Cluster Expansion and Resurgence in the Polyakov Model

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In the Polyakov model, a nonperturbative mass gap is formed at leading-order semiclassics by instanton effects. By using the notions of critical points at infinity, cluster expansion, and Lefschetz thimbles, we show that a third-order effect in semiclassics gives an imaginary ambiguous contribution to the mass gap, which is supposed to be real and unambiguous. This is troublesome for the original analysis, and it is difficult to resolve this issue directly in quantum field theory (QFT). However, we find a new compactification of the Polyakov model to quantum mechanics, by using a background 't Hooft flux. The compactification has the merit of remembering the monopole instantons of the full QFT within Born-Oppenheimer approximation, while the periodic compactification does not. In the quantum mechanical limit, we prove the resurgent cancellation of the ambiguity in three-instanton sector against ambiguity in the Borel resummation of the perturbation theory around one instanton. Assuming that this result holds in QFT, we provide a large-order asymptotics of perturbation theory around perturbative vacuum and instanton.

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Introduction.—The Polyakov model is a prototypical example of nonperturbatively calculable weakly coupled quantum field theory [1]. It is by now standard textbook material in quantum field theory (QFT) and condensed matter physics, and it is also intimately tied with statistical field theory of Coulomb gases [2–5]. Despite the fact that some fundamental facts about the theory have been known for more than four decades now, after the advent of resurgence [6–8] and Lefschetz thimbles [9], many subtle issues emerged concerning this and other calculable theories. An important issue is the following. It is known that the mass gap in the theory is sourced by monopole instantons on $\mathbb{R}^3$ and is of the order of $m_g^2 \sim e^{-S_0}$, where S_0 is monopole-instanton ($\mathcal{M}_{\alpha_i}$) action, with magnetic charge $\alpha_i \in \Delta^0$ in the simple root system. How does one incorporate saddles with higher action $(nS_0, n \geq 2)$? Should one care about them? Do they contribute to mass gap? This class of questions is usually not addressed and swept under the rug by assuming that these are higher-order quantitative corrections, and not important [1,4], even in more mathematical treatments of the theory [2]. Of course, there are also theories in which mass gap is induced at higher-order effect in monopole expansion due to Berry phase [10] or topological theta angle [11] induced destructive interference between leading monopole events or as an effect of index theorem [12]. But here we address the above questions in a simple Polyakov model, where higher-order effect just seem like a nuisance, by using the concepts of critical point at infinity, quasi-zero-mode Lefschetz thimbles, and cluster expansion [13] systematically.

Here is the main point of our analysis. The mass gap in semiclassical expansion in the Polyakov model is of the following form (ignoring inessential factors to lessen the clutter):

$$(m_g^2)_\pm \sim (e^{-S_0}P_1 + e^{-2S_0}P_2 + e^{-3S_0}P_3 + \cdots) \pm i(e^{-3S_0} + \cdots), \tag{1}$$

where P_i denotes perturbative expansions around the relevant saddle. It is reasonable to drop $O(e^{-2S_0})$ terms in the real part of this analysis, as they provide only minor quantitative corrections. But, as we emphasize, there is a new effect in third order in semiclassics, which renders the semiclassical expansion multifold ambiguous and void of meaning. It is actually not correct to ignore $\pm i(e^{-3S_0})$, because it is an effect of a different nature, giving mass an imaginary ambiguous part. Therefore, one is entitled to ask whether Polyakov's analysis is rigorous enough even within semiclassics. In particular, for the famous result on mass gap to be justified, one needs a mechanism for the cancellation of imaginary ambiguity on $\mathbb{R}^3$.

The type of ambiguities that appear in Eq. (1) is, in fact, expected. The reason for this is because $\mathrm{Arg}(\hbar) = 0$ and $\mathrm{Arg}(\hbar) = \pi$ are, in general, Stokes lines. On Stokes lines, contributions of a subclass of saddles can indeed be multifold ambiguous. In fact, there are infinitely many critical points at infinity, and, generically, there are multifold ambiguities. It is desirable to resolve these pathological features in order to make Polyakov's solution meaningful in this new light.

According to resurgence, there is another ambiguity in the story. Perturbation theory around a perturbative vacuum saddle, one-instanton saddle, etc., are all expected to be divergent asymptotic expansions [14–18]. They are also expected to be non-Borel summable, meaning that Borel resummation of perturbation theory around each saddle is multifold ambiguous. Resurgence implies that, for the theory to be meaningful, these two types of ambiguities must cancel around each sector of the theory. However, demonstrating this in a generic QFT is hard. It is possible to take mileage on this problem using the idea of adiabatic continuity and turning on background fields [6–8], by working with QFTs with special properties such as integrability [19,20], or working with rather special QFTs in which one has a good knowledge of perturbation theory [21]. In this work, we employ a 't Hooft flux background (couple the theory to a one-form symmetry background) to tackle this problem in the Polyakov model.

Basic.—The Polyakov model is given as an SU(N) non-Abelian gauge theory coupled to an adjoint scalar field in a 3D Euclidean space:

$$S = \int d^3x \frac{1}{2g_3^2}[\mathrm{tr}F^2 + \mathrm{tr}(D\varphi)^2 + \lambda V(\varphi)], \tag{2}$$

where the potential $V(\varphi)$ leads to the Abelianization of gauge dynamics down to $\mathrm{U}(1)^{N-1}$. We assume without loss of generality that the eigenvalues of ϕ are uniformly separated: $v_i - v_{i+1} = v$. In the $\lambda \ll 1$ limit, the theory has saddles which are solutions to (anti-)self-duality equations $F = \pm\star_3 D\varphi$ with topological (magnetic) charges $Q_{M_i} = (2\pi/g_3)\alpha_i$ and actions $S_0^{(i)} = (4\pi v/g_3^2) \equiv (s_0/g^2)$, where $\alpha_i \in \Delta^0$ are $N-1$ simple roots, where $g^2 = g_3^2/v$ is a dimensionless expansion parameter. The monopole operators are $\mathcal{M}_{\alpha_i} \sim (S_0)^2 e^{-S_0} e^{-(4\pi/g^2)\tilde{\boldsymbol{\phi}}(x)\cdot\boldsymbol{\alpha}_i + i\boldsymbol{\alpha}_i\cdot\boldsymbol{\sigma}(x)}$, where $\tilde{\boldsymbol{\phi}}(x)$ and $\sigma(x)$ are fluctuations of Cartan components of adjoint scalar and dual photon, respectively, and $(S_0)^2$ arise from the four zero modes of the monopole. The former can be set to zero in the description of the long-distance theory. The proliferation of monopoles generates a mass gap for gauge fluctuations as discovered by Polyakov [1]; see also [2,22].

Critical points at infinity and cluster expansion.—The model, apart from regular saddles, also possesses critical points at infinity [23–25]. These critical points are very likely one of the most important concepts in semiclassics; yet, there is a minuscule amount of work on systematizing them or a heavy burden of misunderstandings emanating from 1970s related to them; see [23,24]. These configurations, to our knowledge, are not addressed at all in the context of Polyakov model and will play an important role below.

Consider a monopole-monopole or monopole-antimonopole pair. The interaction between the two in the $\lambda \to 0$ limit is

$$V_{\mathrm{int}}(r) = \begin{cases} 0 & \text{for } (\mathcal{M}_i, \mathcal{M}_j), \\ \frac{2\pi}{g^2}\frac{\alpha_i\cdot(-\alpha_j)}{r}, & \text{for } (\mathcal{M}_i, \bar{\mathcal{M}}_j). \end{cases} \tag{3}$$

Thus, $(\mathcal{M}, \mathcal{M})$ do not interact [26,27], but $(\mathcal{M}, \bar{\mathcal{M}})$ pairs interact, attractively for $i = j$ and repulsively for $j = i \pm 1$. At any finite separation, since $V'_{\mathrm{int}}|_{r<\infty} \neq 0$, these pairs are not exact solutions. But at $r = \infty$, they become exact solutions, hence the name. Such configurations are genuine critical points, but they are non-Gaussian, i.e., $V''_{\mathrm{int}}|_{r=\infty} = 0$, unlike Gaussian saddles. Because of this property, one needs to integrate over the whole steepest descent cycle to find the effect of such pairs.

The integrals that give the contribution of the second-order effects in semiclassics are of the form

$$Z_2 = [\mathcal{M}_i][\bar{\mathcal{M}}_j]\int d^3r_1 d^3r_2 e^{-V_{\mathrm{int}}(|\mathbf{r_1}-\mathbf{r_2}|)}. \tag{4}$$

The measure can be expressed in terms of the center-of-action coordinate, the integral over which gives a space-time volume factor $\mathcal{V}$, and the relative coordinate, which corresponds to quasi-zero-mode direction: $[\mathcal{M}_i][\bar{\mathcal{M}}_j]\mathcal{V}4\pi\int dr r^2 e^{-V_{\mathrm{int}}(r)}$. This type of integral appears in the standard cluster expansion in statistical field theory [13].

In semiclassics, steepest descent cycles are not necessarily real. Let us call $r \to z \in \mathbb{C}$. The steepest descent cycles can end up at points where $e^{-f(z)/g^2} \to 0$. For polynomial $f(z)$, this gives a homology cycle decomposition of the integration [9], in terms of cycles that end at infinity in certain wedges. For the Coulomb potential, the end point is a pole at $z = 0$ similar to Ref. [28], and the critical point is at $z = \infty$. For $e^{-1/(g^2 z)}$, the cycle must enter to the pole in the $\arg(z) = 0$ direction, while for $e^{+1/(g^2 z)}$, the cycle must enter to the pole in the $\arg(z) = \pi$ direction for steepest descent. So, the steepest descent directions for the attractive and repulsive potentials are different.

Steepest descent cycles are easiest to visualize if we map $r \equiv z \in \mathbb{C}$ complex domain to a Riemann sphere by using one-point compactification. To see the thimbles more clearly, we can introduce a regulator for the integral, find the steepest descent, and ultimately remove the regulator. For the two interaction types, the cycles for $\arg(g^2) = 0^\pm$ are given by

$$\begin{aligned} &\mathcal{J}_1(0)\colon z \in [0, \infty], \quad \text{repulsive}, \\ &\mathcal{J}_2(0^\pm)\colon z \in [0, -\infty] \cup [C_\infty^{\circlearrowleft} \quad \text{or} \quad C_\infty^{\circlearrowright}], \quad \text{attractive}, \end{aligned} \tag{5}$$

and are shown in Fig. 1. The integrations are given by

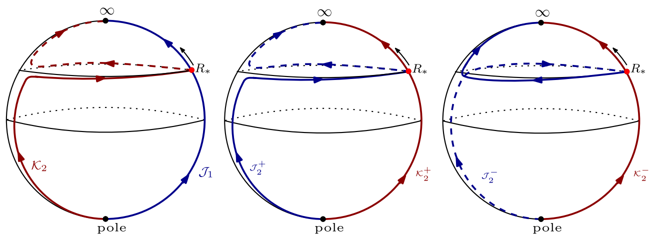

FIG. 1. $\mathcal{J}_1$ and $\mathcal{J}_2^\pm$ are the (regularized) steepest descent cycles for repulsive and attractive interaction. The former is unique, and the latter is twofold ambiguous at $\arg(g^2) = 0$. The point R^* tends to infinity as the cutoff is removed, but the loops at infinity remain.

$$\begin{aligned} Z_2 &\sim \xi^2 \mathcal{V}^2 \quad \text{noninteracting}, \\ Z_2 &\sim \xi^2 \mathcal{V}[\mathcal{V} + I(g^2)] \quad \text{repulsive}, \\ Z_{2,\pm} &\sim \xi^2 \mathcal{V}[\mathcal{V} + I(g^2 e^{\pm i\pi})] \quad \text{attractive}, \end{aligned} \tag{6}$$

where the extensive part corresponds to free (noninteracting) monopole gas with fugacity ξ and $I(g^2)$ in the subextensive term is called the second virial coefficient, capturing the effect of interactions. It is given by

$$I(g^2) = \frac{4\pi}{6}\left(\frac{2\pi|\alpha_i \cdot \alpha_j|}{g^2}\right)^3 \left[\ln\left(\frac{2\pi|\alpha_i \cdot \alpha_j|}{g^2}\right) + \gamma - \frac{11}{6}\right]. \tag{7}$$

In the repulsive case, the subextensive part is usually called magnetic bion, and its amplitude is $[\mathcal{M}_{\alpha_i}\bar{\mathcal{M}}_{\alpha_{i\pm1}}] = I(g^2)[\mathcal{M}_{\alpha_i}][\bar{\mathcal{M}}_{\alpha_{i\pm1}}]$ [29]. In statistical physics, this is a configuration in two-cluster, $\mathcal{C}_2$.

Attractive case is more interesting. First, note that

$$I(g^2 e^{\pm i\pi}) = e^{i\pi} I(g^2) \pm i\frac{2\pi^2}{3}\left(\frac{2\pi|\alpha_i \cdot \alpha_j|}{g^2}\right)^3, \tag{8}$$

which implies two different remarkable phenomena for this element of $\mathcal{C}_2$. First, we identify the subextensive part in the attractive case with neutral bions: $[\mathcal{M}_{\alpha_i}\bar{\mathcal{M}}_{\alpha_i}]_\pm = I(g^2 e^{\pm i\pi})[\mathcal{M}_{\alpha_i}][\bar{\mathcal{M}}_{\alpha_i}]$ [29]. The contribution is twofold ambiguous. This is expected, because we are formulating the path integral on a Stokes line, and the configurations with attractive interactions are expected to lead to twofold ambiguous results. At least in some limit of QFT, we will prove that this twofold ambiguity cancels against another ambiguity.

The overall phase in front of Eq. (8) is equally interesting. It tells us that the fugacity of the two-cluster elements can be complex:

$$\mathrm{Arg}([\mathcal{M}_{\alpha_i}\bar{\mathcal{M}}_{\alpha_{i\pm1}}]) = \mathrm{Arg}(\mathrm{Re}[\mathcal{M}_{\alpha_i}\bar{\mathcal{M}}_{\alpha_i}]_\pm) + \pi. \tag{9}$$

This is in some sense similar to Refs. [10,11], where there is a relative topological phase (sourced by Berry phase or θ angle) between monopole events. This relative phase between the two contributions is now sourced by the phase associated with the thimble and is called the hidden topological angle [30]. It is known to play a crucial role in semiclassics. For example, in pure supersymmetric gauge theory on $\mathbb{R}^3 \times S^1$ and supersymmetric quantum mechanics, the vanishing of the vacuum energy is due to the relative phase between these two types of nonperturbative saddle. For other aspects of bions, see Refs. [8,12,18,31–38].

At third order in semiclassics, events such as $[\mathcal{M}_{\alpha_i}\bar{\mathcal{M}}_{\alpha_i}\mathcal{M}_{\alpha_i}]_\pm = O(e^{-3S_0}) \pm iO(e^{-3S_0})$ provide a twofold ambiguous contribution to mass gap, which is supposed to be real and unambiguous. The fact that the ambiguity in mass first appears in the third order comes from the structure of the resurgence triangle [6]. As it stands, this is quite disturbing for Polyakov's well-known solution.

We can anticipate that these ambiguities in two-event and three-event contributions must be related with the non-Borel summability of the perturbation theory around perturbative vacuum and one-instanton sector, respectively. The left-right Borel resummation is twofold ambiguous as well, and these two types of ambiguities are expected to cancel.

However, it is difficult to test this scenario in full QFT. As we describe, it is also not possible to address this question by using a naive dimensional reduction of QFT to quantum mechanics within Born-Oppenheimer approximation. However, we propose a compactification with discrete 't Hooft flux in which monopole actions remain the same and such cancellations in quantum mechanical limit can be shown.

Periodic T^2 compactification does not work.—One may consider compactification on $T^2 \times \mathbb{R}$ and study the interplay of instantons and perturbation theory in quantum mechanical reduction. However, a problem awaits us here. The states in quantum mechanics are described in terms of magnetic flux through T^2, $|\Phi\rangle$. The lowest-energy state is zero flux state $|\mathbf{0}\rangle$, and magnetic flux states with nonzero flux $|\boldsymbol{\alpha_a}\rangle$ are parametrically separated in energy: $E_0 = 0$ and $E_{\boldsymbol{\alpha_a}} = (\Phi^2/2A_{T^2})$, where A_{T^2} is the area of the torus [3]. Therefore, the naively reduced quantum mechanics possesses a unique perturbative vacuum and does not have instantons. It is not possible to obtain knowledge concerning QFT from naive dimensional reduction with periodic compactification in Born-Oppenheimer approximation.

However, despite being correct, this is a little bit oversimplified. For example, for SU(2) gauge theory, the flux states are $|0\rangle$, and perturbatively degenerate pairs of states, $|\pm 1\rangle, |\pm 2\rangle$, etc., $|+1\rangle$ mix up with $|-1\rangle$ due to two instanton effects with action $2S_0$, where S_0 is the monopole action in QFT. Therefore, we can try to engineer a vacuum structure by turning on background fluxes, such that the instantons of QFT survive in the ground state description of quantum mechanics.

Reducing to quantum mechanics with 't Hooft flux.—The 3D theory has a $\mathbb{Z}_N^{[1]}$ 1-form symmetry but no mixed anomalies. We can turn on a discrete flux to examine the dynamics of the theory [39–52]. Since the model is Abelianized at long distances, we can replace our way of thinking in terms of discrete flux with a magnetic flux in coweight lattice thanks to the relation $\mathbb{Z}_N \cong \Gamma_w^\vee/\Gamma_r^\vee$. Turning on the background flux, $\boldsymbol{\mu}_1 \in \Gamma_w^\vee$, we end up with N degenerate states:

$$|\boldsymbol{\nu}_1\rangle \underbrace{\longrightarrow}_{-\boldsymbol{\alpha}_1} |\boldsymbol{\nu}_2\rangle \underbrace{\longrightarrow}_{-\boldsymbol{\alpha}_2} \cdots \underbrace{\longrightarrow}_{-\boldsymbol{\alpha}_{N-1}} |\boldsymbol{\nu}_N\rangle \tag{10}$$

connected to each other via monopole events $\alpha_a \in \Delta^0$. Below, we argue that these instanton events have the same action as in $\mathbb{R}^3$. Assume T^2 size L obeys

$$r_m \ll L \ll d_{mm}, \tag{11}$$

where r_m is the monopole core size and d_{mm} is the characteristic distance between monopoles on $\mathbb{R}^3$. This guarantees that, at a fixed Euclidean time slice τ and per T^2 size, there will typically be at most one monopole. We also choose $L \gg r_m$ so that the theory is locally 3D, and the action, which receives its contribution from the core region of monopole, remains unchanged relative to $\mathbb{R}^3$. At distances $\tau \gtrsim L$, the theory is correctly described by a simple quantum mechanical system with instantons, whose action are same as in the original 3D theory.

Here, we focus on $N = 2$. The insertion of $\Phi_{\mathrm{bg}} = \frac{1}{2}$ modifies the energetics of the setup. It turns the flux states into $|n + \frac{1}{2}\rangle$ with perturbative energies $E_n = (n + \frac{1}{2})^2/2A_{T^2}$, and the fractional flux states $|\pm\frac{1}{2}\rangle$ become perturbatively degenerate vacua. The tunneling between them is an instanton effect with $\Delta\Phi = 1$ and action S_0, the same as the instanton in full QFT. Note that the energy of the states $|\pm(3/2)\rangle$ is higher, and we are justified to drop them, and the SU(2) Polyakov with 't Hooft flux reduces to a simple double-well potential.

This story sounds almost identical to a particle on a circle in the presence of magnetic flux and a potential $\cos(2q)$ leading to two harmonic minima. At $\theta = e\Phi_{\mathrm{bg}} = \pi$, all states are twofold degenerate even nonperturbatively because of mixed anomaly between $\mathbb{Z}_2$ translation symmetry and time reversal symmetry [51,53]. In fact, SU(2) Polyakov model and SU(2) deformed Yang-Mills theories reduce to a double-well potential with configuration space $\mathbb{R}$ and S^1 [52,54], respectively. The former does not have a mixed anomaly, and the latter does. The twofold degeneracy in quantum mechanics is the remnant of CP broken vacua of Yang-Mills theory.

In our reduced Polyakov model with flux, the twofold degeneracy is lifted nonperturbatively, and the ground and first excited states are separated by a single monopole-instanton effect:

$$|\Psi_0\rangle = \frac{1}{\sqrt{2}}\left(\left|\frac{1}{2}\right\rangle + \left|-\frac{1}{2}\right\rangle\right), \quad |\Psi_1\rangle = \frac{1}{\sqrt{2}}\left(\left|\frac{1}{2}\right\rangle - \left|-\frac{1}{2}\right\rangle\right),$$
$$\Delta E = 2Ke^{-S_0}, \tag{12}$$

where S_0 is monopole action. Therefore, we claim that the resurgence properties in the quantum mechanical limit of the Polyakov model with flux are dictated by the same instanton action as in $\mathbb{R}^3$.

In quantum mechanics, for the double-well potential, the following resurgent cancellations are already proven by multiple methods [16,55–57]:

$$\begin{aligned} &\mathrm{Im}[\mathcal{S}_\pm P_0 + [\mathcal{M}_{\alpha_i}\bar{\mathcal{M}}_{\alpha_i}]_\pm + \cdots] = 0, \\ &\mathrm{Im}[[\mathcal{M}_{\alpha_i}]\mathcal{S}_\pm P_1 + [\mathcal{M}_{\alpha_i}\bar{\mathcal{M}}_{\alpha_i}\mathcal{M}_{\alpha_i}]_\pm + \cdots] = 0. \end{aligned} \tag{13}$$

Here,

$$P_0(g^2) = \sum_{k=0}^{\infty} b_k^{(0)} g^{2k}, \qquad P_1(g^2) = \sum_{k=0}^{\infty} b_k^{(1)} g^{2k} \tag{14}$$

are divergent asymptotic expansions around a perturbative vacuum and one-instanton saddle, and $\mathcal{S}_\pm$ indicate the lateral Borel resummations. These series are non-Borel summable; i.e., the sum has an imaginary twofold ambiguous part indicated by subscript $\pm$. As described above, the two-events and three-events in Eq. (13) also have twofold ambiguities. For the combination of the perturbation theory and the semiclassical analysis to be meaningful and ambiguity free, these two types of ambiguities must cancel, and they do. In fact, exact WKB analysis proves that these cancellations and their generalizations hold true in all nonperturbative sectors of quantum mechanics [56,57]. Note that we work in the $\lambda \to 0$ limit and investigate resurgent structure in g^2 only. This make sense provided $\lambda \ll g^2$. At finite λ, one needs a double series, and, relatedly, the action acquire a λ dependence, $S_0 = (4\pi/g^2)f(\lambda)$ [58].

Back to $\mathbb{R}^3$.—Resurgent cancellations (13) are not easy to prove in full QFT on $\mathbb{R}^3$. But they are proven in the small $T^2 \times \mathbb{R}$ with discrete flux, a construction in which instantons of infinite volume theory survive. In QFT, what we know rigorously is the existence of ambiguity in the correlated events. The imaginary ambiguous parts at the second and third order in QFT on $\mathbb{R}^3$ are given by

$$\begin{aligned} &\mathrm{Im}[\mathcal{M}_{\alpha_i}\bar{\mathcal{M}}_{\alpha_i}]_\pm \sim \pm i\left(\frac{s_0}{g^2}\right)^7 e^{-2s_0/g^2}, \\ &\mathrm{Im}[\mathcal{M}_{\alpha_i}\bar{\mathcal{M}}_{\alpha_i}\mathcal{M}_{\alpha_i}]_\pm \sim \pm i\left(\frac{s_0}{g^2}\right)^{12} \ln\left(\frac{s_0}{g^2}\right) e^{-3s_0/g^2}. \end{aligned} \tag{15}$$

Here, the power of $(\frac{s_0}{g^2})$ is $2\nu + 3(\nu - 1)$ for $\nu = 2, 3, \ldots$ is the number of instantons that enter to correlated events. Recall that each monopole has four bosonic zero modes, and each zero mode induces $(s_0/g^2)^{1/2}$ in the prefactor, explaining 2ν. Each quasi-zero-mode direction gives a factor of 3 and, hence, $3(\nu-1)$. For a general ν-instanton configuration, the power of $\ln\frac{s_0}{g^2}$ is given by $\nu-2$.

In order for Polyakov's result for mass gap to be meaningful (real, unambiguous), the counterpart of Eq. (13) *must* hold in full QFT. With this assumption and using dispersion relations such as $b_k^{(0)} = (1/\pi)\int_0^\infty d(g^2)(\mathrm{Im}\mathcal{E}_{\mathrm{np}}(g^2)/(g^2)^{k+1})$, we can determine the large-order growth of perturbation theory around a perturbative vacuum and monopole saddle as

$$b_k^{(0)} \sim \frac{\Gamma(k+7)}{(2s_0)^k}, \qquad s_0 = 4\pi,$$
$$b_k^{(1)} \sim \frac{\Gamma(k+10)\ln(k+10)}{(2s_0)^k}. \tag{16}$$

A few remarks are in order. Relative to the standard quantum mechanical result for $b_k^{(0)}$ and $b_k^{(1)}$, where the factorial growth appears generally as $\Gamma(k+1)$ for ground state, we obtain an enhancement. This is due to the difference of the number of zero and quasizero modes in the two setups. In $b_k^{(1)}$, there is an extra $\ln(k+10)$ enhancement as well. The log enhancement also appears in the context of quantum mechanics around instanton sectors [59–61]. It is there because the three-instanton amplitude has a $[\ln(-s_0/g^2)]^2$ in it, coming from integrating out of two quasizero modes, which leads to a log-dependent imaginary part in Eq. (15). It is quite curious to note that $b_k^{(1)} \sim (d/d\nu)b_{k-2}^{(0)}|_{\nu=2}$ asymptotically, reminiscent of an exact relation in quantum mechanics [17,61]. The analogous relation in quantum mechanics tells us that perturbation theory around an instanton is dictated by perturbation theory around a perturbative vacuum via a simple formula. It would be remarkable if such a relation also holds in QFT. The appearance of this enhancement is a relatively new effect in QFT, an example of which also appeared in Ref. [62]. But, in retrospect, it is inevitable and generic. Application of stochastic perturbation theory on a lattice can be useful to check these predictions [63–65], especially by modifying the scalar potential in Ref. [64] to generate adjoint Higgsing and monopole confinement.

Conclusions.—In order for Polyakov's famous analysis for mass gap to be justified, one needs Eq. (13) to be true on $\mathbb{R}^3$. We showed that the relation (13) is true in a special quantum mechanical reduction of the Polyakov model with discrete 't Hooft flux. The reduction has the merit that it remembers the instanton of the theory on $\mathbb{R}^3$ on small $T^2\times\mathbb{R}$ in the description of the ground state. We emphasize that the use of 't Hooft flux or twisted boundary conditions is not a nuisance regardless of the absence or presence of a mixed anomaly. To the contrary, it is necessary to make the instantons (or fractional instantons in gauge theory on $\mathbb{R}^3\times S^1$) transparent in a quantum mechanical reduction [44,49,52,66]. Assuming that Eq. (13) continues to be valid in the decompactification limit provides estimates of large-order structure of perturbation theory around perturbative vacuum and instanton (16). We hope that these relations can be tested using stochastic perturbation theory, putting Polyakov's analysis on a firmer ground.

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