D) NH lattices with SIBC, there are infinitely many localized (normalizable) topological skin edge states that can reconstruct their shape after being scattered off by a rather arbitrary space-time potential.

Wave self-healing.—Let us consider the time-dependent dynamics of a wave function $|\psi(t)\rangle$ described by the Schrödinger-like wave equation

$$i\frac{d}{dt}|\psi\rangle = (\hat{H}+\hat{V})|\psi\rangle, \tag{1}$$

where $\hat{H}$ is the time-independent Hamiltonian of the system, which is assumed rather generally NH, and $\hat{V} = \hat{V}(t)$ describes a space-time local scattering potential (the "obstacle"), which vanishes for $t > T$ and with compact support in space [Fig. 1(a)]. At initial time $t = 0$, the system is prepared in the state $|\psi(0)\rangle = |\phi(0)\rangle$, and let $|\phi(t)\rangle$ be the evolved wave function in the absence of the scattering potential $\hat{V}$, i.e., $|\phi(t)\rangle = \exp(-i\hat{H}t)|\phi(0)\rangle$. Clearly, the presence of the scattering potential destroys the unperturbed evolution of the wave function, so that after interaction with the potential, i.e., for $t > T$, $|\psi(t)\rangle$ can largely deviate for ever from the unperturbed solution $|\phi(t)\rangle$. The wave function $|\phi(t)\rangle$ is dubbed self-healing if the deviation $|\xi(t)\rangle \equiv |\psi(t)\rangle - |\phi(t)\rangle$ is asymptotically much smaller than $|\phi(t)\rangle$ as $t \to \infty$ regardless of the form of $\hat{V}$, i.e., provided that [Fig. 1(a)] $\lim_{t\to\infty} \epsilon(t) = 0$, where

$$\epsilon(t) = \frac{\langle\xi(t)|\xi(t)\rangle}{\langle\phi(t)|\phi(t)\rangle}. \tag{2}$$

Note that the above condition corresponds to $\|\tilde{\psi}(t) - \tilde{\phi}(t)\| \to 0$ for the normalized wave functions $|\tilde{\psi}(t)\rangle = |\psi(t)\rangle/\|\psi(t)\|$ and $|\tilde{\phi}(t)\rangle = |\phi(t)\rangle/\|\phi(t)\|$. Clearly, in a Hermitian system owing to norm conservation, any normalizable wave function is not strictly self-healing, though it can approximate an extended (non-normalizable) wave function at some extent [6]. For example, for a freely

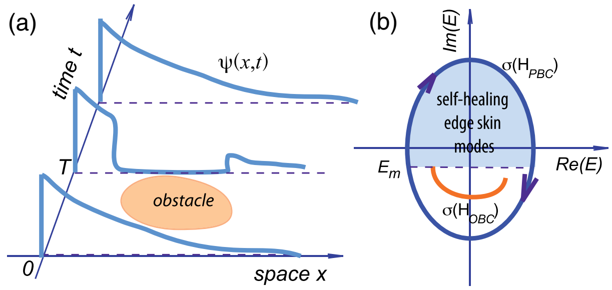

FIG. 1. (a) Sketch of wave function propagation and self-healing property. After being scattered off by a space-time localized potential (the obstacle), the wave function $\psi(x,t)$ can reconstruct its shape, as if the scattering potential was not present. (b) In a NH semi-infinite lattice with a left boundary, any topological edge skin mode at energy E with $W(E)<0$ and $\mathrm{Im}(E)>E_m$ (shaded area in the figure) is a self-healing wave function. In the figure, the outer closed loop describes the energy spectrum $\sigma(H_{\mathrm{PBC}})$, whereas the inner open arc is the energy spectrum $\sigma(H_{\mathrm{OBC}})$.

moving quantum particle in a one-dimensional space, $\hat{H}=-\partial^2/\partial x^2$, the self-accelerating Airy solutions to the time-dependent Schrödinger equation [5] are non-normalizable self-healing waves [3]. Other non-normalizable self-healing modes include Bessel waves, parabolic cylinder waves, Weber and Mathieu beams, Bloch surface waves, and others (see, e.g., [2,9,101]). However, in a NH system, propagation-invariant normalizable waves can be found [102].

Energy spectra, topological skin modes, and the bulk-edge correspondence.—We consider a one-dimensional NH lattice with short-range hopping with Hamiltonian $\hat{H}$ in physical space given by

$$\hat{H}=\sum_{n,l=1}^{N} H_{n,l}|n\rangle\langle l|, \tag{3}$$

where $H_{n,l}$ is a $N\times N$ banded matrix and N is the number of lattice sites. We indicate by H_{PBC} and H_{OBC} the $N\times N$ matrix Hamiltonians under PBC and OBC, respectively, in the large (thermodynamic) N limit. For a single-band model, H_{OBC} is a banded Toeplitz matrix, i.e., $(H_{\mathrm{OBC}})_{n,l}=t_{n-l}$ with $t_n=0$ for $n>s$ and $n<-r$ ($t_{-r},t_s\neq 0$), where $t_{\pm l}$ are the left or right hopping amplitudes among sites distant $\pm l$ in the lattice and $r,s\geq 1$ are the largest orders of left or right hopping. H_{PBC} is a circulant matrix with the same form as H_{OBC}, except for the top right and bottom left corners of the matrix. Finally, we indicate by H_{SIBC} the infinite-dimensional matrix Hamiltonian under SIBC with a boundary on the left but not on the right, i.e., $(H_{\mathrm{SIBC}})_{n,l}=t_{n-l}$ for $n,l=1,2,3,\ldots$. The central result in the band theory of NH systems is that the energy spectra $\sigma(H_{\mathrm{PBC}})$, $\sigma(H_{\mathrm{OBC}})$ and $\sigma(H_{\mathrm{SIBC}})$ are rather generally distinct, which implies the emergence of the NH skin effect, topological NH edge states, and the need for a non-Bloch band theory. These results, studied in several recent Letters [30,38,40,55,59,60] and briefly reviewed in Sec. 1 of [103], are basically rooted in the spectral theory of non-self-adjoint Toeplitz matrices and operators [104–107]. Specifically, for a single-band lattice: (i) $\sigma(H_{\mathrm{PBC}})$ is a closed loop in a complex energy plane described by the Bloch Hamiltonian $H(k)=P[\beta=\exp(ik)]$, where $P(\beta)=\sum_{l=-r}^{s} t_l\beta^l$ is the Laurent polynomial associated with the Toeplitz matrix and $-\pi\leq k<\pi$ is the Bloch wave number. (ii) $\sigma(H_{OBC})$ is the set of complex energies $E=P(\beta)$, where β varies on the generalized Brillouin zone (GBZ) C_β. $\sigma(H_{\mathrm{OBC}})$ is always topological trivial in terms of a point gap [55]. The definition and calculation of C_β is discussed in [30,38,59,60], and briefly reviewed in [103]. (iii) $\sigma(H_{\mathrm{SIBC}})=\sigma(H_{\mathrm{PBC}})\cup B$, where B is the interior of the PBC energy spectrum loop such that for $E\in B$ the winding number $W(E)$, defined by

$$W(E)=\frac{1}{2\pi i}\int_{-\pi}^{\pi} dk\frac{d}{dk}\log\det\{H(k)-E\}, \tag{4}$$

is nonvanishing. If $W(E)<0$, then E is an eigenvalue of H_{SIBC} of multiplicity $|W(E)|$, and the corresponding (right) eigenvectors are exponentially localized at the left edge. Such a result provides a bulk-boundary correspondence for NH systems, relating the appearance of skin edge states in a semi-infinite lattice to the topology of the PBC energy spectrum [55].

Self-healing of topological skin modes.—The central result of this Letter is that, in NH lattices displaying the NH skin effect, there are infinitely many skin edge modes that are self-healing. Specifically, let us consider a one-dimensional NH lattice with SIBC, with a boundary on the left but no boundary on the right, and with a GBZ C_β that is, at least partly, external to the unit circle (to ensure the existence of left-edge skin states). The local scattering potential is assumed to have a compact support both in space and time, i.e., $\hat{V}=V_n(t)|n\rangle\langle n|$ with $V_n(t)=0$ for $t>T$ and $n>L$. Let us indicate by E_{m_1} the largest imaginary part of the energies in the set $\sigma(H_{\mathrm{OBC}})$, i.e., $E_{m1}=\max_{\beta\in C_\beta}\mathrm{Im}\{P(\beta)\}$; E_{m_2} the largest imaginary part of the energies E in the set B defined by $\{E\in B|W(E)>0\}$; and $E_m=\max(E_{m_1},E_{m_2})$. Note that the set B is empty if the GBZ is entirely external to the unit circle $|\beta|=1$, i.e., if there are not Bloch points [38]; in this case, one should assume $E_m=E_{m1}$ [as in Fig. 1(b)]. The following theorem can then be proven, which is illustrated in Fig. 1: any topological skin edge state $|\phi(t)\rangle=|\phi_0\rangle\exp(-iE_0t)$ with energy E_0 and $W(E_0)<0$ is self healing if and only if $\mathrm{Im}(E_0)>E_m$. A simple corollary of this theorem is that any topological skin edge state belonging to H_{OBC} is not self-healing, because, in this case, one has $\mathrm{Im}(E_0)\leq E_{m1}\leq E_m$.

Here, we provide a sketch of the proof of the theorem (technical details are given in [103]). Let us indicate by $|\psi(t)\rangle$ the wave function satisfying Eq. (1) with the initial condition $|\psi(0)\rangle = |\phi_0\rangle$, and let $|\xi(t)\rangle = |\psi(t)\rangle - |\phi(t)\rangle$ be the deviation of the wave function $|\psi(t)\rangle$ from the unperturbed (skin edge eigenstate) solution. The proof consists of two main steps. In the first step, one shows that, after interaction with the scatting potential, the deviation $\xi_n(T) = \langle n|\xi(T)\rangle$ vanishes as $n \to \infty$ faster than exponentially, i.e., for any $h > 0$ one has $\lim_{n\to\infty} \xi_n(T)\exp(hn) = 0$. Physically, this result stems from the fact that, since the hopping in the lattice is finite (short range), and the scattering potential has a limited support in space ($V_n = 0$ for $n > L$), the speed of excitation spreading in the lattice arising from the interaction with the scattering potential is bounded (according to the Lieb-Robinson bound [27]), and thus, after interaction, $\xi_n(T)$ remains basically unperturbed, i.e., very close to zero, for large enough n. The fast decay of ξ_n with n mathematically justified by the asymptotic form of the exponential of a banded matrix [108] (Sec. 2 of [103]). Then, let us indicate by $|\beta\rangle$ the set of eigenfunctions of H_{OBC} (skin modes) with energy $P(\beta)$ belonging to $\sigma(H_{\mathrm{OBC}})$, i.e., $H_{\mathrm{OBC}}|\beta\rangle = P(\beta)|\beta\rangle$ with $\beta \in C_\beta$. Note that $|\beta\rangle$ is also an eigenstate of H_{SIBC} when $|\beta| > 1$ in the $N \to \infty$ limit. For large n, $\langle n|\beta\rangle$ behaves as $\langle n|\beta\rangle \sim \beta^{-n}[1 + A_\beta \exp(-i\theta_\beta n)]$ with some β-dependent constants A_β and θ_β. Since $|\xi(T)\rangle$ is bounded with a localization higher than any exponential, one can decompose $|\xi(T)\rangle$ as a superposition (integral) of $|\beta\rangle$ skin states, i.e., one can write (Sec. 1 of [103]) $|\xi(T)\rangle = \oint_{C_\beta} d\beta F(\beta)|\beta\rangle$ with $F(\beta)$ nonsingular on C_β. Since $\hat{V} = 0$ for $t > T$, after the scattering event, the wave function $|\xi(t)\rangle$ evolves according to the Schrödinger equation $i\partial_t|\xi\rangle = \hat{H}_{\mathrm{SIBC}}|\xi\rangle$, so that for $t > T$, one has $|\xi(t)\rangle = \oint_{C_\beta} d\beta F(\beta)\times \exp[-iP(\beta)(t-T)]|\beta\rangle$. The second step is to calculate the growth rate of $\|\xi(t)\|^2 = \langle\xi(t)|\xi(t)\rangle$. To this aim, one has to distinguish two cases (Sec. 3 of [103]). If C_β is entirely external to the unit circle, i.e., $|\beta| > 1$ for any $\beta \in C_\beta$, the growth rate of $\|\xi(t)\|$ is $E_{m1} = \max_{\beta\in C_\beta} \mathrm{Im}[P(\beta)]$, which is attained at the value $\beta_s \in C_\beta$ corresponding to the most unstable saddle point of $P(\beta)$. Since $\|\phi(t)\|$ grows in time as $\sim \exp[\mathrm{Im}(E_0)t]$, one has $\lim_{t\to\infty} \epsilon(t) = 0$ if and only if $\mathrm{Im}(E_0) > E_m$, where $\epsilon(t)$ is defined by Eq. (2) and $E_m = E_{m_1}$. On the other hand, if a portion of C_β is internal to the unit circle, the asymptotic analysis shows that the growth rate of $\|\xi(t)\|$ is the larger number between E_{m_1} and E_{m_2}, where E_{m_2} is the largest imaginary part of energies in the set B [103]. This proves the theorem. ■

As an illustrative example, let us consider a lattice with nearest- and next-nearest-neighbor hopping ($r = s = 2$). Figure 2 shows the energy spectra $\sigma(H_{\mathrm{PBC}})$, $\sigma(H_{\mathrm{OBC}})$ and $\sigma(H_{\mathrm{SIBC}})$ and corresponding GBZ, which is entirely external to the unit circle with $E_m = E_{m_1} \simeq 0.5$. In the wide light shaded region of Fig. 2(a), for each complex energy E, there is a single topological skin edge state ($W = -1$), while when E is internal to the narrow dark shaded region encircling the origin, there are two linearly independent skin edge states ($W = -2$). To show the self-healing property of skin edge states, we consider a strongly absorbing potential $V_n(t) = -10i$ which is nonvanishing in the interval $2 < t < 4$ and in the spatial region $1 \leq n \leq L = 10$. The initial state $|\phi_0\rangle$ is chosen to be a skin edge state with an energy E_0 in the stable ($\mathrm{Im}(E_0) > E_m$) or unstable ($\mathrm{Im}(E_0) < E_m$) regions. The self-healing property is measured by the long-time behavior of $\epsilon(t)$ [Eq. (2)]. Figure 3 illustrates the typical numerical results of wave propagation in the lattice, corresponding to the self-healing of the skin mode for $\mathrm{Im}(E_0) > E_m$ [Fig. 3(a)], and to the disruption of the skin mode for $\mathrm{Im}(E_0) < E_m$ [Fig. 3(b)]. The results are obtained by solving Eq. (1) in Wannier (real-space) basis by an accurate fourth-order Runge-Kutta method on a finite-sized lattice with OBC and with a size wide enough ($N = 300$ sites) to avoid right-edge effects over the largest propagation time ($t \sim 20$), which would destroy the SIBC skin state [27,109]. A strategic method to selectively prepare the system in a self-healing SIBC edge state is discussed in [109] and in Sec. 5 of [103]. As is clearly shown in the left panel of Fig. 3(a), the strongly absorbing potential cuts the excitation at lattice sites $n \leq L$, however, after the scattering process the skin edge state can restore its original shape, corresponding to a vanishing of $\epsilon(t)$ [right panel of Fig. 3(a)]. A different behavior is observed in Fig. 3(b), where the skin edge state cannot restore its original shape, and $\epsilon(t)$ does not decay toward zero. We checked [103] that the self-healing property can also be observed when there

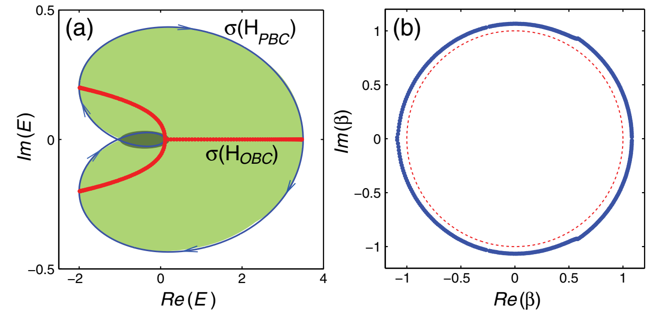

FIG. 2. (a) Energy spectrum of H_{PBC} (outer thin closed loop with one self-intersection), H_{OBC} (inner bold open arcs) and H_{SIBC} (shaded areas) of a NH lattice with nearest- and next-to-nearest neighbor hopping amplitudes $t_{-2} = 1$, $t_{-1} = 1$, $t_0 = 0$, $t_1 = 0.7$, and $t_2 = 0.8$. In the light shaded area, $W(E) = -1$, corresponding to simple (nondegenerate) skin edge state, whereas in the dark shaded area, $W(E) = -2$, corresponding to the existence of two energy-degenerate skin edge states of H_{SIBC}. The largest value E_{m_1} of $\mathrm{Im}[\sigma(H_{\mathrm{OBC}})]$ is $E_{m_1} = 0.2$. (b) The numerically computed GBZ C_β, corresponding to a deformed circle with $|\beta| > 1$ all along C_β. The thin dashed curve depicts the reference unit circle $|\beta| = 1$.

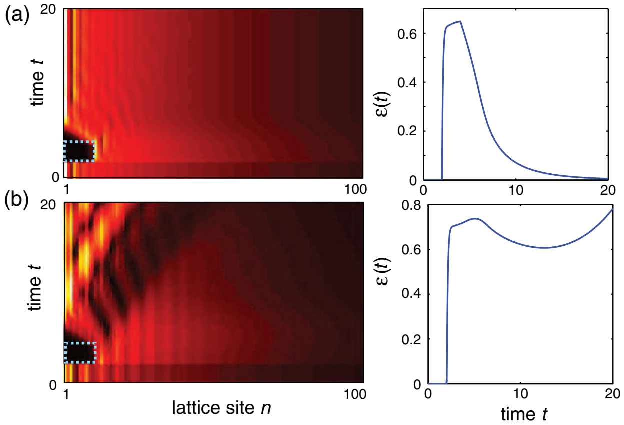

FIG. 3. Self-healing of topological skin edge states. The left panels show the temporal evolution of the modulus of the normalized amplitudes $\tilde{\psi}_n(t)$ on a pseudocolor map, in a semi-infinite lattice with parameter values as in Fig. 2 and with an absorbing scattering potential (obstacle) localized in the dotted rectangular region of the space-time plane ($n \leq 10$ and $2 < t < 4$). The initial state $\psi_n(0)$ is the skin edge mode with energy $E_0 = 0.35i$ in (a), and $E_0 = -1 + 0.05i$ in (b). The right panels show the corresponding temporal evolution of the function $\epsilon(t)$, defined by Eq. (2), which measures the deviation of the evolved wave function from the skin state.

are Bloch points (the GBZ crosses the unit circle) and for different types of scattering potentials, including inhomogeneous Hermitian and non-Hermitian amplifying potentials.

Multiband systems.—The previous analysis has been focused to single band models, however, the self-healing property of topological skin edge states can be extended to multiband systems. As an illustrative example, we consider a quasi-1D lattice composed of two side-coupled Hatano-Nelson chains [110] [Fig. 4(a)], which displays the critical NH skin effect [64]. The Bloch Hamiltonian of the system reads

$$H(k) = \sigma_0 d_0 + t_0 \sigma_x + [V + i(\delta_b - \delta_a)\sin k]\sigma_z, \quad (5)$$

where $d_0 = 2t_1 \cos k - i(\delta_a + \delta_b)\sin k$, σ_l are the Pauli matrices, $(t_1 \pm \delta_{a,b})$ are the asymmetric left or right hopping amplitudes in the upper (a) and lower (b) chains, $\pm V$ their on-site energy offset, and t_0 is the side coupling constant. Figures 4(b) and 4(c) show a typical behavior of GBZ and energy spectra (PBC, OBC, and SIBC) for $\delta_a > 0$, $\delta_b < 0$, with the shaded region corresponding to topological skin edge states localized at the left boundary under SIBC. Self-healing skin edge states are those with energy E satisfying the condition $\mathrm{Im}(E) > E_m$, with $E_m = \max(E_{m_1}, E_{m_2}) = E_{m1} \simeq 0.255$. The self-healing property is illustrated in Fig. 4(d), where a skin edge state is scattered off by a complex absorbing potential in both chains [$V_n(t) = 10i$ for $4 < t < 8$ and $1 \leq n \leq 10$, $V_n = 0$, otherwise].

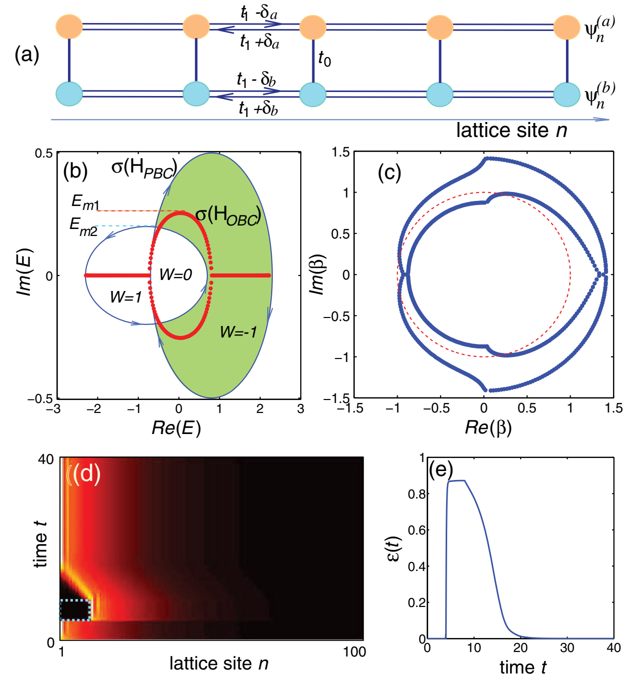

FIG. 4. (a) Scheme of two side-coupled Hatano-Nelson chains. (b) PBC (thin solid curves), OBC (solid dots), and SIBC (shaded area) energy spectra for $t_1 = 0.75$, $\delta_a = 0.25$, $\delta_b = -0.15$, $t_0 = 0.05$, and $V = 0.8$. The two PBC Bloch bands form two closed loops which are traveled in opposite directions, leading to three possible values $0, \pm 1$ of the winding W in their interior. For any energy E in the shaded area ($W = -1$) there is one topological edge state at the left boundary of the lattice. (c) Diagram of the GBZ (solid dots). The thin dashed curve shows the unit circle as a reference. (d),(e) Self-healing of the topological edge state with energy $E = 1 + 0.4i$. (d) Evolution of the normalized amplitudes $\sqrt{|\psi_n^{(a)}|^2 + |\psi_n^{(b)}|^2} / \sum_n \sqrt{|\psi_n^{(a)}|^2 + |\psi_n^{(b)}|^2}$, where $\psi_n^{(a)}$ and $\psi_n^{(b)}$ are the wave amplitudes at site n in the two chains (a) and (b), respectively. (e) Temporal behavior of $\epsilon(t)$. The absorbing scattering potential is localized in the dotted rectangular region of the space-time plane.

Conclusion.—In summary, we have demonstrated that infinitely many topological edge skin modes in semi-infinite NH lattices can exhibit self-healing properties, i.e., they can reconstruct their shape after being scattered off by a rather arbitrary space-time potential. Contrary to self-healing waves known in Hermitian systems, such as Bessel and Airy waves, the topological skin edge states are truly normalizable eigenstates of the underlying Hamiltonian. Our results unravel a fascinating fundamental property of recently discovered topological skin modes, extend the idea of self-healing waves beyond the diffraction-free paradigm of Hermitian physics, and, thus, could be of potential relevance in different areas of physics and for future applications of self-healing NH waves.

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